<u>Corrigendum — Corrections to</u> Local Water Slamming Impact on Sandwich Composite Hulls, *Journal of Fluids and Structures* **27**, **4**, 523-551, 2011.

K. Das and R. C. Batra, Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061

Email: Kaushik.DAS@3ds.com; rbatra@vt.edu

We erred in labeling curves in Fig. 26 of (Das and Batra, 2011) because of our inability to decipher incorrect labels of strain components in LS-PREPOST software. Figure 1a-d shows a picture of computer screen of LS-PREPOST software exhibiting the variation of different strain components on the hull geometry. Figs. 1a and 1b exhibiting the variation of the yy- and xy- strains are similar, respectively, to Figs. 27(a) and 27b of (Das and Batra, 2011). The x- and the y- axes, used to compute strain and stress components, are transverse and parallel, respectively, to the hull length and not along the global axes shown in the bottom left corner of Fig. 1. *It seems that labels "y-strain" and "xy-strain" in LS-PREPOST software (Fig. 1a and 1b) are interchanged.* Figs. 1c and 1d show the yy- and the xy- components of the stress, respectively. It is clear from Fig. 1d that the distribution of the xy-stress component on the hull geometry is remarkably similar to that of the "yy-strain" component shown in Fig. 1a. Unfortunately, we missed making this connection earlier. The strain energies reported in Fig. 26 of (Das and Batra, 2011) computed from strain components exported to a data file and using the constitutive relation are incorrect. Accordingly, we mistakenly reported that the transverse normal strain in the core significantly contributed to the total strain energy.

The energies reported here in Figs. 2 and 3 below have been computed by exporting the stress components shown in Fig. 1c, d to data file and using the constitutive relation. Figs. 2a and 2b show the strain energy densities along the span of the hull at an early stage of slamming (t = 2.74 ms), and Figs. 3a and 3b exhibit the same at a late stage of slamming (t = 6.02 ms). Energies computed when shell elements are used to represent the hull have also been plotted in these Figs. The energy due to the normal strain in the core is found to be negligible and is not plotted in these Figs. In Figs. 2 and 3, curves labeled "Present (3D)" represent results of finite element simulations described in (Das and Batra, 2011). It is clear from results plotted in Figs. 2 and 3 that the strain energy density due to the shear strain in the core is comparable to the strain energy density stored in the face sheets.



Fig. 1a: Picture of computer screen of LS-PREPOST software. The yy-strain component is plotted on the hull geometry



Display front view - ZY

Fig. 1b: Picture of computer screen of LS-PREPOST software. The xy-strain component is plotted on the hull geometry



Set rotation angle to +ve or -ve, right click to change value

Fig. 1c: Picture of computer screen of LS-PREPOST software. The yy-stress component plotted on the hull geometry.



Fig. 1d: Picture of computer screen of LS-PREPOST software. The xy-stress component is plotted on the hull geometry.





Fig. 2: Variation of strain energy density in (a) face sheets, and (b) the core at an early stage of slamming, i.e., t = 2.74 ms.





Fig. 3: Variation of strain energy density in (a) face sheets, and (b) the core at a late stage of slamming, i.e., t = 6.02 ms.

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Local water slamming impact on sandwich composite hulls

Kaushik Das^{a,*}, Romesh C. Batra^b

^a Department of Aerospace Engineering, Texas A&M University, College Station, TX 77843, USA ^b Department of Engineering Science and Mechanics, M/C 0219, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

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ABSTRACT

The local water slamming refers to the impact of a part of a ship hull on stationary water for a short duration during which high local pressures occur on the hull. We simulate slamming impact of rigid and deformable hull bottom panels by using the coupled Lagrangian and Eulerian formulation included in the commercial software LS-DYNA. We use the Lagrangian formulation to describe plane-strain deformations of the hull panel and consider geometric nonlinearities. The Eulerian formulation is used to analyze deformations of the water. Deformations of the hull panel and of the water are coupled through the hydrodynamic pressure exerted by water on the hull, and the velocity of particles on the hull wetted surface affecting deformations of the water. The continuity of surface tractions and the inter-penetrability of water into the hull are satisfied by using a penalty method. The computer code is verified by showing that the computed pressure distributions for water slamming on rigid panels agree well with those reported in the literature. The pressure distributions computed for deformable panels are found to differ from those obtained by using a plate theory and Wagner's slamming impact theory. We have also delineated jet flows near the edges of the wetted hull, and studied delamination induced in a sandwich composite panel due to the hydroelastic pressure.

