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ON THE INTERFACE STABILITY OF A NECK PROPAGATING IN A SHEET REINFORCED WITH SHAPE-MEMORY FIBERS†

C. Q. RU and R. C. BATRA

Department of Mechanical and Aerospace Engineering and Engineering Mechanics, University of Missouri-Rolla, Rolla, MO 65401-0249, U.S.A.

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Abstract—We propose a simple model of necking in a sheet reinforced with straight fibers made of a shape-memory alloy and examine conditions under which the interface between the necked and the unnecked region is morphologically stable. We use the Mullins and Sekerka (1963, *J. Appl. Phys.* **34**, 323-330; 1964, *J. Appl. Phys.* **35**, 444-450) method, established for studying the stability of a moving interface in a solidification problem, to investigate the interface stability of a propagating neck. It is found that the moving straight interface is morphologically stable for several typical cases even in the absence of surface-tension effects, and the stationary interface is always stable.

1. INTRODUCTION

One-dimensional phase-transition problems in shape-memory alloys and certain polymers modeled as nonlinear elastic solids have been extensively studied (Ericksen, 1975; James, 1979; Hutchinson and Neale, 1983; Coleman, 1985; Falk and Seibel, 1987), and the solution corresponding to a phase transition between two stable uniform states has been referred to as necking (Hutchinson and Neal, 1983; Coleman, 1985). Physicists (Barsch and Krumhansl, 1988; Jacobs, 1985, 1992) working in the area of ferroelasticity have examined phase transformations in two-dimensional problems. In two-dimensional solidification problems, it has been found (Langer, 1980; Delves, 1975; Godreche, 1992; Davis *et al.*, 1990) that the straight interface between the solid and liquid phases will become unstable and deform into a cellular one under certain conditions, and may even develop into a more complicated dendritic pattern.

Here we study the necking phenomenon in a sheet reinforced with a large number of straight shape-memory alloy fibers aligned in the longitudinal direction. It is assumed that the nonlinear stress-strain curve for the material of the fibers exhibits two stable branches. We model the sheet as an elastic material, consider a simple two-dimensional model for its deformations, and use Mullins and Sekerka's (1963, 1964) method to study the stability of the straight interphase boundary between the necked and unnecked regions. The model predicts that unlike solidification problems wherein surface tension forces are essential for the stability of the interface, the propagating straight boundary of the necked region can be stable even when there are no surface tension forces.

2. FORMULATION OF THE PROBLEM

We consider a fiber-reinforced flat sheet of unit thickness with fibers aligned along the x-direction, and the end surfaces of the sheet subjected to uniform surface tractions in the x-direction. We assume that the sheet undergoes infinitesimal deformations so that linear kinematic relations apply, and the fibers are densely packed so that lateral deformations of the sheet are negligible. Thus every point of the sheet undergoes a time-dependent displacement u in the x-direction, and u is in general also a function of x and y. Whereas both the fibers and the matrix material are taken to be elastic, the stress-strain curve for the former is taken to be nonlinear with two stable branches and

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that for the latter, linear. The material of the fibers is presumed to exhibit phase transformations. Also, fibers are assumed to carry all of the tensile load and the matrix material supports the shear stress. Thus

$$\sigma_{xx} = f(\varepsilon_{xx}) = \frac{\partial W}{\partial \varepsilon_{xx}}, \qquad \varepsilon_{xx} = u_{,x}; \qquad \sigma_{xy} = 2G\varepsilon_{xy} = G\frac{\partial u}{\partial y} = Gu_{,y}. \tag{1}$$

Here σ_{xx} and σ_{xy} are the nonzero components of the Cauchy stress tensor, ε_{xx} and ε_{xy} are nonvanishing components of the infinitesimal strain tensor, W is the strain-energy density for the fiber material normalized to vanish for null value of ε_{xx}, and G is the shear modulus of the matrix material. The axial displacement u is governed by

$$\rho u_{,tt} = f'(\varepsilon_{xx})u_{,xx} + Gu_{,yy} \tag{2}$$

which is hyperbolic if $f'(\varepsilon_{xx}) > 0$ and the mass density ρ is positive. Henceforth we assume that $\rho > 0$ and $f'(\varepsilon_{xx}) > 0$ for the two stable branches. Here f' denotes the derivative of f with respect to its argument. We note that eqn (2) differs from that in a solidification problem wherein the governing equation is parabolic. The second term on the right-hand side of eqn (2) describes the coupling effect between adjacent fibers.

