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Stress Concentration in an Elastic Cosserat Plate Undergoing Extensional Deformations

We formulate and solve the boundary value problem of a linearly elastic, infinite Cosserat plate which contains a circular hole and which is loaded in tension at infinity. The effect of hole radius, plate thickness, and material parameters on the stress concentration at the hole is discussed. We also discuss the stress concentration when the plate is subjected to pure shear.

1 Introduction

The problem of stress concentration around a circular hole in the Kirchhoff-Love theory for flexural deformations of a plate was investigated by Goodier (1936). This bending theory does not consider the effect of transverse shear deformation; the paper by Reissner (1945) addresses the stress concentration problem within the context of a theory which accounts for transverse shear deformation. The corresponding problem for extensional deformations of a classical plate is well known; it is precisely the generalized plane-stress problem of linear elasticity, where the unbounded domain containing a circular hole is loaded in far-field tension (see, e.g., Sokolnikoff, 1956). However, the generalized plane stress theory does not account for transverse normal strain effects. We remark that in the context of a dynamical problem of plates, Kane and Mindlin (1956) have proposed a set of equations for extensional motions which include the effect of transverse normal strain. They have indicated the importance of this effect in the context of high-frequency extensional vibrations of plates. However, the stress concentration problem in an extensional theory of plates accounting for transverse normal strain does not appear to have been studied.

In the present work, we study the stress concentration problem using the linear theory of an elastic Cosserat plate. This theory is unified in the sense that it encompasses the set of equations for both extension as well as bending of a plate. This is in contrast to the theory of Kane and Mindlin (1956) which is specifically designed to address the high-frequency extensional behavior of plates, so much so that the value of a constant in their theory is tailored to accurately reflect three-dimensional high-frequency behavior. This is the main reason we choose the Cosserat theory to model our plate.

In Section 2, we record the basic equilibrium equations of the linear theory of a Cosserat plate. We also indicate the relationship between the extensional equations arising in this theory and those utilized by Kane and Mindlin (1956). In Section 3, we formulate and solve the stress concentration problem for the case of uniaxial tension. We also discuss in this section the stress concentration when the plate is subjected to pure shear at infinity.

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2 Basic Equations for a Linearly Elastic Cosserat Plate

We recall that a Cosserat plate is an initially flat material surface embedded in a three-dimensional Euclidean space and to each point of which is attached deformable vector fields called directors. Here, we confine ourselves to the case of a single director. These models have been discussed extensively by Naghdi (1972).

The linear version of the theory of a Cosserat plate has been discussed in detail by Green and Naghdi (1967) and by Green, Naghdi, and Wenner (1971). The basic kinematic ingredients in the theory are the displacement $\mathbf{u}(x_\alpha)$ of surface particles and the director displacement $\boldsymbol{\delta}(x_\alpha)$, where x_α ($\alpha = 1, 2$) denote rectangular Cartesian coordinates on the midsurface of the plate.[1] The fundamental kinematic assumption in the theory is the following condition relating the displacement $\mathbf{u}^*(x_\alpha, x_3)$ of a particle in the three-dimensional plate-like continuum to $\mathbf{u}$ and $\boldsymbol{\delta}$:

$$\mathbf{u}^*(x_\alpha, x_3) = \mathbf{u}(x_\alpha) + x_3\boldsymbol{\delta}(x_\alpha), \tag{2.1}$$

where x_3 is the third Cartesian coordinate which is normal to the midsurface.[2] We denote the fixed orthonormal basis associated with x_i by $\{\mathbf{e}_i\}$ and write

$$\mathbf{u} = u_i\mathbf{e}_i, \quad \boldsymbol{\delta} = \delta_\alpha\mathbf{e}_\alpha + \delta\mathbf{e}_3. \tag{2.2}$$

The kinematical variables which enter the theory of a Cosserat plate are the quantities $e_{\alpha\beta}$, γ_i, and $\kappa_{i\alpha}$ defined by

$$2e_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha}, \quad \gamma_\alpha = \delta_\alpha + u_{3,\alpha},$$
$$\gamma_3 = \delta, \quad \kappa_{\alpha\beta} = \delta_{\alpha,\beta}, \quad \kappa_{3\alpha} = \delta_{,\alpha}, \tag{2.3}$$

where a comma followed by the index α denotes partial differentiation with respect to x_α. From a physical standpoint, $e_{\alpha\beta}$ represents the strain at a point on the midsurface, γ_α is the transverse shear strain, δ is the transverse normal strain, and $\kappa_{i\alpha}$ represents the variation of δ_i on the midsurface.

