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Saint-Venant's Principle for a Micropolar Helical Body

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With 2 Figures

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Summary

The analog of Toupin's version of Saint-Venant's principle is proved for an isotropic, linear elastic micropolar body of arbitrary length and of uniform cross-section which in the unstressed state is helical. That is, when such a body is loaded by self-equilibrated stresses and couple stresses at one end only, we show that the elastic strain energy stored in the part of the body beyond a certain distance from the loaded end, decreases exponentially with the distance.

Introduction

In 1965 Toupin [1] gave a precise mathematical formulation and proof of Saint-Venant's principle for non-polar elastic bodies. He showed that for a linear elastic homogeneous cylindrical body of arbitrary length and cross-section loaded on one end only by an arbitrary system of self-equilibrated forces, the elastic energy $U(s)$ stored in that part of the body which is beyond a distance s from the loaded end satisfies the inequality

$$U(s) \leqq U(0) \exp\left(-(s-l)/s_c\right). \tag{1}$$

The characteristic decay length $s_c(l)$ depends upon the maximum and the minimum elastic moduli for the material and the smallest nonzero characteristic frequency of free vibration of a slice of the cylinder of length l. For isotropic materials, he proved that the inequality (1) implies the exponential decay of stresses with the distance from the loaded end. An inequality of the type (1) for a homogeneous isotropic micropolar linear elastic cylindrical body has been obtained by Berglund [2]. By using an estimate, due to Ericksen [1, p. 88], for the norm of the stress-tensor in terms of the strain-energy density, Berglund showed that $s_c(l)$ depends on the maximum elastic modulus.

The statements and proofs of other mathematical versions of Saint-Venant's principle due to Sternberg, Knowles, Zanaboni, and Robinson and of Toupin's version of the Saint-Venant principle are given in Gurtin's monograph [3].

Recently, Batra [4] proved an inequality similar to (1) for a linear elastic anisotropic helical spring of arbitrary but constant cross-section. Herein we generalize that to the case when the helical body is made of a micropolar material

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but is isotropic. We assume that the cross sections are materially uniform in the sense that one cross section can be obtained from the other by a rigid body motion. Thus the material properties do not depend upon the axial coordinate of the point. This idea of material uniformity is due to Ericksen [5] who has discussed this concept in more general terms.

We describe the deformation of the helical body with respect to suitably selected coordinate axes and use equilibrium equations in the form of Euler-Lagrange equations derived by extremising a functional. The characteristic decay length is found to depend upon the maximum elastic modulus and the characteristic frequency of free vibration of a slice of the helical body of axial length l. This agrees with the result obtained by Berglund for a straight prismatic micropolar body. Thus relatively large elasticities associated with the microdeformation will reduce the rate of decay of the energy.

Formulation of the Problem

Consider an isotropic linear elastic micropolar body B of arbitrary but constant cross-section which in the unstressed state is a clockwise helix. Introduce two coordinate systems, one a fixed rectangular coordinate system $\boldsymbol{X}$ with X^3-axis coincident with the axis of the helix, the plane $X^3 = 0$ containing one end cross section of the helix with $X^3 \geqq 0$ for points in the body and the other a curvilinear coordinate system $\boldsymbol{Y}$ related to the former by the transformation

$$\begin{bmatrix} Y^1 \\ Y^2 \\ Y^3 \end{bmatrix} = \begin{bmatrix} \cos bX^3 & -\sin bX^3 & 0 \\ \sin bX^3 & \cos bX^3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X^1 \\ X^2 \\ X^3 \end{bmatrix}. \tag{2.1}$$

The coordinate transformation (2.1) is invertible, the inverse being

$$\begin{bmatrix} X^1 \\ X^2 \\ X^3 \end{bmatrix} = \begin{bmatrix} \cos bY^3 & \sin bY^3 & 0 \\ -\sin bY^3 & \cos bY^3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y^1 \\ Y^2 \\ Y^3 \end{bmatrix}. \tag{2.2}$$