Since fiber material is presumed to exhibit phase transformation, it is conceivable that there is a surface of discontinuity, or a singular surface, in the sheet in the sense that fibers are in different phases on the two sides of this surface. Across this surface, the displacements must be continuous, i.e.

$$[\![u]\!] = 0 \tag{3}$$

where $[\![u]\!] = u^+ - u^-$ equals the difference in the values of u on the positive and negative sides of the surface; the positive side of the surface being the one on which the outward normal points in the direction of propagation of the surface. If this surface propagates with speed V in the x-direction, then the Rankine–Hugoniot (Hutchinson and Neale, 1983; Falk and Seibel, 1987; Smoller, 1983) condition requires that

$$-\rho V[\![u_{,t}]\!] = [\![f(u_{,x})]\!]. \tag{4}$$

The balance of total energy across the singular surface gives

$$-V[\![W + \tfrac{1}{2}\rho(u_{,t})^2]\!] = [\![f(u_{,x})u_{,t}]\!]. \tag{5}$$

3. STEADY-STATE NECKING SOLUTION

We consider a necking solution that is independent of y, and with

$$z = x - Vt \tag{6}$$

can be expressed as

$$u(x, t) = U(z). \tag{7}$$

Thus

$$u_{,x} = U', \qquad u_{,t} = -VU', \qquad U' \equiv U_{,z}. \tag{8}$$

Substitution from (8) into (2) gives

$$\rho V^2 U'' = f'(U')U'' \tag{9}$$

which holds when either

$$U' = \text{const.} \qquad \text{or} \qquad \rho V^2 = f'(U'). \tag{10}$$

As shown below [cf. eqn (17)], the second relation need not always holds, so we take $U' =$ const. as the solution of eqn (9). Thus the steady-state solution in the z, t frame of reference with interface $z = 0$ consists of

