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Decay of the Kinetic and the Thermal Energy of Compressible Micropolar Fluids

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Summary — Zusammenfassung

Decay of the Kinetic and the Thermal Energy of Compressible Micropolar Fluids. We consider a heat conducting compressible micropolar fluid at rest and filling a closed stationary rigid container. We show that the energy of arbitrary disturbances of the fluid eventually decays.

Abnahme der kinetischen und der thermischen Energie kompressibler mikropolarer Flüssigkeiten. Betrachtet wird eine ruhende, wärmeleitende, kompressible, mikropolare Flüssigkeit und das Füllen eines geschlossenen stationären starren Behälters. Es wird gezeigt, daß die Energie beliebiger Störungen der Flüssigkeit unter Umständen abnimmt.

1. Introduction

In 1949, Kampé de Fériet [1] showed that the energy of arbitrary disturbances of a rest state of an incompressible viscous fluid filling a closed rigid container decays to zero exponentially. Since then, similar results have been proved for heat conducting incompressible fluids [2], [3], fluids whose thermomechanical deformations are governed by Boussinesq equations [4], [5], heat conducting compressible fluids [6], incompressible micropolar fluids [7] and compressible micropolar fluids [8]. In [7] and [8] the linear theory of micropolar fluids developed by Eringen [9] is used. Here, using the same linear theory of micropolar fluids, we show that the total energy of arbitrary disturbances of the rest state of a heat conducting compressible micropolar fluid filling a closed rigid container eventually decays to zero. Thus the present work generalizes the work of Shahinpoor and Ahmadi [8] on compressible micropolar fluids to heat conducting compressible micropolar fluids.

2. Formulation of the Problem

The thermo-mechanical deformations of a linear micropolar fluid are governed by the following equations [9].

$$
\begin{aligned}
&\dot{\varrho} + \varrho v_{i,i} = 0, \\
&\varrho \dot{v}_i = t_{ki,k} - \varrho \Omega_{,i}, \\
&\varrho j \dot{\nu}_r = m_{kr,r} + \varepsilon_{rks} t_{ks} + \varrho l_r, \\
&\varrho \dot{\varepsilon} = t_{ks} b_{ks} + m_{ks} \nu_{s,k} + q_{k,k} + \varrho h,
\end{aligned} \tag{1}
$$

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where

$$
\begin{aligned}
t_{ks} &= (-p + \lambda v_{r,r})\,\delta_{ks} + \mu(v_{k,s} + v_{s,k}) + \bar{\eta} b_{ks}, \\
m_{ks} &= \alpha \nu_{r,r}\delta_{ks} + \beta \nu_{k,s} + \gamma \nu_{s,k} + \bar{\alpha}\varepsilon_{ksr}\theta_{,r}, \\
q_k &= -\varkappa\theta_{,k} + \bar{\beta}\varepsilon_{krs}\nu_{r,s}, \\
b_{ks} &= v_{s,k} - \varepsilon_{ksr}\nu_r .
\end{aligned} \tag{2}
$$

Throughout this paper, we use rectangular cartesian coordinates and the Cartesian tensor notation wherever convenient. In the preceeding equations, ϱ is the present mass density, $\boldsymbol{v}$ is the velocity of a material particle $\boldsymbol{X}$ that presently is at place $\boldsymbol{x}$, a superimposed dot indicates material time derivative, p is the pressure field, a comma followed by an index k indicates partial differentiation with respect to x_k, t_{ki} is the Cauchy stress tensor, Ω is the potential of body forces and is assumed to be a non-negative bounded function of position $\boldsymbol{x}$ only, j is the microinertia, $\boldsymbol{\nu}$ is the microrotation of the flow, $\boldsymbol{l}$ is the supply density of the microrotation, ε is the internal energy density, $\boldsymbol{q}$ is the heat flux per unit present area, h is the supply density of the internal energy, and $\theta > 0$ is the absolute temperature of a material particle that currently is at place $\boldsymbol{x}$. The viscosity coefficients λ, μ, $\bar{\eta}$, α, β, γ and the heat conduction coefficient $\varkappa$, $\bar{\alpha}$ and $\bar{\beta}$ are functions of ϱ and θ. This implies that the fluid is homogeneous.