Here b equals the angle of twist of the helix. Under the coordinate transformation (2.1), $Y^3 = X^3$ and the Y^1, Y^2-coordinate curves are obtained by rotating clockwise the X^1, X^2-coordinate axes through an angle bX^3, the axis of rotation being parallel to X^3-axis. In index notation, we write (2.1) as

$$Y^i = R_\alpha{}^i X^\alpha. \tag{2.3}$$

Throughout this paper we use a mixture of direct and index notation. Repeated indices imply summation over the range of indices. $\delta_{ij} = \delta^{ij} = \delta_j{}^i$ is the Kronecker delta. The Greek indices refer to components with respect to $\boldsymbol{X}$-axes and both the upper case and lower case Latin indices refer to components with respect to $\boldsymbol{Y}$-axes. The upper case Latin indices take values 1, 2; other indices assume values 1, 2, 3. A comma followed by an index j indicates partial derivatives with respect to Y^j.

We note that when the helix angle b equals zero, the body is straight prismatic and the coordinate systems $\boldsymbol{X}$ and $\boldsymbol{Y}$ coincide with each other. In the $\boldsymbol{Y}$ coordinate system, the helical body of axial length L occupies the cylindrical region $C_0 X[0, L]$

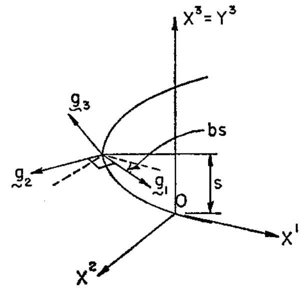

Fig. 1

where C_0 is the end cross-section of the body contained in the plane $X^3 = 0$. The covariant base vectors $\boldsymbol{g}_i$ directed tangentially along the Y^i-coordinate curves are given by

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{X}}{\partial Y^i} = (R_i^{\cdot\alpha} + \delta_{i3} R^{\alpha}_{K,3} Y^K)\, \boldsymbol{e}_\alpha ,$$

in which $\boldsymbol{e}_\alpha$ are base vectors for the Cartesian coordinate axes $\boldsymbol{X}$. The base vectors at a typical point are shown in Fig. 1. The base vectors $\boldsymbol{g}_i$ do not form an orthogonal set. This becomes obvious when we look at the explicit expression, given below, for the metric tensor $\boldsymbol{G}$ defined as

$$G_{ij} = \frac{\partial X^\alpha}{\partial Y^i} \frac{\partial X^\alpha}{\partial Y^j} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j ,$$

$$[G_{ij}] = \begin{bmatrix} 1 & 0 & -bY^2 \\ 0 & 1 & bY^1 \\ -bY^2 & bY^1 & 1 + b^2[(Y^1)^2 + (Y^2)^2] \end{bmatrix}. \tag{3}$$

The metric tensor G^{ij} is obtained by inverting (3). One can raise or lower Latin indices by using $\boldsymbol{G}$. Since det $[G_{ij}] = 1$, the volume element dV given by $dX^1\, dX^2 \cdot dX^3$ equals $dY^1\, dY^2\, dY^3$.

In micropolar continuum mechanics [6], to each point P of the continuum, we have a director $\overrightarrow{PQ}$ attached (see Fig. 2) that can only rotate, its rotation being