$$U' = E_1, \qquad U = U_1 = E_1 z \qquad \text{for } z < 0 \tag{11}$$

$$U' = E_2, \qquad U = U_2 = E_2 z \qquad \text{for } z > 0, \tag{12}$$

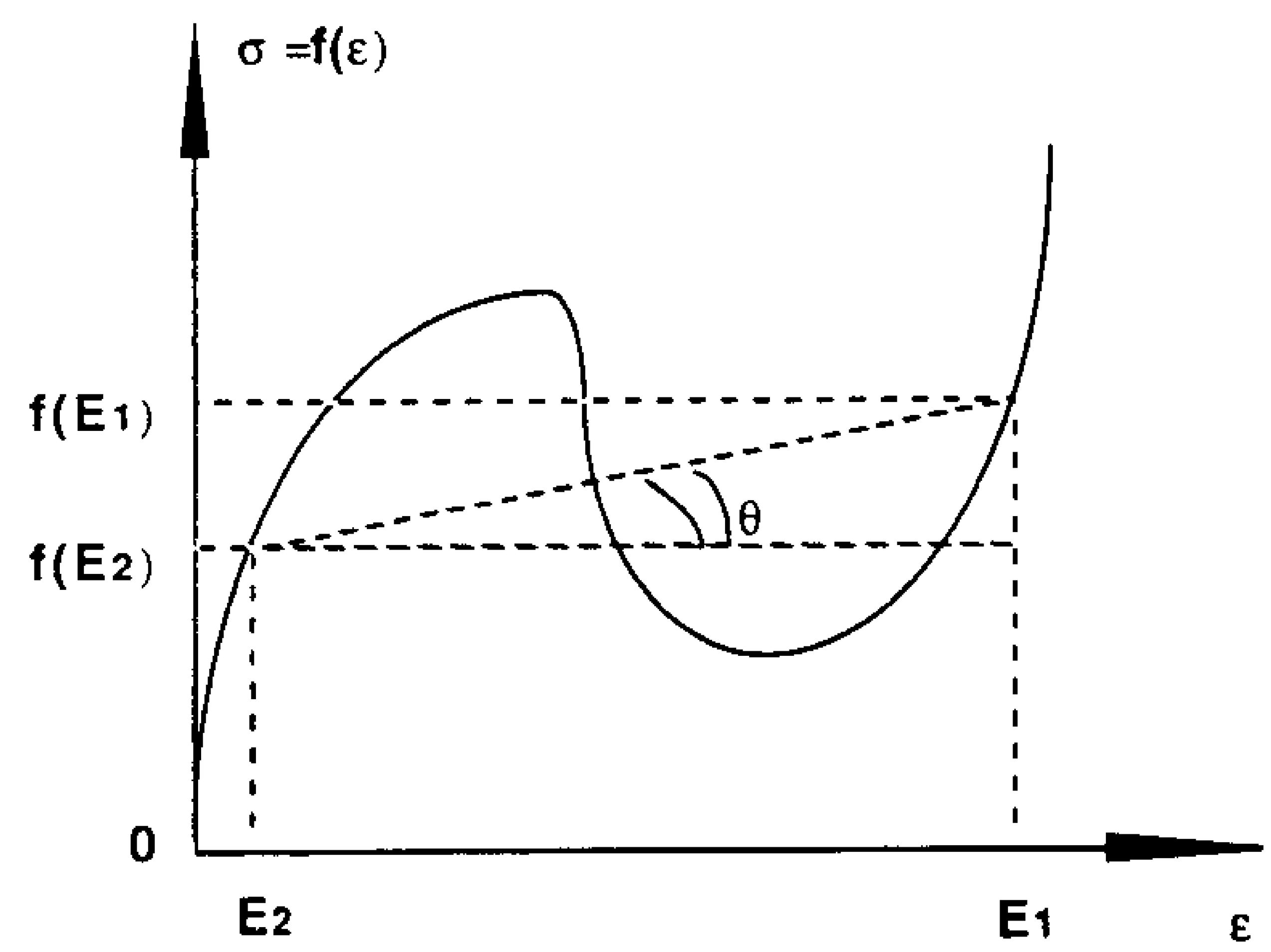

Fig. 1. The stress–strain curve for the shape-memory fibers, and geometric interpretation of eqn (13).

where E_1 and E_2 are positive constants. Without any loss of generality, we take $E_1 > E_2$. The jump conditions (4) and (5) reduce to

$$\rho V^2(E_1 - E_2) = f(E_1) - f(E_2), \tag{13}$$

$$[\![W(U') + \tfrac{1}{2}\rho V^2(U')^2]\!] = [\![f(U')U']\!]. \tag{14}$$

Since $E_1 > E_2$, therefore, relation (13) gives $f(E_1) > f(E_2)$. Relations (13) and (14) also imply the so called equal-area rule, i.e.

$$W(E_1) - W(E_2) = (f(E_1) + f(E_2))(E_1 - E_2)/2. \tag{15}$$

On the stress–strain curve shown in Fig. 1, let the straight line joining the states $(E_1, f(E_1))$, $(E_2, f(E_2))$ subtend an angle θ with the horizontal axis. Then

$$\operatorname{tg}\theta = \rho V^2 \tag{16}$$

follows from eqn (13). Thus

$$\rho V^2 < f'(E_1), \qquad \rho V^2 < f'(E_2), \tag{17}$$

and the maximum value V_m of V occurs when $E_2 = 0$. V_m is given by

$$\rho V_m^2 E_1 = f(E_1), \tag{18}$$

$$W(E_1) = f(E_1)E_1/2. \tag{19}$$

However, when ρV^2 is negligible, we have

$$f(E_1) = f(E_2) = \sigma_0, \qquad \text{the axial traction applied at the ends,} \tag{20}$$

$$W(E_1) - W(E_2) = \sigma_0(E_1 - E_2), \tag{21}$$

which is the classical "Maxwell rule", e.g. see Ericksen (1975), Hutchinson and Neal (1983) and Coleman (1985).

