Torsion of a Viscoelastic Cylinder

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Finite torsional deformations of an incompressible viscoelastic circular cylinder are studied with its material modeled by two constitutive relations. One of these is a linear relation between the determinate part of the second Piola-Kirchhoff stress tensor and the time history of the Green-St. Venant strain tensor, and the other a linear relation between the deviatoric Cauchy stress tensor and the left Cauchy-Green tensor, its inverse, and the time history of the relative Green-St. Venant strain tensor. It is shown that the response predicted by the latter constitutive relation is in better agreement with the test data, and this constitutive relation is used to compute energy dissipated during torsional oscillations of the cylinder. [S0021-8936(00)00502-X]

Batra and Yu [1] recently studied the stress relaxation in an isotropic, incompressible, and homogeneous viscoelastic body deformed either in finite simple shear or finite simple extension. The material response was modeled by two constitutive relations, one linear in the history of the Green-St. Venant strain tensor $\mathbf{E}$ (e.g., see Christensen [2]) and the other linear in the history of the relative Green-St. Venant strain tensor $\mathbf{E}_t$ (e.g., see Bernstein, Kearsley, and Zapas [3] and Fosdick and Yu [4]). For each one of the two deformations studied, the former constitutive relation predicted that the tangent modulus (i.e., the slope of the stress-strain curve) is an increasing function of the strain but according to the latter constitutive relation, the tangent modulus is a nonincreasing function of the strain which agrees with the behavior observed experimentally for most materials (e.g., see Bell [5]). A similar result had been obtained by Batra [6] for two linear constitutive relations in isotropic finite elasticity. We note that both simple shearing and simple extension are homogeneous deformations and are universal in the sense that they can be produced by surface tractions alone in every elastic or viscoelastic body. Batra [7] has recently compared the response predicted by four linear constitutive relations for finite shearing, finite extension, biaxial loading, and triaxial loading of an isotropic elastic body.

Here we study finite torsional deformations of an incompressible, homogeneous, and isotropic viscoelastic circular cylinder. Even though these deformations are inhomogeneous, Ericksen [8] and Carroll [9] have shown that they are universal for elastic and viscoelastic bodies, respectively. In cylindrical coordinates, torsion of a circular cylinder is described by $r=R$, $\theta=\Theta+\kappa Z$, $z=Z$, where (r,θ,z) denote cylindrical coordinates of a point in the present configuration that occupied the place (R,Θ,Z) in the stress-free reference configuration, and κ is the angle of twist per unit length of the cylinder. Relative to an orthonormal set of bases, the physical components of the deformation gradient $\mathbf{F}$, the left Cauchy-Green tensor $\mathbf{B}$, and tensors $\mathbf{E}$ and $\mathbf{E}_t$ are given by

$$\mathbf{F}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & \kappa r\\ 0 & 0 & 1\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1+\kappa^2 r^2 & \kappa r\\ 0 & \kappa r & 1\end{bmatrix}, \qquad (1)$$

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$$\mathbf{E}=\frac{1}{2}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \kappa r\\ 0 & \kappa r & \kappa^2 r^2\end{bmatrix},$$

$$\mathbf{E}_t(\tau)=\frac{1}{2}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & (\kappa(\tau)-\kappa(t))r\\ 0 & (\kappa(\tau)-\kappa(t))r & (\kappa(\tau)-\kappa(t))^2 r^2\end{bmatrix}. \quad (2)$$

We model the material by the following two constitutive relations (e.g., see Christensen [10], Bernstein et al. [3], and Fosdick and Yu [4])

$$\bar{\mathbf{T}}=-p\mathbf{1}+\rho\mathbf{F}\dot{\bar{\psi}},_{\mathbf{E}}\mathbf{F}^T, \quad (3a)$$

$$\mathbf{T}=-p\mathbf{1}+\rho\mathbf{F}(\psi,_{\mathbf{F}})^T, \quad (3b)$$

where a superimposed dot indicates the material time-derivative $\mathbf{C}=\mathbf{F}^T\mathbf{F}$, $\mathbf{B}=\mathbf{F}\mathbf{F}^T$, $2\mathbf{E}=(\mathbf{C}-\mathbf{1})$, $\bar{\psi},_{\mathbf{E}}\equiv\partial\bar{\psi}/\partial\mathbf{E}$, $\psi,_{\mathbf{F}}\equiv\partial\psi/\partial\mathbf{F}$,