Relative to the basis $\{\mathbf{e}_i\}$, components of the contact force are denoted by $N_{\alpha i}$, those of the contact director couple by $M_{\alpha i}$ and those of the intrinsic director couple by k_i.[3] We assume that there are no body forces and body couples. The kinetic

[1] Greek indices range from 1 to 2 while Latin ones from 1 to 3. Also, the usual summation convention is adopted for both Greek and Latin indices.

[2] We refer to the surface defined by $x_3 = 0$ as the midsurface and take it to also be the material surface defining the Cosserat plate.

[3] The director displacement $\boldsymbol{\delta}$ is chosen here to be dimensionless and hence the quantities k_i have physical dimensions of couple per unit area. For an elaboration of this point, see p. 482 of Naghdi (1972).

quantities just introduced are integrated versions of the three-dimensional stress tensor $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$:

$$N_{\alpha i} = \int_{-h/2}^{h/2} T_{\alpha i} dx_3, \quad M_{\alpha i} = \int_{-h/2}^{h/2} x_3 T_{\alpha i} dx_3,$$

$$k_i = \int_{-h/2}^{h/2} T_{3i}\, dx_3, \tag{2.4}$$

where h is the uniform initial thickness of the plate. The equilibrium equations governing the infinitesimal deformations of the Cosserat plate are

$$N_{\alpha\beta,\alpha} = 0, \quad M_{\alpha 3,\alpha} = k_3, \tag{2.5}$$

$$M_{\alpha\beta,\alpha} = k_\beta, \quad N_{\alpha 3,\alpha} = 0, \tag{2.6}$$

along with the restrictions

$$N_{\alpha\beta} = N_{\beta\alpha}, \quad N_{\alpha 3} = k_\alpha. \tag{2.7}$$

The basic idea behind the constitutive theory is to first define a two-dimensional complementary energy function by integrating its three-dimensional counterpart through the thickness of the plate. The constitutive equations for the plate then follow from appropriate relationships involving the two-dimensional complementary energy. We refer the reader to Green, Naghdi, and Wenner (1971) and Section 20 of Naghdi (1972) for details and simply record the following constitutive relations:

$$N_{\alpha\beta} = (1-\nu)C\left[\frac{\nu}{1-2\nu}\, e_{\gamma\gamma}\delta_{\alpha\beta} + e_{\alpha\beta} + \frac{\nu}{1-2\nu}\,\delta_{\alpha\beta}\gamma_3\right], \tag{2.8}$$

$$M_{(\alpha\beta)} = B[\nu\delta_{\alpha\beta}\delta_{\gamma\delta} + (1-\nu)\delta_{\alpha\gamma}\delta_{\beta\delta}]\kappa_{(\gamma\delta)}, \tag{2.9}$$

$$M_{[\alpha\beta]} = 0, \tag{2.10}$$

$$N_{\alpha 3} = \frac{5}{12}\frac{Eh}{(1+\nu)}\,\gamma_\alpha, \tag{2.11}$$

$$M_{\alpha 3} = \frac{7}{20}(1-\nu)B\kappa_{3\alpha}, \tag{2.12}$$

$$k_3 = \frac{1-\nu}{1-2\nu}\, C[(1-\nu)\gamma_3 + \nu e_{\alpha\alpha}], \tag{2.13}$$

where $\delta_{\alpha\beta}$ is the two-dimensional Kronecker delta, ν is Poisson's ratio, E is Young's modulus, the symbols ( ) and [ ] in (2.9) and (2.10) denote the symmetric and skew-symmetric parts of $M_{\alpha\beta}$, respectively, and

$$B = \frac{Eh^3}{12(1-\nu^2)}, \quad C = \frac{Eh}{1-\nu^2}. \tag{2.14}$$

Equations (2.5), (2.8), (2.12), and (2.13) are associated with extensional deformations of the plate while Eqs. (2.6), (2.9), (2.10), and (2.11) characterize flexural deformations of the plate. In the linear theory, the extensional equations are decoupled from the flexural ones. The flexural Eqs. (2.6) along with constitutive Eqs. (2.9), (2.10), and (2.11) are identical to those of Reissner's plate theory (1945).