4. LINEARIZED ANALYSIS OF THE STABILITY OF THE INTERFACE

According to Mullins and Sekerka's method (Langer, 1980; Delves, 1975; Godreche, 1992; Davis *et al.*, 1990), we consider a perturbation of the interface geometry in the reference frame $\{z = x - Vt, y, t\}$ moving with the speed V (cf. Fig. 2). Thus, let the interface geometry be perturbed to that given by

$$z = z^*(y, t) = \delta \cos ky\, e^{\omega t}, \tag{22}$$

and the steady-state solution be replaced by

$$u(z, y, t) = E_1 z + u_1(z) \cos ky\, e^{\omega t}, \qquad z < z^*(y, t), \tag{23}$$

$$u(z, y, t) = E_2 z + u_2(z) \cos ky\, e^{\omega t}, \qquad z > z^*(y, t), \tag{24}$$

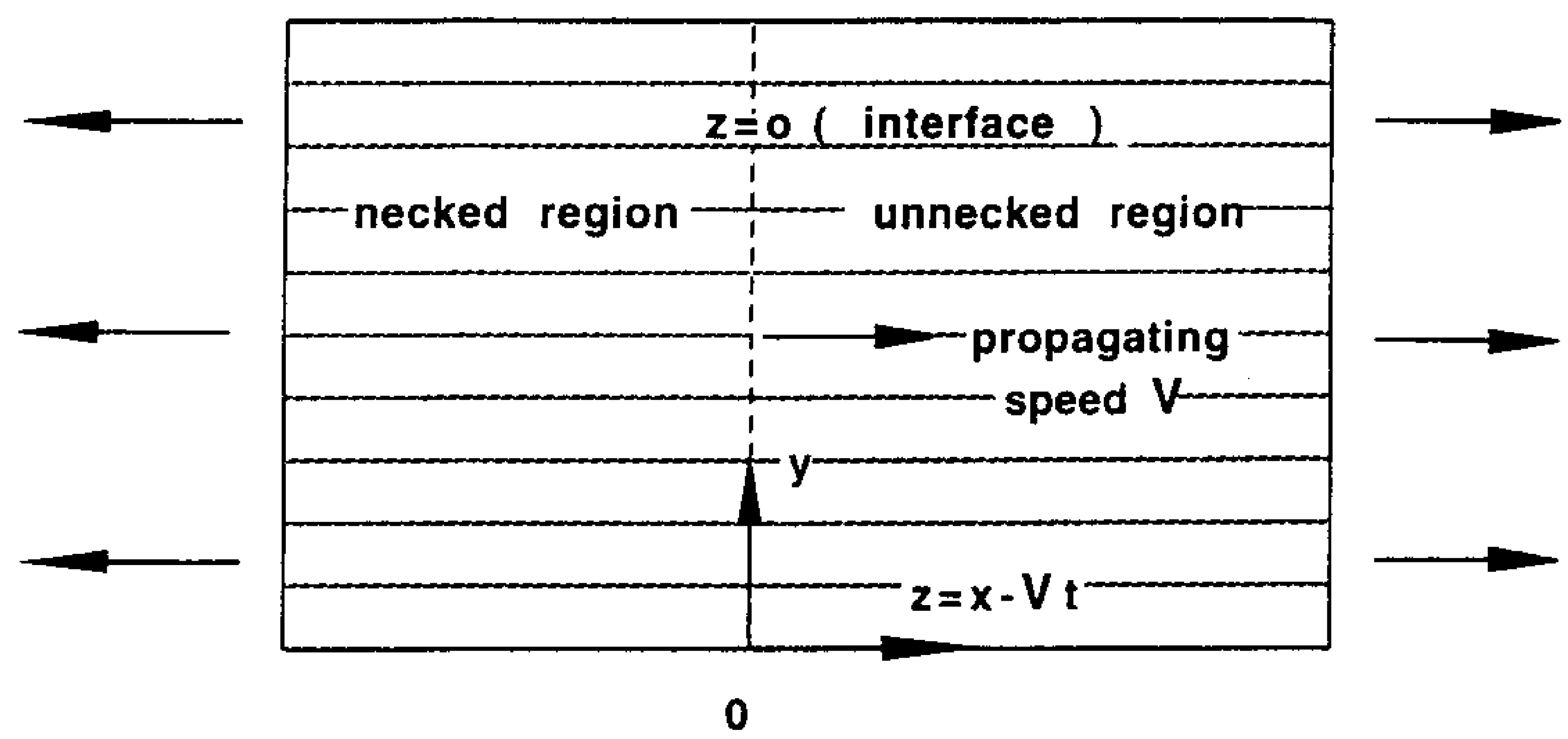

Fig. 2. A schematic sketch of a propagating neck in a uniaxially reinforced sheet.

where δ is an infinitesimal number, and $u_1(z)$ and $u_2(z)$ are of order δ. The balance of linear momentum (2) in the frame $\{z, y, t\}$ takes the form

$$\rho(V^2 u_{,zz} - 2Vu_{,zt} + u_{,tt}) = f'(u_{,z})u_{,zz} + Gu_{,yy}. \tag{25}$$

Keeping with the spirit of the Mullins and Sekerka method, we take

$$u_1(z) = \alpha\, e^{pz}, \qquad z < z^*(y, t), \tag{26}$$

$$u_2(z) = \beta\, e^{-qz}, \qquad z > z^*(y, t), \tag{27}$$

where α and β are of order δ, and p and q are two complex numbers with non-negative real parts (due to the localized characteristics of the perturbation) determined by