We assume that, in the reference configuration in which the fluid is at rest, the fluid occupies a bounded region R with a boundary ∂R which is smooth enough to apply the divergence theorem [10], the Poincaré inequality [11] and the Korn inequality [11]. Once the fluid is disturbed from its rest state, we assume that the container is subsequently held stationary and that the following boundary conditions are maintained during the deformations of the fluid.

$$
\begin{aligned}
\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{0}, \qquad \boldsymbol{\nu}(\boldsymbol{x}, t) = \boldsymbol{0}, \qquad & (\boldsymbol{x}, t) \in \chi(\partial R, t) \times (0, t), \\
\theta(\boldsymbol{x}, t) = \theta_0, \qquad & (\boldsymbol{x}, t) \in \chi(\partial_1 R(t), t) \times (0, t), \\
q_i(\boldsymbol{x}, t)\, n_i(\boldsymbol{x}, t) = -b(\theta, \theta_0)\,(\theta - \theta_0), \qquad & (\boldsymbol{x}, t) \in \chi(\partial_2 R(t), t) \times (0, t).
\end{aligned} \tag{3}
$$

Here $\partial_1 R \subset \partial R$, $\partial_2 R = \partial R - \partial_1 R$ and $\chi(\boldsymbol{X}, t) = \boldsymbol{x}(\boldsymbol{X}, t)$. The seboundary conditions correspond to the case in which the fluid adheres to the walls of the container, the fluid is in perfect thermal contact with the walls in the sense that the temperature of the fluid particle and the point of the container to which it is presently adhering is same, a part or all of the boundary of the container is maintained at a uniform temperature θ_0 and the rest, if any, is exchanging heat with the surroundings according to the law $(3)_4$. In order that heat may flow from the container walls into the surroundings when the former is at a higher temperature, b should be positive.

The various coefficients λ, μ etc. appearing in (2) cannot assume all possible values. Rather, these and the pressure function p should satisfy the following in order that every solution of (1) and (2) satisfy the Clausius-Duhem inequality [9].

$$\begin{gathered}3\lambda + 2\mu + \bar{\eta} \geqq 0, \qquad 2\mu + \bar{\eta} \geqq 0, \qquad \bar{\eta} \geqq 0, \qquad \varkappa \geqq 0,\\ 3\alpha + \beta + \gamma \geqq 0, \qquad \gamma - |\beta| > 0,\\ p = \varrho^2 \frac{\partial \phi}{\partial \varrho}, \qquad \phi \equiv \varepsilon - \eta\theta, \qquad \phi = \phi(\varrho, \theta),\\ \varepsilon = \varepsilon(\varrho, \theta), \qquad \eta = -\frac{\partial \phi}{\partial \theta}.\end{gathered} \tag{4}$$

Here ϕ is the Helmholtz free energy and η is the entropy density. Using $(4)_{7-11}$, we obtain

$$\dot{\varepsilon} - \theta_0 \dot{\eta} = \left(1 - \frac{\theta_0}{\theta}\right) \dot{\varepsilon} + \frac{\theta_0}{\theta} \frac{p}{\varrho^2} \dot{\varrho}. \tag{5}$$

To evaluate the left-hand side of (5), we introduce a finite Taylor expansion in the temperature for ϕ, obtaining thereby

$$\begin{gathered}\varepsilon - \theta_0 \eta = \psi(\varrho) + K(\theta - \theta_0)^2,\\ \psi(\varrho) \equiv \phi(\varrho, \theta_0), \qquad K = -\frac{1}{2} \frac{\partial^2 \phi}{\partial \theta^2} (\varrho, \theta^*) = \frac{C(\varrho, \theta^*)}{2\theta^*},\end{gathered} \tag{6}$$

C is the specific heat and θ^* is a value of temperature between θ and θ_0. Combining (5) and (6), we arrive at

$$\frac{d}{dt} \int [\psi + K(\theta - \theta_0)^2] \, \varrho \, dV = \int \varrho \left(1 - \frac{\theta_0}{\theta}\right) \dot{\varepsilon} \, dV + \int \frac{\theta_0}{\theta} \frac{p}{\varrho} \dot{\varrho} \, dV. \tag{7}$$

In (7), and henceforth, the integration is over the region occupied by the fluid. The function ψ is normalized so as to assume only non-negative values.

Taking the inner product of $(1)_2$ with v_i, of $(1)_3$ with ν_r, integrating the resulting scalar equations over the region occupied by the fluid, simplifying the right-hand sides of these equations by using the divergence theorem, boundary conditions (3), and adding these equations, we obtain

$$\dot{E}_1 + \frac{d}{dt} \int \varrho \Omega \, dV = -\int (t_{ki} b_{ik} + m_{kr} \nu_{r,k}) \, dV, \tag{8}$$

where we have set $\boldsymbol{l} = \mathbf{0}$ and

$$E_1 = \frac{1}{2} \int \varrho (v^2 + j\nu^2) \, dV. \tag{9}$$

Adding (7) and (8), substituting for $\varrho\dot{\varepsilon}$ from $(1)_4$ into the resulting equation, and simplifying by using the divergence theorem, boundary conditions (3) and the constitutive relation $(2)_3$, we get