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1. Introduction

Local water slamming refers to the impact of a part of a ship hull on water for a brief duration during which high peak pressure acting on the hull can cause significant local structural damage (Faltinsen, 1990). Initial research focused on the problem of a rigid body touching at time t=0 the free surface of a stationary fluid with a known velocity and finding for t > 0, the velocity of the fluid, the pressure exerted by it on the rigid body, the wetted length, and the position and the velocity of the rigid body (Fig. 1). An early work on water entry of a rigid v-shaped wedge with small deadrise angle β is due to von Kármán (1929). Wagner (1932) also considered a v-shaped wedge of small deadrise angle and generalized von Kármán's solution by including the effect of water splash-up on the body; however, the effect of the jet flow (Fig. 1) during the impact was not considered.

The effect of the jet flow was included in the analysis by Armand and Cointe (1986). Wagner as well as Armand and Cointe assumed that the depth of penetration of the rigid body into the fluid region is small. Zhao et al. (1996) generalized Wagner's solution to wedges of arbitrary deadrise angles, solved the problems numerically by using a boundary-integral equation method, and ignored effects of the jet flow. The variation of the hydrodynamic pressure on the rigid hull

* Corresponding author. Tel.: +1 979 845 6965.

E-mail addresses: k.das@tamu.edu (K. Das), rbatra@vt.edu (R.C. Batra).

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Fig. 1. Schematic sketch of the problem studied depicting slamming upon the bottom surface of a hull (Ω_1); top: at t=0, position of the hull touching the free surface of the fluid region; and bottom: for t > 0, the deformed fluid and solid regions. The problem domain is symmetric about the x_2 -axis or the centerline.

from Zhao et al.'s. (1996) solution agrees well with that found experimentally implying that the jet flow does not significantly affect the pressure variation on a rigid wedge. By neglecting effects of the jet flow, Mei et al. (1999) analytically solved the general impact problems of cylinders and wedges of arbitrary deadrise angles, and numerically solved problems by considering effects of the jet flow.

In practical slamming impact problems, the hull is deformable and its deformations affect the motion of the fluid and the hydroelastic pressure on the solid–fluid interface. In early attempts of analyzing water slamming problems, hulls have been idealized as rigid to estimate the hydrodynamic pressure (Bereznitski, 2001). Sun (2007) has numerically analyzed, using the boundary element method (BEM), the potential flow problem during slamming impact of a 2-D rigid body of arbitrary geometry. Sun and Faltinsen (2006–2009) have considered hydroelastic effects in analyzing deformations of circular shells made of steel and aluminum by studying deformations of the fluid by the BEM and those of shells by the modal analysis. Qin and Batra (2009) have analyzed the slamming problem by using the {3, 2}-order plate theory for a sandwich wedge and modified Wagner's slamming impact theory to account for wedge's infinitesimal elastic deformations. The plate theory incorporates the transverse shear and the transverse normal deformations of the core, but such deformations of the face sheets are not considered since they are modeled with the Kirchhoff plate theory. Greco et al. (2009a, 2009b) have studied slamming incident on the bottom of large floating structures by using a domain-decomposition strategy, which combines a linear global analysis with a nonlinear local analysis, respectively, for computation of the global motion of the structure and the local deformation and hydrodynamic pressure due to the slamming.

In the present study, the commercial finite element (FE) software, LS-DYNA, is used to study finite transient deformations of an elastic fiber-reinforced composite sandwich panel due to slamming impact with the water modeled as an inviscid fluid. The problem formulation accounts for geometric nonlinearities and inertia effects in the fluid. Furthermore, the fluid is assumed to be compressible and its deformations need not be irrotational. The rest of the paper is organized as follows: Section 2 summarizes governing equations and the numerical technique used to solve the initial-boundary-value problem (IBVP). Subsequent to verifying the software by solving slamming problems for a rigid hull, we report in Section 3 results for water slamming on deformable panels. Conclusions of the work are summarized in Section 4.

2. Mathematical model

2.1. Balance laws for deformations of the hull

A schematic sketch of the problem studied is shown in Fig. 1 that also exhibits the rectangular Cartesian coordinate axes used to describe finite deformations of the solid and the fluid bodies; the coordinate axes are fixed in space. At time t=0, let $\Omega_1 \subset \Re^3$ and $\Omega_2 \subset \Re^3$ be regions occupied by the hull and the fluid, respectively; $\Omega_3 \subset \Re^3$ is the region surrounding Ω_1 , situated above the fluid body Ω_2 , and is modeled as vacuum. $\Gamma_1(\Gamma_2)$ is the boundary of $\Omega_1(\Omega_2)$ with disjoint parts Γ_{1u} (Γ_{2u}) and $\Gamma_{1t}(\Gamma_{2t})$. After deformation, bodies occupying regions Ω_i (i=1, 2 and 3) in the reference configuration occupy regions $\omega_i \subset \Re^3$ in the deformed or the present configurations. Γ_{1u} and Γ_{2u} deform to γ_{1u} and γ_{2u} , respectively; Γ_{1t} and Γ_{2t} deform to γ_{1t} and γ_{2t} , and γ_{12} , where γ_{12} is a priori unknown interface between ω_1 and ω_2 . The free water surface γ_{2t} and the wetted surface γ_{12} vary with time t and are to be determined as a part of the solution of the problem. We note that at