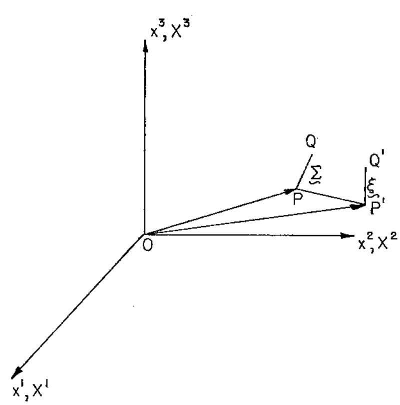

Fig. 2

independent of the deformations of the continuum surrounding P. Setting

$$\begin{aligned}\overrightarrow{OP} &= \boldsymbol{X} = X^\alpha \boldsymbol{e}_\alpha = (R_i{}^\alpha Y^i)\, \boldsymbol{e}_\alpha, \\ \overrightarrow{PQ} &= \boldsymbol{\Sigma} = \hat{\Sigma}^\alpha \boldsymbol{e}_\alpha = (R_i{}^\alpha \Sigma^i)\, \boldsymbol{e}_\alpha, \\ \overrightarrow{OQ} &= \hat{\boldsymbol{X}} = \hat{X}^\alpha \boldsymbol{e}_\alpha,\end{aligned} \tag{4}$$

we have

$$\hat{X}^\alpha = X^\alpha + \hat{\Sigma}^\alpha = R_i{}^\alpha (Y^i + \Sigma^i). \tag{5}$$

Due to the application of loads to the body, let the point P deform into P' and the director $\overrightarrow{PQ}$ into $\overrightarrow{P'Q'} = \boldsymbol{\xi}$. Letting

$$\begin{aligned}\overrightarrow{PP'} &= \hat{u}^\alpha \boldsymbol{e}_\alpha = (R_i{}^\alpha u^i)\, \boldsymbol{e}_\alpha, \\ \overrightarrow{P'Q'} &= \hat{\xi}^\alpha \boldsymbol{e}_\alpha = (R_i{}^\alpha \xi^i)\, \boldsymbol{e}_\alpha, \\ \overrightarrow{OP'} &= x^\alpha \boldsymbol{e}_\alpha = (X^\alpha + \hat{u}^\alpha)\, \boldsymbol{e}_\alpha \\ \overrightarrow{OQ'} &= \hat{x}^\alpha \boldsymbol{e}_\alpha = (x^\alpha + \hat{\xi}^\alpha)\, \boldsymbol{e}_\alpha,\end{aligned} \tag{6}$$

$$\hat{\xi}^\alpha = \hat{\Sigma}^\alpha + \hat{\Phi}_\beta{}^\alpha \hat{\Sigma}^\beta, \quad \hat{\Phi}_\beta{}^\alpha = -\hat{\Phi}_\alpha{}^\beta, \tag{6.9, 6.10}$$

we have

$$\hat{x}^\alpha = X^\alpha + \hat{u}^\alpha + \hat{\xi}^\alpha = R_i{}^\alpha (Y^i + u^i + \Sigma^i + \Phi_j{}^i \Sigma^j), \tag{7}$$

with

$$\Phi_j{}^i \equiv R_\alpha{}^i \hat{\Phi}_\beta{}^\alpha R_j{}^\beta = -\Phi_i{}^j. \tag{8}$$

Note that u^3 equals the displacement of a point along the axis of the helix and, u^1 and u^2 equal components of displacement $\hat{\boldsymbol{u}}$ along Y^1 and Y^2 coordinate curves. Thus u^3 is not a component of $\hat{\boldsymbol{u}}$ along the Y^3-coordinate curve. A similar interpretation applies to $\hat{\xi}^3$. The use of $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{\xi}}$ instead of $\boldsymbol{u}$ and $\boldsymbol{\xi}$ simplifies considerably the algebraic work. The assumptions (6.9) and (6.10) imply that the director can undergo infinitesimal rotation independent of the deformation of the continuum around P.