$$\begin{aligned}\dot{E} + \dot{E}_2 = &-\int \frac{\theta_0}{\theta} \left[t_{ks} b_{ks} + m_{ks} \nu_{s,k} - \frac{p}{\varrho} \dot{\varrho}\right] dV - \int \frac{b}{\theta} (\theta - \theta_0)^2 \, dA\\ &- \int \frac{\theta_0}{\theta^2} \varkappa \theta_{,k} \theta_{,k} dV + \int \frac{\varrho h}{\theta} (\theta - \theta_0) \, dV,\end{aligned} \tag{10}$$

in which

$$E = E_1 + \int \varrho K(\theta - \theta_0)^2 \, dV, \qquad E_2 = \int (\psi + \Omega) \varrho \, dV. \tag{11}$$

Thus E equals the sum of the kinetic energy, the energy of microrotation and the temperature deviation of the fluid from that of its environment. E equals zero if and only if the fluid is at rest, there is no microrotation, and the temperature is uniform throughout the fluid and has a value equal to the temperature of the environment.

We assume that the initial disturbances $\boldsymbol{v}(\boldsymbol{x}, 0) = \overline{\boldsymbol{v}}(\boldsymbol{x})$, $\boldsymbol{\nu}(\boldsymbol{x}, 0) = \bar{\boldsymbol{\nu}}(\boldsymbol{x})$, $\varrho(\boldsymbol{x}, 0) = \bar{\varrho}(\boldsymbol{x})$ and $\theta(\boldsymbol{x}, 0) = \bar{\theta}(\boldsymbol{x})$ belong to the set S and are such that there exists a classical solution for $t > 0$ of (1) satisfying these initial conditions and the boundary conditions (3). The problem of existence of solutions of (1) seems not to have been studied so far. Lange [12] recently studied the existence of solutions of initial-boundary value problems for the equations which describe the homothermal flow of incompressible micropolar fluids.

We now state the theorem we wish to prove below.

Theorem: Every solution of (1) under the boundary conditions (3) and initial conditions belonging to the set S exhibits the behavior

$$E \to 0 \quad \text{as} \quad t \to \infty, \qquad \lim_{t \to \infty} E_2(t) \quad \text{exists}, \tag{12}$$

provided that

$$\begin{aligned} C_1 &\equiv \inf_{\varrho,\theta} \frac{\theta_0}{\theta} \left\{ \left(\lambda + \frac{2}{3} \mu \right), 2\mu \right\} > 0, \\ C_2 &\equiv \inf_{\varrho,\theta} \frac{\theta_0}{\theta} \left\{ \alpha + \frac{\beta}{3} + \frac{\gamma}{3}, \gamma + \beta \right\} > 0, \\ C_3 &\equiv \inf_{\varrho,\theta} \frac{\theta_0}{\theta^2} \varkappa > 0 \\ C_4 &\equiv \inf_{\theta} \frac{b(\theta, \theta_0)}{\theta} > 0, \\ C_5 &\equiv \sup_{\varrho,\theta} K < \infty, \\ \varrho_0 \equiv \sup \varrho < \infty, &\qquad \gamma - \beta \geqq 0, \qquad \bar{\eta} \geqq 0, \qquad h(\theta - \theta_0) \leqq 0. \end{aligned} \tag{13}$$

It should be noted that some of the inequalities in (13) are stronger than those in (4). Also the last inequality in (13) is satisfied by the choice $h = 0$, that is, there is no supply of the internal energy. Otherwise, it requires that the supply density h of the internal energy depend upon θ and θ_0 in a specific way. The definitions of various constants in (13) can be sharpened by taking the infimum or the supremum over those values of ϱ and θ which are ever realized at any fluid particle.

3. Proof of the Theorem

Recalling (2), $(1)_1$ and using the definitions

$$v_{(k,s)} \equiv \frac{1}{2}(v_{k,s} + v_{s,k}),$$

$$v_{[k,s]} \equiv \frac{1}{2}(v_{k,s} - v_{s,k}),$$

$$v^d_{k,s} \equiv v_{k,s} - \frac{1}{3} v_{r,r}\delta_{ks},$$

we note that

$$\begin{aligned}\frac{\theta_0}{\theta}\left[t_{ks}b_{ks} + m_{ks}\nu_{s,k} - \frac{p}{\varrho}\dot{\varrho}\right] &= \frac{\theta_0}{\theta}\left[\left(\lambda + \frac{2}{3}\mu\right) v_{k,k}v_{r,r} + 2\mu v^d_{(s,k)}v^d_{(s,k)}\right.\\ &\quad + \left(\alpha + \frac{\beta}{3} + \frac{\gamma}{3}\right)\nu_{r,r}\nu_{s,s} + (\gamma + \beta)\,\nu^d_{(k,s)}\nu^d_{(k,s)}\\ &\quad \left. + (\gamma - \beta)\,\nu_{[k,s]}\nu_{[k,s]} + \bar{\alpha}\varepsilon_{ksr}\theta_{,r}\nu_{s,k}\right],\\ &\geqq C_1 v_{(s,k)}v_{(s,k)} + C_2\nu_{(k,s)}\nu_{(k,s)} + \bar{\alpha}\,\frac{\theta_0}{\theta}\,\varepsilon_{ksr}\theta_{,r}\nu_{s,k}.\end{aligned} \tag{14}$$