The extensional Eqs. (2.5), (2.8), and (2.13) correspond to the static counterpart of those used by Kane and Mindlin (1956) provided δ in (2.2) is identified with $2w/h$ of their paper and the ratio of resultants in (2.4) to the corresponding ones in Kane and Mindlin's work is identified as the plate thickness. As regards (2.12), Naghdi (1972, p. 574) indicates the procedure by which the constant 7/20 is obtained and also comments on its reasonableness. Jin and Hwang (1989) have indicated that for static problems, the value of the constant κ in Kane and Mindlin's paper be taken as 1; this corresponds to a value 1/2 in (2.12) rather than 7/20. The slight difference in our final results from those gotten by utilizing the static counterpart of the Kane-Mindlin theory (see Fig. 3) can be attributed to the value of the constant in (2.12). In a recent paper, Naghdi and Rubin (1995) utilize a third procedure which results in a value 1/2 for the aforementioned constant. This indicates that the choice 1/2 in (2.12) may perhaps be more accurate. However, we shall see in Section 3 that the two choices produce almost identical results for both the stress concentration problem of uniaxial tension and pure shear. We note that $(2.5)_1$ and (2.8) are the equations of generalized plane stress or alternatively, they may be regarded as those governing extensional deformations in the classical Kirchhoff-Love plate theory.

We now obtain the equations of extensional deformations of the Cosserat plate in terms of an Airy function φ. The extensional equations for the static counterpart of the Kane-Mindlin theory have been given by Jin and Hwang (1989) in terms of an Airy function. In the Cosserat theory, the procedure is identical and we briefly describe it below. From Eq. $(2.5)_1$, it may be deduced that there exists a scalar function $\varphi(x_\alpha)$ such that

$$N_{\alpha\beta} = \nabla^2\varphi\,\delta_{\alpha\beta} - \varphi_{,\alpha\beta}, \tag{2.15}$$

where $\nabla^2\varphi$ denotes the two-dimensional Laplacian of φ. Now, we use the compatibility condition[4]

$$e_{11,22} + e_{22,11} - 2e_{12,12} = 0 \tag{2.16}$$

of the linear theory of a Cosserat plate and the constitutive Eqs. (2.8) and (2.13) to deduce that the Airy function φ satisfies the differential equation

$$\frac{(1-\nu^2)}{E\nu}\nabla^4\varphi - \nabla^2\delta = 0, \tag{2.17}$$

where ∇^4 is the two-dimensional biharmonic operator. The remaining constitutive Eq. (2.12) may be used in conjunction with (2.13) to obtain

$$\frac{7}{240}\frac{h^2}{(1+\nu)}\nabla^2\delta - \delta - \frac{\nu}{Eh}\nabla^2\varphi = 0. \tag{2.18}$$

The two differential Eqs. (2.17) and (2.18) for φ and δ may be rewritten as

$$\nabla^4\delta - G\nabla^2\delta = 0, \tag{2.19}$$

$$\nabla^2\varphi = G_1\nabla^2\delta - G_2\delta, \tag{2.20}$$

where

$$G_1 = \frac{7Eh^3}{240\nu(1+\nu)}, \quad G_2 = \frac{Eh}{\nu}, \quad G = \frac{240}{7h^2}\frac{1}{(1-\nu)}. \tag{2.21}$$

Once (2.19) has been solved for δ, we can solve (2.20) for φ.

3 A Boundary Value Problem

Consider an infinite plate with uniform initial thickness h. The plate contains a circular hole of radius R and is loaded in far-field tension as shown in Fig. 1. We introduce polar coordinates (r, θ) and record the boundary conditions below.

$$\text{As } r \to \infty, \quad N_{rr} = \frac{N}{2}(1 + \cos 2\theta), \quad N_{r\theta} = -\frac{N}{2}\sin 2\theta,$$

$$N_{\theta\theta} = \frac{N}{2}(1 - \cos 2\theta), \tag{3.1}$$

$$\text{At } r = R, \quad N_{rr} = 0, \quad N_{r\theta} = 0, \quad M_{r3} = 0. \tag{3.2}$$

The quantities appearing in (3.1) and (3.2) represent the physi-

[4] See Section 6 of Naghdi (1972).