$$\rho\bar{\psi}=\int_{-\infty}^{t}\int_{-\infty}^{t}G_1(t-\tau,t-\eta)\frac{\partial tr\mathbf{E}(\tau)}{\partial\tau}\frac{\partial tr\mathbf{E}(\eta)}{\partial\eta}d\tau d\eta+\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t}G_2(t-\tau,t-\eta)tr\left(\frac{\partial\mathbf{E}(\tau)}{\partial\tau}\frac{\partial\mathbf{E}(\eta)}{\partial\eta}\right)d\tau d\eta, \quad (4)$$

$$\rho\psi=\frac{1}{2}\beta_1 tr\mathbf{B}+\frac{1}{2}\beta_{-1}tr\mathbf{B}^{-1}+\int_{-\infty}^{t}g(t-\tau)\frac{\partial tr\mathbf{E}_t(\tau)}{\partial\tau}d\tau.$$

Here, $\mathbf{T}$ is the Cauchy stress tensor; p the hydrostatic pressure not determined by the deformation; ρ the mass density; $\bar{\psi}$ and ψ are specific (per unit mass) strain energy functionals; and $g(\cdot)$, $G_1(\cdot,\cdot)$, and $G_2(\cdot,\cdot)$ are material relaxation functions which are smooth, positive, and monotonically decreasing functions of time t. $G_1(\cdot,\cdot)$ and $G_2(\cdot,\cdot)$ satisfy $G_1(x,y)=G_1(y,x)$. The constants β_1 and β_{-1} satisfy $\beta_1>0$, $\beta_{-1}<0$. Substituting (4) into (3) yields, in physical components,

$$\bar{T}_{ij}=-p\delta_{ij}+\left(\delta_{KL}\int_{-\infty}^{t}2G_1(t-\tau,0)\frac{\partial E_{MM}(\tau)}{\partial\tau}d\tau+\int_{-\infty}^{t}G_2(t-\tau,0)\frac{\partial E_{KL}(\tau)}{\partial\tau}d\tau\right)F_{iK}F_{jL}, \quad (5a)$$

$$T_{ij}=-p\delta_{ij}+\beta_1 B_{ij}+\beta_{-1}B_{ij}^{-1}+\int_{-\infty}^{t}g(t-\tau)\frac{\partial E_{tij}(\tau)}{\partial\tau}d\tau, \quad (5b)$$

where δ_{ij} is the Kronecker delta. Here and below, quantities for the constitutive relation (3a) are indicated by a superposed bar. Constitutive relations (5a) and (5b) are more general than those studied by Batra and Yu [1].

Christensen [10] has analyzed the torsional deformations of a homogeneous viscoelastic cylinder made of material (5a). Following the same procedure or that given by Truesdell and Noll [11] for the torsion of an isotropic elastic cylinder, we determine the hydrostatic pressure and the components of the Cauchy stress tensor that satisfy the balance of linear momentum without body and inertia forces, and the boundary condition of null tractions on the mantle of the cylinder.

The stress components, T_{zz} and $T_{\theta z}$, have the expressions

$$\bar{T}_{zz}(t)=-\left(\frac{1}{4}\kappa^2(t)(a^4-r^4)(2F_1(t)+F_2(t))+\frac{1}{2}\kappa(t)(a^2-r^2)F_3(t)\right)+r^2F_2(t), \quad (6a)$$

$$T_{zz}(t)=\left(\frac{\beta_1}{2}+\beta_{-1}\right)r^2\kappa^2(t)+\frac{1}{2}\int_{-\infty}^{t}g(t-\tau)r^2\frac{\partial}{\partial\tau}(\kappa(\tau)-\kappa(t))^2d\tau\beta_1\frac{a^2}{2}\kappa^2, \quad (6b)$$

$$\bar{T}_{\theta z}(t)=(2F_1(t)+F_2(t))\kappa(t)r^3+\frac{1}{2}rF_3(t), \quad (7a)$$