$$(\rho V^2 - f'(E_1))p^2 - 2V\omega\rho p + (\rho\omega^2 + Gk^2) = 0, \tag{28}$$

$$(\rho V^2 - f'(E_2))q^2 + 2V\omega\rho q + (\rho\omega^2 + Gk^2) = 0. \tag{29}$$

Thus

$$p = \{V\omega\rho \pm (Gk^2(f'(E_1) - \rho V^2) + \rho\omega^2 f'(E_1))^{1/2}\}/(\rho V^2 - f'(E_1)), \tag{30}$$

$$q = \{-V\omega\rho \pm (Gk^2(f'(E_2) - \rho V^2) + \rho\omega^2 f'(E_2))^{1/2}\}/(\rho V^2 - f'(E_2)). \tag{31}$$

The sign in eqns (30) and (31) is selected to ensure that the real parts of p and q are non-negative. That such roots exist, at least in some special cases, is illustrated below. For the case of real ω, inequalities (17) guarantee that the desired values of p and q are obtained by selecting the minus sign in expressions (30) and (31).

Let

$$\Delta \equiv (f'(E_2) - \rho V^2)q - (f'(E_1) - \rho V^2)p. \tag{32}$$

In the neighborhood of the critical point $\omega = 0$ of instability, if any, and for $|\omega| \ll 1$, conditions (28) and (29) reduce to

$$(f'(E_1) - \rho V^2)p^2 \approx Gk^2, \tag{33}$$

$$(f'(E_2) - \rho V^2)q^2 \approx Gk^2, \tag{34}$$

and

$$\Delta \approx (Gk^2(f'(E_2) - \rho V^2))^{1/2} - (Gk^2(f'(E_1) - \rho V^2))^{1/2}. \tag{35}$$

For infinitesimal perturbations (22) of the interface and the corresponding solution (23) and (24) we make the following observations.