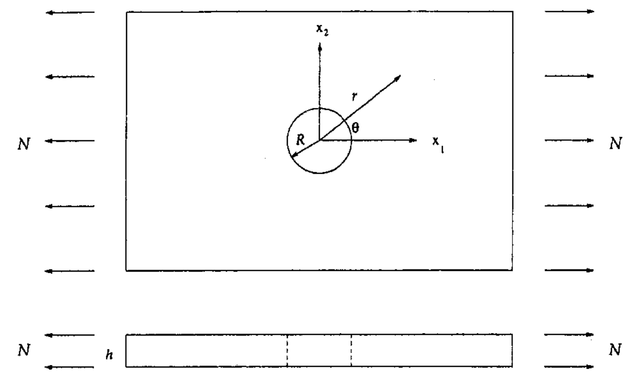

Fig. 1 Plate loaded in tension at infinity

cal components $N_{\langle\alpha\beta\rangle}$ and $M_{\langle\alpha 3\rangle}$. That the solution to the boundary value problem defined by (2.6)–(2.8) and (3.1)–(3.2) is unique (upto a rigid motion) follows from the discussion in Section 26 of Naghdi (1972).

Keeping in mind the form of the boundary conditions (3.1), we seek a solution to (2.19) of the form

$$\delta(r, \theta) = \delta_a(r) + \delta_b(r) \cos 2\theta. \tag{3.3}$$

Substitution of (3.3) into (2.19) results in a pair of ordinary differential equations for δ_a and δ_b, whose solutions involve modified Bessel functions I_0, K_0 of order zero and I_2, K_2 of order 2. The solution procedure being straightforward, we merely present the result for δ below:

$$\delta(r, \theta) = a_1 + a_2 \ln r + a_3 I_0(\sqrt{G}r) + a_4 K_0(\sqrt{G}r) + \left[b_1 r^2 + \frac{b_2}{r^2} + b_3 I_2(\sqrt{G}r) + b_4 K_2(\sqrt{G}r) \right] \cos 2\theta, \tag{3.4}$$

where a_1, a_2, a_3, a_4, b_1, b_2, b_3, and b_4 are constants to be determined from the boundary conditions. Since the director displacement δ must remain bounded at infinity, the constants a_2, a_3, b_3, and b_1 vanish so that

$$\delta(r, \theta) = a_1 + a_4 K_0(\sqrt{G}r) + \left[\frac{b_2}{r^2} + b_4 K_2(\sqrt{G}r) \right] \cos 2\theta. \tag{3.5}$$

We now substitute (3.5) into (2.20) and observe that φ satisfies the Poisson equation. The solution for φ is

$$\varphi(r, \theta) = p_1 + p_2 \ln r + \frac{\bar{a}}{4} r^2 + \frac{\bar{b}}{G} K_0(\sqrt{G}r) + \left[a_1 r^2 + \frac{q_2}{r^2} - \frac{\bar{c}}{4} + \frac{\bar{d}}{G} K_2(\sqrt{G}r) \right] \cos 2\theta, \tag{3.6}$$

where

$$\bar{a} = -a_1 G_2, \quad \bar{b} = a_4(GG_1 - G_2),$$
$$\bar{c} = -b_2 G_2, \quad \bar{d} = b_4(GG_1 - G_2), \tag{3.7}$$

and p_1, p_2, q_1, q_2 are constants to be determined from the boundary conditions. Expressions for the resultants in polar coordinates are

$$N_{rr} = \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}, \quad N_{r\theta} = N_{\theta r} = \frac{1}{r^2}\frac{\partial \varphi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \varphi}{\partial r \partial \theta}, \tag{3.8}$$
$$N_{\theta\theta} = \frac{\partial^2 \varphi}{\partial r^2}, \quad M_{r3} = \frac{7}{20}(1 - \nu) B \frac{\partial \delta}{\partial r},$$

$$M_{\theta 3} = \frac{7}{20}(1 - \nu) B \frac{\partial \delta}{\partial \theta}. \tag{3.9}$$

Using (3.5) and (3.6) to calculate the derivatives appearing above, we record expressions for all the resultants:

$$N_{rr} = \left(\frac{p_2}{r^2} + \frac{\bar{a}}{2} - \frac{\bar{b} K_1}{\sqrt{G}r} \right) + \left[-2q_1 - \frac{6q_2}{r^4} + \frac{\bar{c}}{r^2} - \frac{\bar{d}}{G}\left(\frac{4K_1}{\sqrt{G}r^3} + \frac{\sqrt{G}K_1}{r^2} + \frac{2K_0}{r^2} - \frac{4K_2}{r^2} \right) \right] \cos 2\theta, \tag{3.10}$$

$$N_{r\theta} = \left[2q_1 - \frac{6q_2}{r^4} + \frac{\bar{c}}{2r^2} - \frac{2\bar{d}}{G}\left(\frac{4K_1}{\sqrt{G}r^3} + \frac{\sqrt{G}K_1}{r} + \frac{2K_0}{r^2} + \frac{K_2}{r^2} \right) \right] \sin 2\theta, \tag{3.11}$$