$$T_{\theta z}(t)=\kappa(t)r(\beta_1-\beta_{-1})+\frac{1}{2}\int_{-\infty}^{t}g(t-\tau)r\frac{\partial}{\partial\tau}(\kappa(\tau)-\kappa(t))d\tau, \quad (7b)$$

where a is the radius of the cylinder, and

$$F_\xi(t)=\frac{1}{2}\int_{-\infty}^{t}G_\xi(t-\tau,0)\frac{d\kappa^2(\tau)}{d\tau}d\tau, \quad \xi=1,2;$$

$$F_3(t)=\int_{-\infty}^{t}G_2(t-\tau,0)\frac{d\kappa(\tau)}{d\tau}d\tau. \quad (8)$$

We now consider a stress-relaxation test with $\kappa(t)=\kappa_0 h(t)$; $h(t)$ being the Heaviside unit step function. Noting that $2F_\xi(t)=G_\xi(t,0)\kappa_0^2$, and $F_3(t)=G_2(t,0)\kappa_0$ (e.g., see Christensen [10]), we obtain the following expressions for the resultant normal force, $N_z(t)$, and the resultant torque, $M_z(t)$, acting on a cross section of the cylinder.

$$\bar{N}_z(t)=-\frac{\pi\kappa_0^4a^6}{6}\left[G_1(t,0)+\frac{1}{2}G_2(t,0)\right],$$

$$N_z(t)=-\frac{\pi\kappa_0^2a^4}{4}[-2\beta_{-1}+\beta_1+g(t)], \quad (9)$$

$$\bar{M}_z(t)=\frac{\pi}{2}\kappa_0a^4\left[\frac{\kappa_0^2a^2}{3}(2G_1(t,0)+G_2(t,0))+\frac{1}{2}G_2(t,0)\right], \quad (10a)$$

$$M_z(t)=\frac{\pi}{2}\kappa_0a^4\left\{(\beta_1-\beta_{-1})+\frac{1}{2}g(t)\right\}. \quad (10b)$$

Recalling that g, G_1, G_2, and β_1 are positive and β_{-1} is negative, each constitutive relation predicts that a compressive axial force must be applied to the end faces of the cylinder in order to maintain its length. The average axial stress is proportional to $\kappa_0^4a^4$ and $\kappa_0^2a^2$ for the constitutive relations (5a) and (5b), respectively. Whereas M_z is a linear function of κ_0 for the constitutive relation (5b), it also depends upon κ_0^3 for the constitutive relation (5a).

We now compare average shear stress versus shear strain curves as predicted from these two constitutive relations without the experimental data of Lenoe et al. ([12], Fig. 3), and set $\kappa(t)=\dot{\kappa}t$, where $\dot{\kappa}$ is the torsional rate. Lenoe et al. assume that $G(t)=\Sigma_{i=0}^3\Psi_ie^{-\gamma_it}$, where Ψ_i is the relaxation modulus and γ_i equals the reciprocal of the relaxation time. For the polyurethane rubber studied, they found that $\Psi_0=2.896$ MPa, $\Psi_1=0.387$ MPa, $\Psi_2=0.152$ MPa, $\Psi_3=0.689$ MPa, and $\gamma_0=0$ s^{-1}, $\gamma_1=0.001316$ s^{-1}, $\gamma_2=0.0050$ s^{-1}, $\gamma_3=0.002631$ s^{-1} provided a good fit to the test data. Recall that the average shear stress, $T_{\theta z}^m=\int_0^a 2rT_{\theta z}dr/a^2$. We assign following values to various material parameters:

$$G_1(t)=\frac{2\Psi_0(1+\nu)}{3(1-2\nu)}, \quad \nu=0.49; \quad G_2(t)=\sum_{i=0}^{3}\Psi_ie^{-\gamma_it};$$

$$\beta_1-\beta_{-1}=\Psi_0; \quad g(t)=\sum_{i=1}^{3}\Psi_ie^{-\gamma_it}. \quad (11)$$