 $t=0, \Gamma_{1t}$ and Γ_{2t} are just about to touch each other, therefore the interface γ_{12} between them is either a point or a line in the reference configuration. We denote coordinates of a point by X_i and x_i (i=1, 2, 3) in the reference and the current configurations, respectively.

The deformations of a continuous body (the hull and the water) are governed by the balance of mass, the balance of linear momentum, and the balance of moment of momentum, given, respectively, by Eqs. (1)-(3) written in the referential description of motion:

$$\rho_0^s J = \rho^s \quad \text{in } \Omega_1, \tag{1}$$

$$\rho_0^{\rm s} \dot{\nu}_i = \frac{\partial \hat{T}_{ji}}{\partial X_j} + \rho_0^{\rm s} f_i \quad \text{in } \Omega_1, \tag{2}$$

$$\hat{T}_{ik}F_{ki} = \hat{T}_{ik}F_{ki} \quad \text{in } \Omega_1.$$
(3)

Here ρ_0^s and ρ^s are mass densities of the material of the hull in the reference and the current configurations, respectively; J the determinant of the deformation gradient $F_{ij} = \partial x_i / \partial X_j$, v_i the velocity field defined as $v_i = \dot{x}_i$, a superimposed dot denotes the material time derivative, \hat{T}_{ij} the first Piola-Kirchhoff stress tensor, f_i the body force per unit mass and a repeated index implies summation over the range of the index. The first Piola-Kirchhoff stress tensor is related to the Cauchy stress tensor T_{ni} by

$$\hat{T}_{ij} = J \frac{\partial X_i}{\partial x_p} T_{pj}.$$
(4)

The coupling between deformations of the hull and the fluid is through the hydrodynamic pressure which acts as tractions on γ_{12} , and is in turn affected by deformations of the hull since fluid particles cannot penetrate through the hull. For a viscous fluid, surface tractions and the velocity must be continuous across the fluid-solid interface, and for an ideal fluid the normal component of velocity and the normal traction (i.e., the pressure) must be continuous across this interface, and the tangential traction vanishes there.

2.2. Balance laws for deformations of the fluid

The motion of the fluid occupying the region ω_2 in the present configuration is governed by Eqs. (5)–(7) written in the spatial description of motion:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v_i}{\partial x_i} + v_i \frac{\partial \rho}{\partial x_i} = 0 \quad \text{in } \omega_2, \tag{5}$$

$$\rho \frac{\partial v_i}{\partial t} + \rho v_k \frac{\partial v_i}{\partial x_k} = \frac{\partial I_{ji}}{\partial x_i} + \rho f_i \quad \text{in } \omega_2, \tag{6}$$

$$T_{ik} = T_{ki} \quad \text{in } \omega_2. \tag{7}$$

2.3. Constitutive relations

We presume that the hull is comprised of an elastic material for which

$$T_{ij} = C_{ijkl} \varepsilon_{kl}, \tag{8}$$

where C_{ijkl} are elastic constants for the material, and ε_{ij} the Almansi–Hamel strain tensor defined as

$$\varepsilon_{ij} = \frac{1}{2} (\delta_{ij} - (F_{ji} F_{il})^{-1}). \tag{9}$$

Note that Eq. (9) considers all geometric nonlinearities, including the von Kármán nonlinearity. With the constitutive assumption (8), the balance of moment of momentum (3) is identically satisfied. Material damping due to viscous effects can be incorporated by modifying the constitutive relation (8) but is not considered here. The number of independent elastic constants equals 2, 5, and 9 for isotropic, transversely isotropic and orthotropic materials, respectively.

We presume that water can be modeled as an inviscid compressible fluid for which

$$T_{ij} = -p\delta_{ij},\tag{10}$$

$$p = C_1 \left(\frac{\rho}{\rho_0} - 1\right),\tag{11}$$

where p is the pressure, C_1 the bulk modulus of water, and ρ_0 its mass density in the reference configuration.