$$N_{\theta\theta} = \left(\frac{\bar{a}}{2} + \bar{b}K_0 + \frac{\bar{b}K_1}{\sqrt{G}r} - \frac{p_2}{r^2} \right) + \left[2q_1 + \frac{6q_2}{r^4} + \frac{\bar{d}}{G}\left(\frac{12K_1}{\sqrt{G}r^3} + \frac{3\sqrt{G}K_1}{r} + \frac{6K_0}{r^2} + GK_0 \right) \right] \cos 2\theta, \tag{3.12}$$

$$M_{r3} = \frac{7}{20}(1 - \nu) B \left\{ -a_4\sqrt{G}K_1 - \left[\frac{2b_2}{r^3} + \frac{4b_4K_1}{\sqrt{G}r^2} + b_4\sqrt{G}K_1 + \frac{2b_4K_0}{r} \right] \cos 2\theta \right\}, \tag{3.13}$$

$$M_{\theta 3} = -\frac{7}{20}(1 - \nu) B \left(\frac{2b_2}{r^3} + \frac{2b_4K_2}{r} \right) \sin 2\theta, \tag{3.14}$$

where the argument $\sqrt{G}r$ of the functions K_0, K_1, and K_2 has been suppressed. The boundary conditions (3.1) and (3.2) yield the following values for the constants:

$$\bar{a} = N, \quad \bar{b} = 0, \tag{3.15}$$

$$\bar{c} = \frac{-2NR^2(2K_2 + \rho K_1)}{16K_2\left(\frac{GG_1}{G_2} - 1 \right) + 12K_1\left(\frac{GG_1}{G_2} - 1 \right) + 2K_2 + \rho K_1}, \tag{3.16}$$

$$\bar{d} = \frac{-4N}{\frac{16K_2}{\rho^2} + \frac{12K_1}{\rho} + \frac{2K_2}{\left(\frac{GG_1}{G_2} - 1 \right)} + \frac{\rho K_1}{\left(\frac{GG_1}{G_2} - 1 \right)}}, \tag{3.17}$$

$$p_2 = -\frac{NR^2}{2}, \quad q_1 = -\frac{N}{4}, \tag{3.18}$$

$$q_2 = \frac{R^4}{6}\left[\frac{N}{2} + \frac{\bar{c}}{R^2} + \bar{d}\left(\frac{K_1}{\rho} - \frac{2K_2}{\rho^2} \right) \right], \tag{3.19}$$

where $\rho = \sqrt{G}R$. We note that the constant p_1 in (3.6) does not influence the value of any of the resultants and hence can have any arbitrary value.

The ratio $N_{\theta\theta}/N$ at $r = R$ is given by

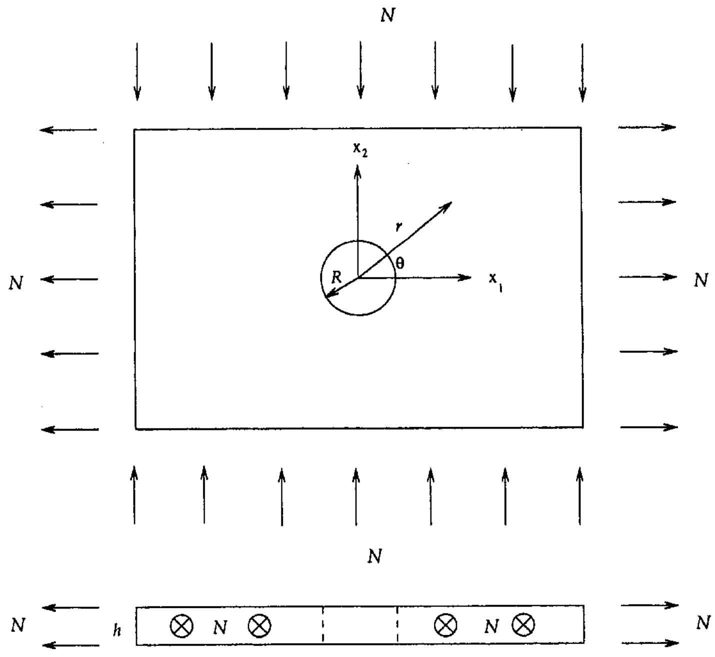

Fig. 2 Plate subjected to pure shear at infinity

$$\frac{N_{\theta\theta}(R,\theta)}{N} = 1 - \left[\frac{\bar{c}}{NR^2} + \frac{\bar{d}}{N}\left(\frac{4K_2}{\rho^2} + \frac{4K_1}{\rho} + K_0\right)\cos 2\theta\right]. \quad (3.20)$$