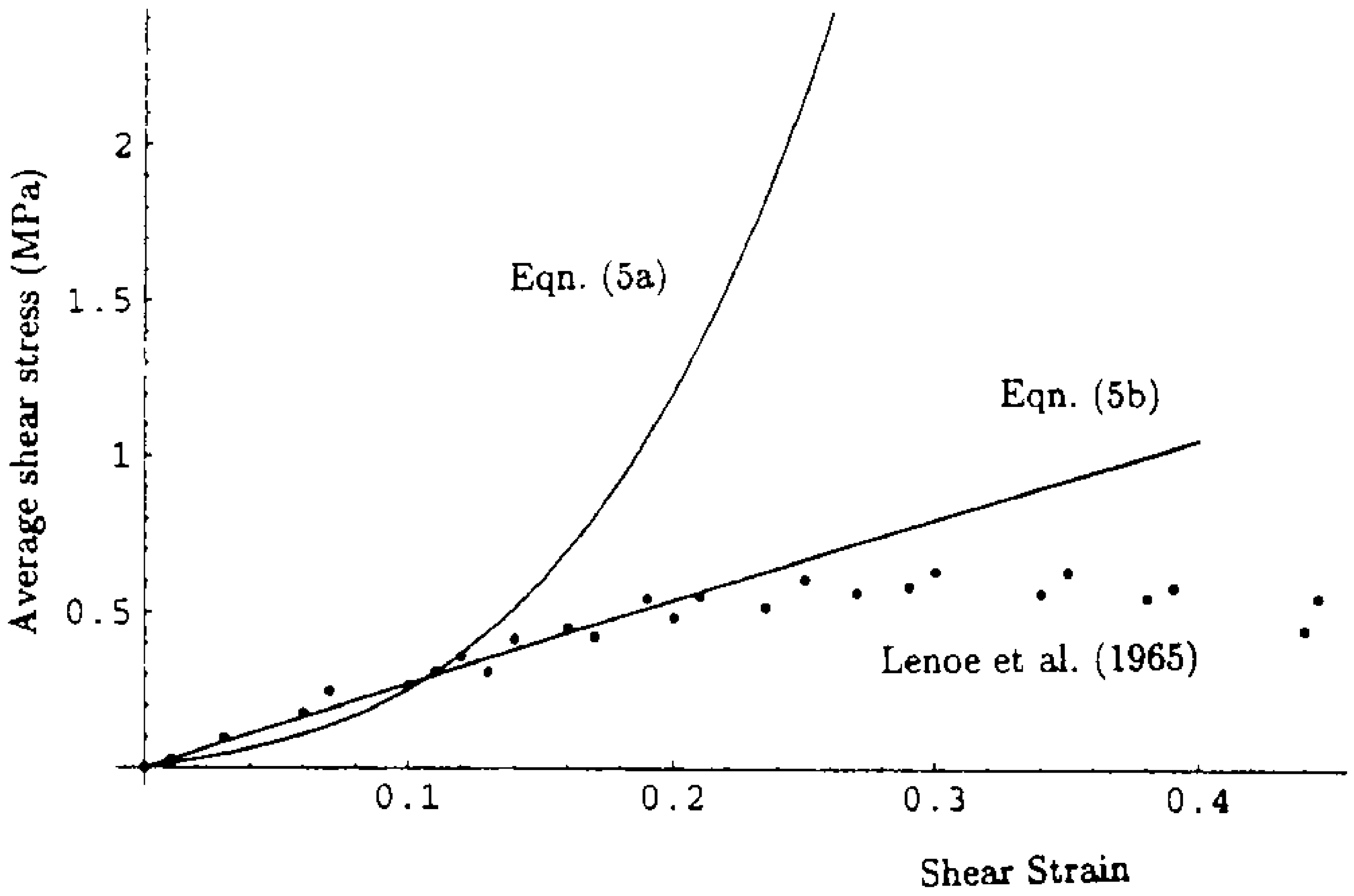

Fig. 1 Average shear stress versus shear strain curves computed from constitutive relations (5*a*) and (5*b*) and the test data of Lenoe et al. [12]. The test data is indicated as dots.