In order to calculate the expressions for appropriate strain tensors, we calculate

$$ds^2 - dS^2 \equiv d\hat{x}^\alpha\, d\hat{x}^\alpha - dX^\alpha\, dX^\alpha$$

and retain terms linear in $\boldsymbol{u}$ and $\boldsymbol{\Phi}$, obtaining thereby

$$\frac{1}{2}(ds^2 - dS^2) = (A_{ij} + B_{ijk}\Sigma^k)\, dY^i\, dY^j + (e_{ij} + D_{ijk}\Sigma^k)\, dY^i\, d\Sigma^j,$$

in which

$$e_{ij} = \delta_{jk} u^k{}_{,i} + b\varepsilon_{jp3}\delta_{3i} u^p + \delta_{ik}\Phi_j{}^k + b\varepsilon_{pq3}\delta_{3i}\Phi_k{}^p \delta_j{}^k Y^q, \tag{9}$$

$$D_{ijk} = \delta_{jp}\Phi^p_{k,i} + b\varepsilon_{jp3}\delta_{3i}\Phi_k{}^p + b\varepsilon_{pk3}\delta_{3i}\Phi_j{}^p. \tag{10}$$

We omit writing the lengthy expressions for A_{ij} and B_{ijk}. ε_{ijk} is the permutation symbol that assumes values 1 or -1 according as i, j, k form an even or an odd permutation of 1, 2, 3, and is zero otherwise. For micropolar media [6], e_{ij} and D_{ijk} are measures of infinitesimal strain. When $b = 0$, their expressions (9) and (10) reduce to the rather familiar expressions used in the linear micropolar theory. Henceforth, we work with the axial vector $\boldsymbol{\Phi}$ related to the skew-symmetric matrix $\Phi_j{}^i$ by

$$\begin{aligned}\Phi_j{}^i &= \delta_{pj}\varepsilon^{ipk}\phi_k,\\ \phi_k &= -\frac{1}{2}\,\varepsilon_{kqn}\Phi_m{}^n\delta^{mq}.\end{aligned} \tag{11}$$

For a linear elastic micropolar body that is stress free in the reference configuration, the strain energy density W per unit volume can be assumed to be a positive definite homogeneous quadratic form in the infinitesimal strains e_{ij} and D_{ijk}. When the body is also isotropic, W can be written explicitly as

$$2W = A^{ijkl}e_{ij}e_{kl} + B^{ijklmn}D_{ijk}D_{lmn}, \tag{12}$$

with

$$2W_1 \equiv A^{ijkl}e_{ij}e_{kl}, \tag{13}$$

$$2W_2 \equiv B^{ijklmn}D_{ijk}D_{lmn}, \tag{14}$$

being positive definite. There is no mixed term in (12) because of the assumption of isotropy. The assumption that in the unstressed reference configuration various cross-sections are materially uniform implies that the elasticities A^{ijkl} and B^{ijklmn} are functions of atmost Y^1, Y^2. The elasticities satisfy the symmetry relations

$$A^{ijkl} = A^{klij}, \qquad B^{ijklmn} = B^{lmnijk}. \tag{15}$$

Substitution from (11) into (9) and (10) and of the resulting expressions for e_{ij} and D_{ijk} into (12) gives

$$W = \overline{W}(u^i{}_{,j}, u^i, \phi^i{}_{,j}, \phi^i, Y^K). \tag{16}$$

$\overline{W}$ is a homogeneous quadratic function of the indicated variables except Y^K. The requirement that W be unaltered by a superimposed rigid body motion yields

$$\overline{W}(u^i{}_{,j}, u^i, \phi^i{}_{,j}, Y^K) = \overline{W}(v^i{}_{,j}, v^i, \Psi^i{}_{,j}, \Psi^i, Y^K), \tag{17}$$

where

$$v^i = u^i + w^i, \tag{18}$$

$$w^i = R_\alpha{}^i(a^\alpha + b_\beta{}^\alpha R_j{}^\beta Y^j), \tag{19}$$

$$\Psi^i = \phi^i - \frac{1}{2}\,R_\alpha{}^i(\varepsilon^{\alpha\beta\gamma}b_{\beta\gamma}), \tag{20}$$

$$b_\alpha{}^\beta = -b_\beta{}^\alpha. \tag{21}$$

The vector $\boldsymbol{w}$ represents the superimposed rigid body motion, its components w^1 and w^2 are measured along Y^1 and Y^2 coordinate curves and w^3 is measured along the axis of the helix. An explicit expression for $\boldsymbol{w}$ is given in [4].