The stress concentration C_{ut} (the subscript ut denotes uniaxial tension) is defined as the maximum value of the expression (3.20), which occurs at both $\theta = \pi/2$ and $\theta = 3\pi/2$:

$$C_{ut} = \frac{N_{\theta\theta}(R,\pi/2)}{N} = 1 - \left[\frac{\bar{c}}{NR^2} + \frac{\bar{d}}{N}\left(\frac{4K_2}{\rho^2} + \frac{4K_1}{\rho} + K_0\right)\right]. \quad (3.21)$$

In the limit when $\rho \to \infty$, which for a given R, corresponds to a vanishingly thin plate, we expect to recover the stress concentration in the generalized plane-stress problem. Indeed, from (3.21), we have

$$\lim_{\rho\to\infty} C_{ut} = 3, \quad (3.22)$$

and the limiting value (3.22) is independent of the material parameters, hole radius and plate thickness, as it should be. The stress concentration factor C_{ut} depends on the material parameters ρ and the ratio GG_1/G_2 occurring in (3.16) and (3.17). From (3.21) and (2.21) it is clear that C_{ut} only depends on Poisson's ratio ν of the material. We plot the variation of C_{ut} as a function of R/h in Fig. 3 for $\nu = 1/3$. The result from the static counterpart of the Kane-Mindlin equations is also depicted here. It may be deduced from (3.21) that in the limit when R/h approaches zero, C_{ut} equals 2.

Finally, we discuss the stress concentration factor C_{ps} when the plate is subjected to pure shear as shown in Fig. 2. It is clear that the solution to this problem may be obtained by suitably superposing the solution for uniaxial tension in the x_1 direction and that for uniaxial compression in the x_2 direction. We do not present expressions for the resultants but note that the stress concentration factor is

$$C_{ps} = \frac{N_{\theta\theta}(R,\pi/2)}{N} = -2\left[\frac{\bar{c}}{NR^2} + \frac{\bar{d}}{N}\left(\frac{4K_2}{\rho^2} + \frac{4K_1}{\rho} + K_0\right)\right]. \quad (3.23)$$

It is clear from (3.23) that the remarks pertaining to C_{ut} apply to C_{ps} as well. In the limiting case when $\rho \to \infty$, C_{ps} equals 4, which corresponds to the generalized plane-stress value. For $\nu = 1/3$, the variation of C_{ps} with R/h is shown in Fig. 3.

The stress concentration factors C_{ut} and C_{ps} do not vary appreciably from their generalized plane stress counterparts of 3 and 4, respectively, for $R/h > 1$. Also, there is very little difference between these factors using the Cosserat versus the Kane-Mindlin theory. However, it is possible that in other equilibrium problems involving extensional deformations, the effect of transverse normal strain may be more pronounced; we mention the paper by Naghdi and Rubin (1989) who have emphasized the importance of this effect for contact problems of beams.

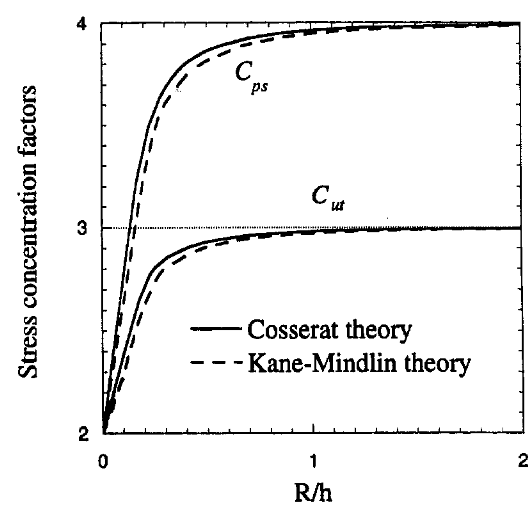

Fig. 3 Variation of stress concentration factors with R/h

The procedure outlined in Section 2 is general and may be used to solve other problems where thickness effects play an important role.

We note that for values of $R/h < 1$, the inclusion of transverse normal strain *lowers* the stress concentration factor. Although we have plotted these factors for very small R/h, we caution the reader that neither the Cosserat theory nor the Kane-Mindlin theory may be very accurate in this regime since these theories, in general, are not valid for very thick plates.

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