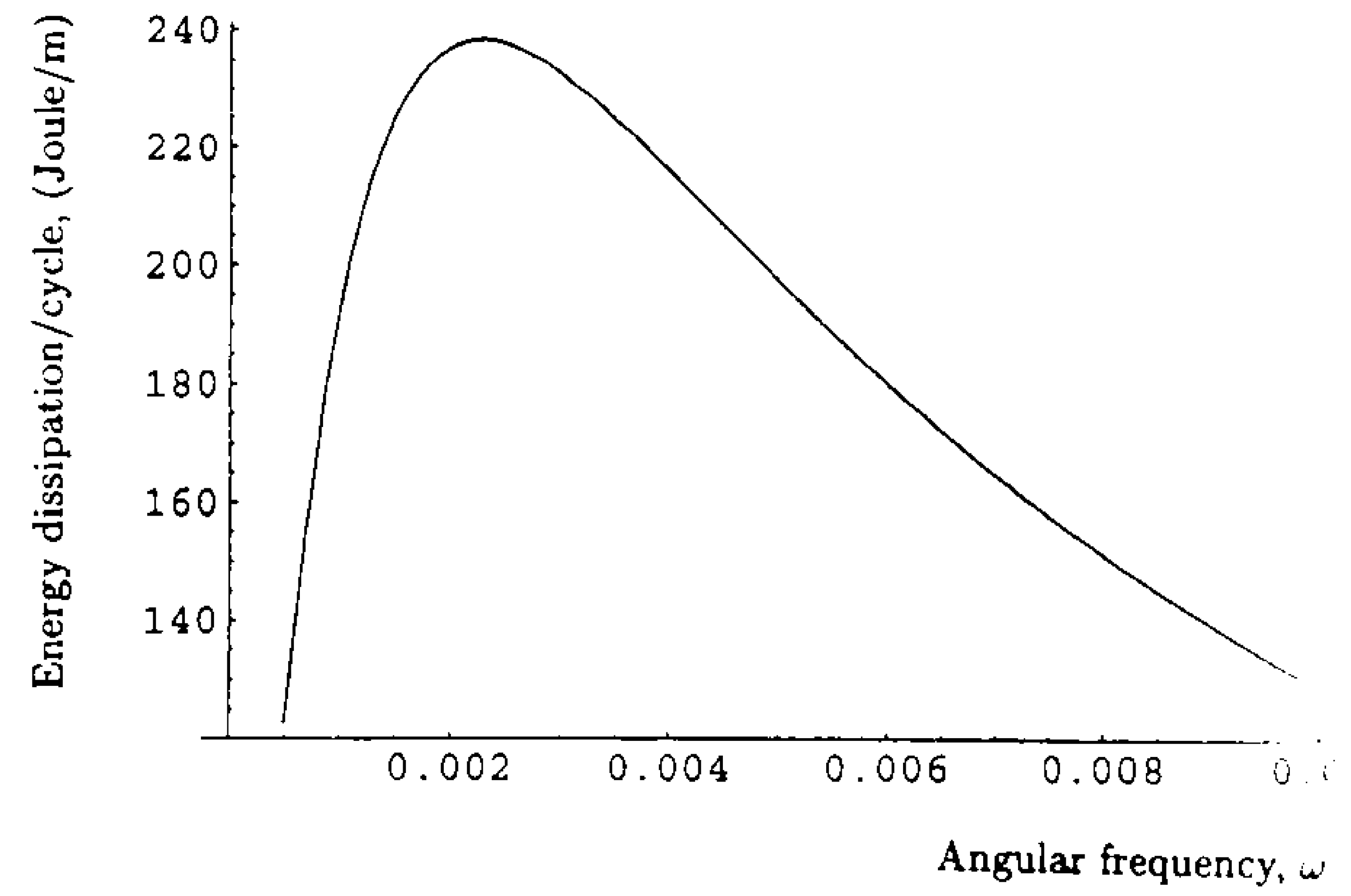

Fig. 3 Energy loss/cycle per unit length of the cylinder as a function of the forcing frequency for the polyurethane rubber tested by Lenoe et al. [12]

Figure 1 exhibits the average shear stress versus the shear strain curves for $\dot{\kappa}=0.0036\ \mathrm{s}^{-1}$ as computed from constitutive relations (5*a*) and (5*b*), and also the experimental data of Lenoe et al. For shear strains up to 0.1, the three curves are close to each other. However, for large shear strains, the experimental curve is concave downwards but the ones obtained from constitutive relations (5*a*) and (5*b*) are concave upwards and nearly linear, respectively. For large shear strains, predictions from the constitutive relation (5*a*) are not even in qualitative agreement with the test data.