(a) Let Θ be the angle that the normal to the interface makes with the x-direction. Then

$$\partial_n = \cos\Theta\,\partial_x + \sin\Theta\,\partial_y. \tag{36}$$

Since $\cos\Theta \sim 1 + O(\delta^2)$, $\sin\Theta \sim O(\delta)$, $\partial_y \sim O(\delta)$, we may approximate (36) by

$$\partial_n \approx \partial_x, \tag{37}$$

and we have ∂_x instead of ∂_n in the jump conditions for the curved interface $z = z^*$.

(b) The local speed v of the interface is given by

$$v = \frac{dx}{dt} = V + z^*_{,t}, \tag{38}$$

which need not equal V.

(c) In solidification problems, the forces due to surface tension have a stabilizing effect. Therefore, we consider them herein. Let η denote the surface tension per unit length along the interface and ψ the local curvature of the interface. Then eqn (4) should be replaced by

$$-\rho v [\![u_{,t}]\!] = [\![f(u_{,x})]\!] + \eta\psi. \tag{39}$$

The influence of η on the energy balance (5) is a second-order effect, and is therefore negligible.

Substitution from (23), (24), (26) and (27) into (3), (5) and (39) yields the following three linear homogeneous equations to determine δ, α and β:

$$\delta(E_1 - E_2) + (\alpha - \beta) = 0, \tag{40}$$

$$f'(E_1)p\alpha + f'(E_2)\beta q + \delta\eta k^2 = \rho V^2(\alpha p + q\beta) + \omega\rho V((\beta - \alpha) + \delta(E_1 - E_2)) \tag{41}$$

$$\begin{aligned} &\omega(f(E_1)(\alpha + \delta E_1) - f(E_2)(\beta + \delta E_2)) - Vf'(E_1)p\alpha E_1 - Vf'(E_2)q\beta E_2 \\ &\quad = \rho V^2(\omega(E_1\alpha - \beta E_2) - V(E_1\alpha p + q\beta E_2)). \end{aligned} \tag{42}$$

In eqns (40)–(42), terms involving the product $\omega\delta$ will disappear once we set $v = V$. The requirement that these equations have a nontrivial solution gives an algebraic equation for the determination of ω whose sign decides the stability of the interface. Alternatively, we derive from eqns (40)–(42) the following two homogeneous equations to solve for $(\alpha\text{–}\beta)$ and β:

$$\begin{aligned} &[(f'(E_2) - \rho V^2)qE_2 + (f'(E_1) - \rho V^2)pE_1]\beta \\ &\quad + [(f'(E_1) - \rho V^2)pE_1 + \omega\rho V(E_1 + E_2)](\alpha - \beta) = 0, \end{aligned} \tag{43}$$

$$\begin{aligned} &[(f'(E_2) - \rho V^2)q + (f'(E_1) - \rho V^2)p](E_1 - E_2)\beta \\ &\quad + [(f'(E_1) - \rho V^2)p(E_1 - E_2) + 2\rho\omega V(E_1 - E_2) - \eta k^2](\alpha - \beta) = 0, \end{aligned} \tag{44}$$

where the common factor V in (43) has been cancelled. In order for eqns (43) and (44) to have a nontrivial solution $((\alpha - \beta), \beta)$, we must have

$$\omega B = A \tag{45}$$

where

$$\begin{aligned} A = &-(E_1 - E_2)^2(f'(E_2) - \rho V^2)(f'(E_1) - \rho V^2)pq - \eta k^2((f'(E_2) - \rho V^2)qE_2 \\ &+ (f'(E_1) - \rho V^2)pE_1), \end{aligned} \tag{46}$$

$$B = \rho V(E_1 - E_2)^2\Delta. \tag{47}$$

When $V = 0^+$, $B = 0$, and eqn (45) gives $A = 0$. Therefore, there does not exist any admissible perturbation with $\mathrm{Re}(\omega) > 0$ and we conclude that the interface $z = 0$ with $V = 0$ is stable in the sense of Mullins and Sekerka's criterion of interface stability.