We have used $(13)_{1,2}$ to obtain the preceeding inequality. By using the divergence theorem and (3), we can show that

$$\int \bar{\alpha}\,\frac{\theta_0}{\theta}\,\varepsilon_{ksr}\theta_{,r}\nu_{s,k}\,dV = 0,$$

and hence by integrating (14) over the region occupied by the fluid and using Poincaré's inequality and Korn's inequality (cf. [2]), we obtain

$$\begin{aligned}\int \frac{\theta_0}{\theta}\left[t_{ks}b_{ks} + m_{ks}\nu_{s,k} - \frac{p}{\varrho}\dot{\varrho}\right] dV &\geqq C_1p_1 \int v^2\,dV + C_2p_1 \int \nu^2\,dV,\\ &\geqq C_6E_1,\end{aligned} \tag{15}$$

where

$$C_6 \equiv \frac{2p_1}{\varrho_0}\min\,(C_1, C_2/j), \tag{16}$$

and p_1 is a positive valued function of R. From $(13)_{3,4,5,9}$ and by using the Poincaré inequality (cf. [2]) we conclude that

$$\int \frac{b}{\theta}(\theta - \theta_0)^2\,dA + \int \frac{\theta_0}{\theta^2}\,\varkappa\theta_{,k}\theta_{,k}\,dV - \int \frac{\varrho h}{\theta}(\theta - \theta_0)\,dV \geqq C_7 \int \varrho K(\theta - \theta_0)^2\,dV, \tag{17}$$

in which $C_7 = p_2 \min\,(C_3, C_4)/(C_5\varrho_0)$ and p_2 is a positive valued function of R and $\partial_1 R$. For the case when $\partial_1 R \neq \partial R$, p_2 varies with time t and we assume that it is bounded and denote its supremum also by p_2.

Substituting from (15) and (17) into (10), we arrive at

$$\dot{E} + \dot{E}_2 \leqq -C_8 E, \tag{18}$$

with $C_8 = \min(C_6, C_7)$. It follows from (18) that

$$E(t) + E_2(t) \leqq E(0) + E_2(0), \qquad \lim_{t\to\infty} [E(t) + E_2(t)] \quad \text{exists.} \tag{19}$$

Integration of (18) over $(0, T)$, T being an arbitrary real positive number, gives

$$C_8 \int_0^T E(t)\, dt \leqq E(0) + E_2(0)$$

which implies that $E(t) \in L^1(0, \infty)$. From (18), by using $E_2(t) < E(0) + E_2(0)$, we conclude that $\dot{E}(t) \in L^1(0, \infty)$. This together with $E(t) \in L^1(0, \infty)$ implies that $E(t) \to 0$ as $t \to \infty$. $(12)_2$ now follows from $(19)_2$.

4. Remarks

The result $(12)_1$ is not as strong as one would like to obtain. It would be desirable to show that the energy of the fluid decays monotonically and obtain the decay rate of the energy. However, for compressible micropolar fluids, (12) is the best I can prove now. In view of $(12)_2$ it seems plausible that there exists a time t_0 such that

$$\dot{E}_2 = 0 \quad \text{for} \quad t \geqq t_0. \tag{20}$$

We remark that for homogeneous incompressible micropolar fluids whose density does not depend upon temperature, (20) holds with $t_0 = 0$. Whenever (20) holds, it follows from (18) and $(19)_1$ that

$$E(t) \leqq \big(E(0) + E_2(0)\big)\, e^{-C_8 t}, \qquad t \geqq t_0. \tag{21}$$

We recall that Shahinpoor and Ahmadi [8] make an assumption analogous to (20) and obtain a result of the type (21) for homothermal deformations of compressible micropolar fluids. It is not quite clear under what circumstances (20) holds. Mainly because of this, we made no attempt to obtain the best possible estimate of the value of C_8. It depends upon the shape of the container and the range of values of viscosity coefficients and heat conduction coefficients.

The assumption that a classical solution of (1) under the boundary conditions (3) and initial conditions belonging to set S is made here to keep the analysis simple. For the purpose of proving the theorem it suffices to assume that a suitably defined weak solution (e.g. see [2]) of (1) exists.

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