Henceforth we only use constitutive relation (5*b*) and analyze the damping of vibrations. We consider steady-state oscillations with $\kappa(t)=\bar{\kappa}_0 \sin \omega t$, where ω, the frequency of oscillations, is such that inertia effects can be neglected (e.g., see Christensen [10]). The energy loss per cycle is given by $\Delta = \int_0^{2\pi/\omega} M(t)\dot{\kappa}(t)dt$ since there is no work done by N_z because of null axial elongation of the cylinder. For $g(t)=g_0 e^{-\gamma t}$ we obtain

$$\Delta(\gamma,\omega)=\frac{\pi g_0 \bar{\kappa}_0^2 a^4}{2}\cdot\frac{\pi\gamma\omega(\gamma^2+\omega^2)+\omega^2\gamma^2(e^{-2\pi\gamma/\omega}-1)}{(\gamma^2+\omega^2)^2}. \tag{12}$$

Whenever the term $e^{-2\pi\gamma/\omega}$ can be neglected, the energy loss will be a symmetric function of γ and ω. Figure 2 depicts the normalized energy loss $\Delta_n = 4\Delta(\gamma,\omega)/\pi g_0 \bar{\kappa}_0^2 a^4$ as a function of γ and ω. For $e^{-2\pi\gamma/\omega} \ll \gamma$ or ω, we see that the energy dissipation per cycle is maximum when $\gamma=\omega$. One possible explanation is that when the material relaxes faster than the frequency of the applied torque, i.e., $\gamma>\omega$, or when the material relaxes very slowly, i.e., $\gamma<\omega$, there will be a larger component of M_z in phase with $\dot{\kappa}$ during a part of the loading cycle which will decrease Δ. For $\gamma=0$, the material takes forever to relax, and there is no energy dissipation.