Equilibrium equations governing the static deformations of a micropolar helical body B in the absence of body forces obtained by taking the extremum of

$$\int_B \overline{W}\,dV - \int_{\partial B} f_i u^i\,dU^1\,dU^2 - \int_{\partial B} m_i\phi^i\,dU^1\,dU^2$$

are

$$\left(\frac{\partial \overline{W}}{\partial u^i,_j}\right)_{,j} - \frac{\partial \overline{W}}{\partial u^i} = 0 \quad \text{in } B, \tag{22}$$

$$\left(\frac{\partial \overline{W}}{\partial \phi^i,_j}\right)_{,j} - \frac{\partial \overline{W}}{\partial \phi^i} = 0 \quad \text{in } B, \tag{23}$$

$$\frac{\partial \overline{W}}{\partial u^i,_j} n_j = f_i, \qquad \frac{\partial \overline{W}}{\partial \phi^i,_j} n_j = m_i \quad \text{on } \partial B, \tag{24}$$

$$n_i = \pm \varepsilon_{ijk} \frac{\partial Y^j}{\partial U^1} \frac{\partial Y^k}{\partial U^2} \tag{25}$$

$$dS_i = n_i \, dU^1 \, dU^2. \tag{26}$$

Here ∂B, the boundary of the body B, is assumed to be given parametrically by $\boldsymbol{Y} = \boldsymbol{Y}(U^1, U^2)$ and the sign in (25) is selected so that $\boldsymbol{n}$ points out of ∂B. Also $\boldsymbol{f}$ is the applied force and $\boldsymbol{m}$ the applied couple stress vector, each being measured per unit coordinate area $dU^1 \, dU^2$; dS_i is the vector element of area on the surface. For a traction boundary value problem in which the body is loaded only at the end $X^3 = 0$, Eqs. (22)—(24) have a solution only if

$$\int\limits_{C_0} \frac{\partial \overline{W}}{\partial u^i,_3} \, dY^1 \, dY^2 = \int\limits_{C_0} \frac{\partial \overline{W}}{\partial \phi^i,_3} \, dY^1 \, dY^2 = 0, \tag{27}$$

i.e., the applied loads must be self-equilibrated.

With the definitions

$$C_s \equiv \{\boldsymbol{Y} : \boldsymbol{Y} \in B, \ Y^3 = s\},$$

$=$ cross-section of the body lying in the plane $Y^3 = s$,

$$C_{s,l} \equiv \{\boldsymbol{Y} : \boldsymbol{Y} \in B, \ s \leqq Y^3 \leqq s + l\},$$

$=$ portion of the body between the planes

$$Y^3 = s \quad \text{and} \quad Y^3 = s + l,$$

$$U(s) \equiv \int \overline{W} \, dY^1 \, dY^2 \, dY^3, \qquad Y^3 \geqq s$$

$a_m \equiv$ the supremum of the eigen values of A^{ijkl} regarded as a linear transformation on the space of second order tensors,

$b_m \equiv$ the supremum of the eigen values of B^{ijklmn} regarded as a linear transformation on the space of third order tensors,

$\lambda_0 \equiv$ the smallest non-zero characteristic value of free vibration of a slice of the helical body of axial length l and mass density as well as the density of microinertia equal to one,

(28)

we state and prove the theorem below.