For the more interesting case of $V \neq 0$, eqn (45) determines ω and the sign of real part of ω decides the stability of the interface.

5. EXAMPLES OF THE INTERFACE STABILITY OF A PROPAGATING NECK

We now consider a few simple cases for which $V \neq 0$ and examine the interface stability of a propagating neck.

(i) $k = 0$

This corresponds to a one-dimensional problem and has also been studied earlier, e.g. see Hutchinson and Neale (1983). For $V \neq 0$ and $\rho \neq 0$, eqns (45)–(47) give

$$\omega = 0. \tag{48}$$

Thus the interface $z = 0$ with $V \neq 0$ is always stable in the sense of Mullins and Sekerka.

(ii) $\eta = 0$ and $f'(E_1) = f'(E_2)$

That is, there is no surface tension and the two modulii are equal. Equation (45) gives

$$\omega^2 = -Gk^2(f'(E_1) - \rho V^2)/(\rho(f'(E_1) + \rho V^2)) < 0, \tag{49}$$

which implies

$$Gk^2(f'(E_1) - \rho V^2) + \rho\omega^2 f'(E_1) > 0. \tag{50}$$

The inequality (50) ensures that $\mathrm{Re}(q) > 0$ and $\mathrm{Re}(p) > 0$. Therefore, we conclude that, when $\eta = 0$ and $f'(E_1) = f'(E_2)$, the admissible perturbation corresponds to $\omega^2 < 0$, and the straight interface $z = 0$ is stable in the sense of Mullins and Sekerka.

(iii) $G = 0$

Such a situation will occur approximately when the matrix material is very weak. Parameters A and B appearing in eqn (45) are given, respectively, by (46) and (47) with Δ defined by (32), and parameters p and q are determined from (30) and (31). The eigenequation (45) gives that either

$$\omega = 0 \tag{51}$$

or

$$\omega = \frac{\eta k^2[(\rho V \pm \sqrt{\rho f'(E_1)})E_1 + (-\rho V \pm \sqrt{\rho f'(E_2)})E_2]}{\rho(E_1 - E_2)^2[\rho V^2 + \sqrt{f'(E_2)f'(E_1)}]}. \tag{52}$$

The value of ω given by (52) contradicts (30) and (31). Therefore, we conclude that, when $G = 0$, the admissible perturbation corresponds to $\omega = 0$ and the straight interface $z = 0$ with $V \neq 0$ is always stable.

(iv) $\rho = 0$

This simulates the situation when fibers are extremely light but are quite strong. From (47) we get $B = 0$ and the eigenequation (45) gives

$$A = 0. \tag{53}$$

It follows from eqn (46) that

$$(E_1 - E_2)^2 f'(E_2)f'(E_1)pqV = -\eta k^2 V[f'(E_2)qE_2 + f'(E_1)pE_1], \tag{54}$$

which cannot be satisfied unless $\eta < 0$, since $p > 0$ and $q > 0$ are determined by (30) and (31). Thus the interface is stable.

6. CONCLUSIONS

We have proposed a simple model for mechanical deformations of a thin sheet reinforced with shape-memory fibers, and have used Mullins and Sekerka's method to analyze the stability of a straight interface between the necked and the unnecked regions. The fibers have been modeled as nonlinear elastic but the matrix as linearly elastic, and the end surfaces of the sheet are subjected to uniform surface tractions in the longitudinal direction. Whereas in solidification problems the surface tension is essential for interface stability, for the present problem, the propagating neck is found to be stable for several cases even in the absence of surface-tension forces. Also, the stationary interface is always stable. We note that the Mullins and Sekerka criterion of interface stability is conceptually different from the so-called "Marginal Stability" approach (Benjacob *et al.*, 1985) developed for studying the stability of a front propagating into an unstable state.

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