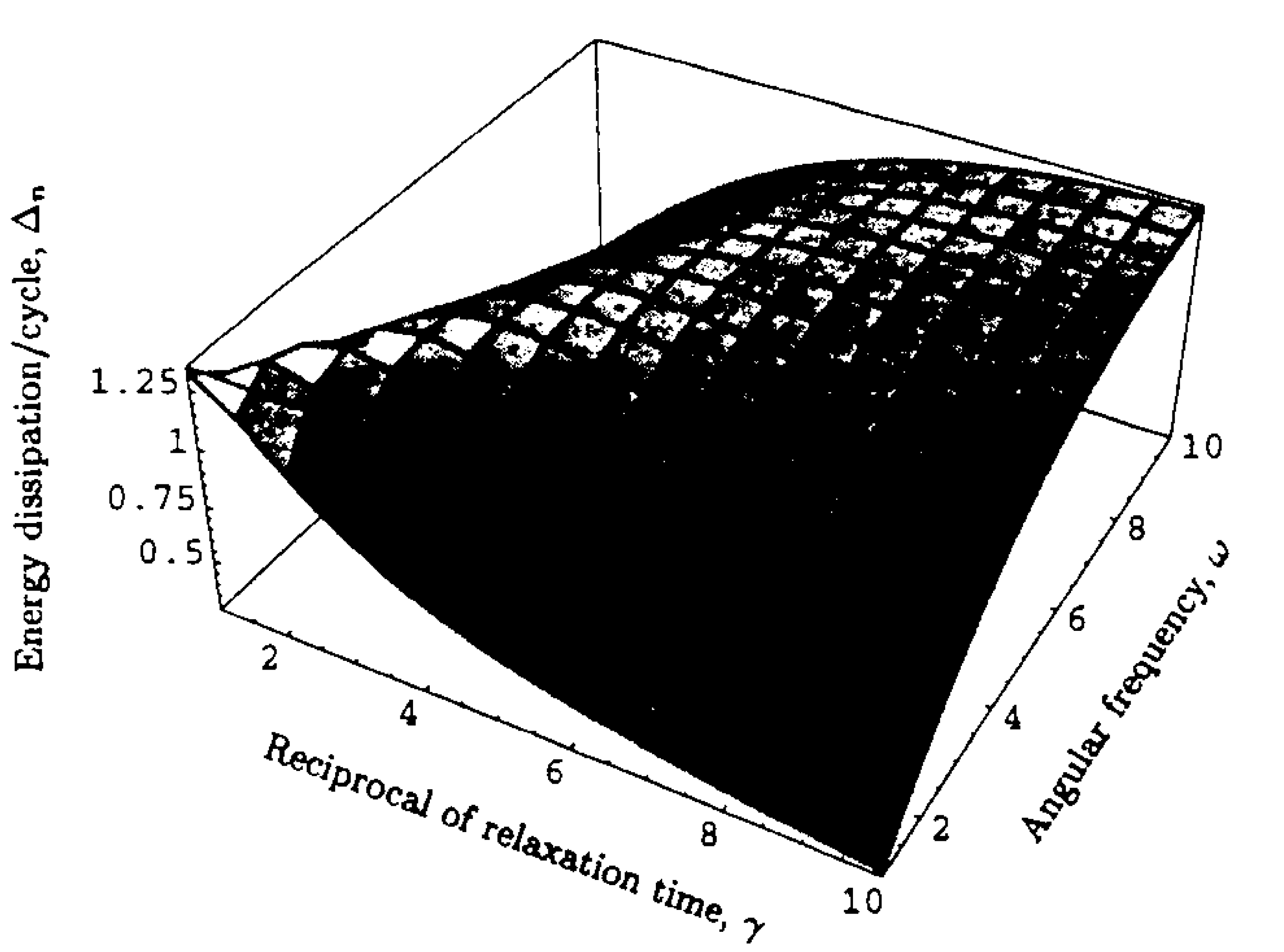

Fig. 2 The normalized energy loss/cycle per unit length of the cylinder as a function of the reciprocal of the relaxation time and the angular frequency

For a viscoelastic material with $g(t)=\Sigma_{i=1}^{3}\Psi_i e^{-\gamma_i t}$, the energy loss per cycle is

$$\Delta(\omega)=\frac{\pi\bar{\kappa}_0^2 a^4}{4}\sum_{i=1}^{3}\Psi_i\cdot\frac{\pi\gamma_i\omega(\gamma_i^2+\omega^2)+\omega^2\gamma_i^2(e^{-2\pi\gamma_i/\omega}-1)}{(\gamma_i^2+\omega^2)^2}. \tag{13}$$

For the aforestated values of material parameters and $\kappa_0=1$, $a=0.1$ m, the energy loss is plotted in Fig. 3. The energy loss is high for $0.002\leq\omega\leq0.004$. One can similarly find the optimum frequency range for other materials.

In conclusion, we note that the predictions from the constitutive relation (5*b*) are in better qualitative agreement with the test observations than those from the constitutive relation (5*a*). A real test of a constitutive relation is its ability to predict results in agreement with test data for configurations other than those used to find the values of material parameters. This arduous task has not been pursued here.

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A Strip Element Method for Analyzing Wave Scattering by a Crack in an Axisymmetric Cross-Ply Laminated Composite Cylinder

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A strip element method is presented for analyzing waves scattered by a crack in an axisymmetric cross-ply laminated composite cylinder. The cylinder is at the outset discretized as axisymmetric strip elements through the radial direction. The application of the Hamilton variational principle develops a set of governing ordinary differential equations. The particular solutions to the resulting equations are found using a modal analysis approach in conjunction with the Fourier transform technique. The complementary solutions are formulated by the superposition of eigenvectors, the unknown coefficients of which are determined from axial stress boundary conditions at the tips of the crack. The summation of the particular and complementary solutions gives the general solutions. Numerical examples are given for cross-ply laminated composite cylinders with radial cracks. The results show that the present method is effective and efficient.
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Introduction