Statement and Proof of the Theorem

Theorem. If for an isotropic linear elastic micropolar body which in the unstressed state is helical, the loads applied at the end $X^3 = 0$ satisfy (27) and

$$f_i = 0 = m_i \qquad \text{on} \qquad \partial B - C_0, \tag{29}$$

then

$$U(s) \leqq U(0) \exp\big(-(s-l)/s_c(l)\big), \tag{30}$$

where

$$\big(s_c(l)\big)^2 = \frac{1}{\lambda_0} \max(a_m, b_m). \tag{31}$$

Proof of the Theorem. Since $\overline{W}$ is a homogeneous quadratic form in $u^i{}_{,j}$, u^i, $\phi^i{}_{,j}$ and ϕ^i, by Euler's theorem,

$$\begin{aligned} U(s) &= \int\limits_{C_{s,l}} \overline{W}\, dV, \\ &= \frac{1}{2} \int\limits_{C_{s,l}} \left[\frac{\partial \overline{W}}{\partial u^i{}_{,j}} u^i{}_{,j} + \frac{\partial \overline{W}}{\partial u^i} u^i + \frac{\partial \overline{W}}{\partial \phi^i{}_{,j}} \phi^i{}_{,j} + \frac{\partial \overline{W}}{\partial \phi^i} \phi^i \right] dA \\ &= -\frac{1}{2} \int\limits_{C_s} \left[\frac{\partial \overline{W}}{\partial u^i{}_{,3}} u^i + \frac{\partial \overline{W}}{\partial \phi^i{}_{,3}} \phi^i \right] dA. \end{aligned} \tag{32}$$

In order to obtain $(32)_3$ from $(32)_2$, we used the divergence theorem, equilibrium Eqs. (22) and (23), the boundary condition (29) and that on C_s,

$$dS_k = -dY^1\, dY^2 \delta_{3k} = -dA\, \delta_{3k}.$$

Because of (17) we can replace $\boldsymbol{u}$ and $\boldsymbol{\phi}$ by $\boldsymbol{v}$ and $\boldsymbol{\Psi}$ respectively. Recalling Eqs. (18) to (20) we see that $\boldsymbol{v}$ and $\boldsymbol{\Psi}$ differ from $\boldsymbol{u}$ and $\boldsymbol{\phi}$ only by a rigid body motion. Thus

$$U(s) = -\frac{1}{2} \int \left[\frac{\partial \overline{W}}{\partial v^i{}_{,3}} v^i + \frac{\partial \overline{W}}{\partial \Psi^i{}_{,3}} \Psi^i \right] dA. \tag{33}$$

Physically this expresses the fact that any self-equilibrated force system does no work during a rigid motion of the body. From (33) and (9) we obtain

$$\begin{aligned} -\int\limits_{C_s} \frac{\partial \overline{W}}{\partial v^1{}_{,3}} v^1\, dA &= -\int\limits_{C_s} \frac{\partial W}{\partial e_{31}} v^1\, dA \\ &\leqq \frac{1}{2} \left[\nu \int\limits_{C_s} \frac{\partial W}{\partial e_{31}} \frac{\partial W}{\partial e_{31}}\, dA + \frac{1}{\nu} \int\limits_{C_s} v^1 v^1\, dA \right], \end{aligned} \tag{34}$$

wherein we used the inequality

$$2 \int fh\, dV \leqq \nu_0 \int f^2\, dV + \frac{1}{\nu_0} \int h^2\, dV \tag{35}$$

which is a consequence of the Schwarz and geometric-arithmetic mean inequalities (e.g. see Toupin [1, p. 93]). In (35) $\nu_0 > 0$ is an arbitrary constant and f and h are scalar fields defined on B. Thus ν in (34) is a positive constant still to be chosen. Writing inequality (34) with the index 1 replaced everywhere in turn by 2 and 3 and adding the respective sides of these three inequalities, we conclude that

$$-\int_{C_s} \frac{\partial \overline{W}}{\partial v^i{}_{,3}} v^i \, dA \leqq \frac{\delta_{ij}}{2} \left[\nu \int_{C_s} \frac{\partial W}{\partial e_{3i}} \frac{\partial W}{\partial e_{3j}} \, dA + \frac{1}{\nu} \int_{C_s} v^i v^j \, dA \right]. \tag{36}$$

Now

$$\begin{aligned} \delta_{ij} \frac{\partial W}{\partial e_{3i}} \frac{\partial W}{\partial e_{3j}} &\leqq \delta_{ij} \frac{\partial W}{\partial e_{ki}} \frac{\partial W}{\partial e_{lj}} \delta_{kl}, \\ &\leqq \delta_{ij} A^{mnki} e_{mn} A^{ljmp} e_{mp} \delta_{kl}, \\ &\leqq 2 a_m W_1. \end{aligned} \tag{37}$$

Details of deducing (37.3) from (37.2) are given by Berglund [2]. Substitution from (37) into (36) results in

$$-\int_{C_s} \frac{\partial \overline{W}}{\partial v^i{}_{,3}} v^i \, dA \leqq \frac{1}{2} \left[2 \nu a_m \int_{C_s} W_1 \, dA + \frac{1}{\nu} \int_{C_s} v^i v^j \delta_{ij} \, dA \right]. \tag{38}$$

Similarly, we can show that

$$-\int \frac{\partial \overline{W}}{\partial \Psi^i{}_{,3}} \Psi^i dA \leqq \frac{1}{2} \left[2 \mu b_m \int_{C_s} W_2 \, dA + \frac{1}{\mu} \int_{C_s} \Psi^i \Psi^j \delta_{ij} \, dA \right] \tag{39}$$

in which $\mu > 0$ in an arbitrary constant. From (38), (39), and (33) and choosing $\nu = \mu$, we obtain

$$\begin{aligned} U(s) \leqq \frac{\nu}{2} &\left(a_m \int_{C_s} W_1 \, dA + b_m \int_{C_s} W_2 \, dA \right) \\ &+ \frac{\delta_{ij}}{4\nu} \left(\int_{C_s} v^i v^j \, dA + \int_{C_s} \Psi^i \Psi^j \, dA \right). \end{aligned} \tag{40}$$

Integrating both sides of this inequality with respect to Y^3 from $Y^3 = s$ to $Y^3 = s + l$ for some $l > 0$ and setting

$$\frac{1}{l} \int_s^{s+l} U(s') \, ds' = Q(s, l) \tag{41}$$

we arrive at the following

$$\begin{aligned} Q(s, l) \leqq \frac{\nu}{2l} &\left(a_m \int_{C_{s,l}} W_1 \, dV + b_m \int_{C_{s,l}} W_2 \, dV \right) \\ &+ \frac{\delta_{ij}}{4\nu l} \left(\int_{C_{s,l}} v^i v^j \, dV + \int_{C_{s,l}} \Psi^i \Psi^j \, dV \right). \end{aligned} \tag{42}$$

In an effort to bound the last two integrals on the right-hand side of (42) by an integral of W_1 and W_2, we consider the free vibration problem of the helical body of unit mass density and unit density of microinertia. Define a characteristic solution (e.g. see Gurtin [3, section 75] and Anderson [7]) as the ordered triplet $[\lambda, \boldsymbol{u}, \boldsymbol{\phi}]$ such that λ is a scalar and $\boldsymbol{u}$ and $\boldsymbol{\phi}$ are fields on B, and

$$\begin{aligned} &\left(\frac{\partial \overline{W}}{\partial u^i{}_{,j}}\right)_{,j} - \frac{\partial \overline{W}}{\partial u^i} + \lambda u_i = 0 \qquad \text{in } B, \\ &\left(\frac{\partial \overline{W}}{\partial \phi^i{}_{,j}}\right)_{,j} - \frac{\partial \overline{W}}{\partial \phi^i} + \lambda \phi_i = 0 \qquad \text{in } B, \\ &\delta_{ij} \int_B (u^i u^j + \Psi^i \Psi^j)\, dV = 1, \\ &u^i \frac{\partial \overline{W}}{\partial u^i{}_{,k}}\, dS_k = \Psi^i \frac{\partial \overline{W}}{\partial \Psi^i{}_{,k}}\, dS_k = 0 \qquad \text{on } \partial B. \end{aligned} \tag{43}$$