Wave propagation in anisotropic media is one of the most fundamental and important subjects in the practice of engineering. Relevant literature is vast. Mal [1] and Nayfeh [2] reviewed it well. Because of the inherent complexities involved in material itself, an analysis of wave propagation in layered composite cylinders needs to resort to numerical techniques. Dealing with propagating waves and edge vibration in anisotropic composite cylinders, Huang and Dong [3] proposed an efficient numerical-analytical method in which a composite cylinder was modeled by finite elements, triangular functions, and wave function expansions in the radial, circumferential, and axial directions, respectively. The salient features of the method are to be capable of reducing the spatial dimensions of a problem by one and to omit tedious pre-processors occupying a substantial part of a finite element method. Rattanwangcharoen et al. [4] utilized the numerical-analytical method to solve the reflection problem of waves at the free edge of a laminated circular cylinder. Recently, Rattanwangcharoen et al. [5] combined the numerical-analytical method and the finite element method to analyze scattering of axisymmetric guided waves by a weldment between two laminated cylinders. In their treatise, the numerical-analytical method was employed to model the cylinders and the finite element method was used to model the weldment. This combinatory procedure was applied to axisymmetric guided wave scattering by cracks in welded steel pipes by Zhuang et al. [6]. The advantage of the combinatory procedure is to be able to treat complex local domains of a cylinder, such as weldment, hole, and imperfection. The disadvantage is to reduce the efficiency of the numerical-analytical method. Therefore, it is interesting to develop a numerical-analytical method for analyzing waves in a composite cylinder containing a crack.

In this paper, a strip element method is formulated for analyzing wave scattering by a crack in an axisymmetric cross-ply laminated composite cylinder, subjected to a harmonic excitation of a line source along the circumferential direction. The method is based on a strip element method proposed by Liu and Achenbach [7,8] for a cracked laminated composite plate as well as the numerical-analytical method proposed by Huang and Dong [3] for a perfect laminated composite cylinder. The cylinder is first modeled using axisymmetric strip elements in the radial direction. Then the Hamilton variational principle is used to derive a system of governing ordinary differential equations for the cylinder in a frequency domain. A particular solution for the resulting equations is found using a modal analysis approach and inverse Fourier transform techniques. A general solution is obtained with axial stress boundary conditions. Lastly, numerical examples are presented for multilayered cylinders with outer surface-breaking and radial interior cracks.

Formulation

Consider an infinitely long cracked cross-ply laminated composite cylinder made of an arbitrary number of linearly elastic cylinder-like laminae. The bonding between plies is perfect except in the region of the crack. Deformations of the cylinder are assumed small under a harmonic excitation. A radial line load of $q=q_0\exp(i\omega t)$ uniformly distributed along the circumferential direction is applied on the outer surface of the cylinder.

Because the geometry of the cylinder and the load are independent of the circumferential direction, the problem is axisymmetric. Let z and r denote, respectively, the axial and radial coordinates, then the strain-displacement relations are given by

$$\boldsymbol{\varepsilon}=\mathbf{L}\mathbf{u} \tag{1}$$

where $\boldsymbol{\varepsilon}[\varepsilon_z\ \varepsilon_\theta\ \varepsilon_r\ \gamma_{rz}]^T$ is the vector of strains and $\mathbf{u}=[u\ w]^T$ is the vector of displacements. Here u and w are the displacement components in the axial and radial directions, respectively. The operator matrix $\mathbf{L}$ is given by

$$\mathbf{L}=\begin{bmatrix} \frac{\partial}{\partial z} & 0 & 0 & \frac{\partial}{\partial r} \\ 0 & \frac{1}{r} & \frac{\partial}{\partial r} & \frac{\partial}{\partial z} \end{bmatrix}^T=\mathbf{L}_1\frac{\partial}{\partial z}+\mathbf{L}_2\frac{\partial}{\partial r}+\mathbf{L}_3\frac{1}{r} \tag{2}$$

where the matrices $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_3$ can be obtained by inspection of Eq. (2).

A lamina under consideration is transversely isotropic, so the stresses are related to strains by

$$\boldsymbol{\sigma}=\bar{\mathbf{Q}}\boldsymbol{\varepsilon} \tag{3}$$

where $\boldsymbol{\sigma}=[\sigma_z\ \sigma_\theta\ \sigma_r\ \tau_{rz}]^T$ is the vector of stresses and

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