By proceeding in the same way as that given in [3] we verify that

$$\lambda = \frac{2 \int_{C_{s,l}} \overline{W}\, dV}{\delta_{ij} \int_{C_{s,l}} (v^i v^j + \Psi^i \Psi^j)\, dV}.$$

Thus the lowest non-zero characteristic value λ_0 corresponding to the free vibration of $C_{s,l}$ satisfies the inequality

$$\lambda_0 \leqq \frac{2 \int_{C_{s,l}} \overline{W}\, dV}{\delta_{ij} \int_{C_{s,l}} (v^i v^j + \Psi^i \Psi^j)\, dV} \tag{44}$$

for every smooth fields $\boldsymbol{v}$ and $\boldsymbol{\Psi}$ on $C_{s,l}$ such that

$$\delta_{ij} \int_{C_{s,l}} v^i v^j\, dV \neq 0, \qquad \int_{C_{s,l}} v^i\, dV = \int_{C_{s,l}} \varepsilon_{ijk} x^j v^k\, dV = 0. \tag{45}$$

Following Toupin [1], for a given $\boldsymbol{u}$, we can choose $\boldsymbol{v}$ such that (45) is satisfied; the corresponding $\boldsymbol{\Psi}$ is obtained from the given $\boldsymbol{\phi}$ by using (20). Inequality (44) when combined with (42) gives

$$Q(s, l) \leqq \frac{\overline{s_c(l)}}{l} \int_{C_{s,l}} \overline{W}\, dV \tag{46}$$

in which

$$\begin{aligned} \overline{s_c(l)} &= \frac{1}{2}\left(\nu\sigma + \frac{1}{\nu\lambda_0}\right), \\ \sigma &= \max\,(a_m, b_m). \end{aligned} \tag{47}$$

We choose $\nu = 1/\sqrt{\lambda_0 \sigma}$ so that $\overline{s_c(l)}$ takes the minimum value

$$s_c(l) = \sqrt{\sigma/\lambda_0}. \tag{48}$$

Differentiation of (41) with respect to s yields

$$\frac{dQ}{ds} = \frac{1}{l}\,[U(s+l) - U(s)] = -\frac{1}{l}\int\limits_{C_{s,l}} \overline{W}\,dV,$$

and this together with (46) and (48) result in

$$s_c(l)\,\frac{dQ}{ds} + Q \leqq 0. \tag{49}$$

Integrating (49) and using

$$U(s+l) \leqq Q(s,l) \leqq U(s)$$

which follows from the observation that $U(s)$ is a nonincreasing function of s, we arrive at

$$\frac{U(s_2+l)}{U(s_1)} \leqq \exp\big(-(s_2 - s_1)/s_c(l)\big).$$

The choice $s_1 = 0$ and $s_2 + l = s$ gives the desired inequality (30).

Remarks

For the micropolar helical body the decay rate (31) is of the same form as that obtained by Berglund for a straight prismatic micropolar body and has no explicit dependence upon the helix angle b. Of course, the value of λ_0, the lowest non-zero characteristic value of a slice of the helical body of axial length l does depend, among other quantities, upon b. For nonpolar helical bodies [4], the decay rate has an explicit as well as implicit dependence upon the helix angle b. Equation (48) implies that the decay rate is inversely proportional to the squareroot of the maximum elasticity. Thus a rather large value of one of the elasticities associated either with macrodeformation or microdeformation will result in a very low decay rate of the energy.

We remark that in the definition of the characteristic solution, we have taken the mass density and the density of microinertia as unity. As was the case with straight prismatic micropolar body, the decay constant neither depends upon mass density nor on the density of microinertia. Whereas for a nonpolar helical body, inorder to prove a similar result, we did not have to assume that the body is isotropic, here the assumption of isotropy is used in writing the expression (12) for the strain energy density.

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