2.4. Initial and boundary conditions

We assume that initially the hull is at rest, and occupies the reference configuration Ω_1 . That is

1	$u_i(X_l, 0) = 0,$	(12)

where u_i is the displacement defined as $u_i = x_i - X_i$ and

$$v_i(X_l, 0) = 0.$$
 (13)

(14)

For the boundary conditions, we take

 $v_i(x_l, t)$ is specified on γ_{1u} for all t and $T_{ii}n_i^1 = 0$ on γ_{lt} for all t.

Here n_i^1 is an outward unit normal vector on γ_{1t} in the current configuration.

We assume that initially the fluid is at rest, and occupies the reference configuration Ω_2 at time *t*=0. Since the fluid problem is formulated in the spatial description of motion, we do not track the motion of each fluid particle. Thus the initial condition is

$$v_i(x_l, 0) = 0.$$
 (15)

Boundary conditions for the fluid are taken to be

$$v_i(x_l, t)n_i^2 = 0, \ e_i^2 T_{ji}n_j^2 = 0 \quad \text{on } \gamma_{2u} \quad \text{for all } t,$$
 (16)

and

$$T_{it}n_i^2 = 0 \quad \text{on } \gamma_{2t} \quad \text{for all } t.$$
⁽¹⁷⁾

Here n_i^2 (e_i^2) is an outward unit normal (tangent) vector on a bounding surface of the fluid in the present configuration; γ_{2t} is the free water surface that is not contacting the hull and is to be determined as a part of the solution of the problem. It is tacitly assumed in writing Eq. (17) that surface tension effects are negligible.

For t > 0, the solid body ω_1 is in contact with the fluid region ω_2 and the interface γ_{12} between the two varies with time t. We assume that following conditions hold on γ_{12} :

$$v_i^s n_i - v_j^t n_i = 0 \quad \text{on } \gamma_{12}, \tag{18}$$

$$T_{ij}^{s}n_{j} = T_{ij}^{f}n_{j} \quad \text{on } \gamma_{l2},$$
⁽¹⁹⁾

where superscripts *s* and *f* on a quantity denote, respectively, its value for the solid and the fluid particles on the fluid–solid interface γ_{12} and n_i is an outward unit normal vector on γ_{12} in the present configuration.

We note that γ_{12} varies with time, is *a priori* unknown, is to be determined as a part of the solution of the problem, and γ_{2t} is given by $F(t, x_i)=0$ such that

$$\dot{F} = v_i^f n_i^2. \tag{20}$$

2.5. Plane strain deformations

We assume that a plane strain state of deformation prevails in the x_1x_2 -plane, and accordingly study a 2-dimensional problem.

2.6. Numerical solution of the initial-boundary-value problem (IBVP)

The commercial code LS-DYNA is used to find an approximate solution of the nonlinear IBVP defined by Eqs. (1)–(20). We use the FEM to solve the IBVP for the hull; the method is described in several references, e.g., see Hallquist (1998) and Zienkiewicz et al. (2005). To solve the IBVP for the fluid, we use the split approach detailed in Benson (1989, 1992), Aquelet and Souli (2003), and Aquelet et al. (2006) and implemented in LS-DYNA.

Fig. 2(a) shows the rectangular Cartesian coordinate system, and a schematic of the FE mesh used to analyze the slamming impact of a hull whose undeformed shape is like a *V*. To analyze deformations in the x_1x_2 -plane, we use 8-node brick elements with one point integration rule and with only one element in the x_3 -direction to discretize the fluid, the solid and the initially void region, and constrain all nodes from moving in the x_3 -direction. The software LS-DYNA rules out spurious modes of deformation by using the hour-glass control algorithm. Fig. 2(b) shows the hull whose material is transversely isotropic with the axis of transverse isotropy along the \overline{x}_1 -direction.

Due to symmetry of the problem geometry and of the initial and the boundary conditions, deformations of bodies occupying regions for which $x_1 \ge 0$ are analyzed (see Fig. 2(a)) and boundary conditions $v_1(x_1=0, t)=0$ and the tangential traction=0 at $x_1=0$ are imposed. In Fig. 2(a), L_4 and (L_5-L_3) are the length and the depth of the fluid region, respectively;



Fig. 2. (a) The FE mesh used to study the slamming problem and (b) undeformed shape of the hull made of a transversely isotropic material with the axis of transverse isotropy along the \bar{x}_1 -axis.

 L_4 and (L_5-L_3) are at least five times the length of the hull. The depth L_3 of the initially void region above water is such that the void region encloses the anticipated deformed shape of the fluid region. The fluid region $(L_2 \times L_1)$ near the hull is meshed with smaller elements; L_1 equals at least the maximum anticipated wetted length c_f (see Fig. 1) and L_2 at least five times the anticipated depth of penetration of the hull into the water. The FE mesh for the hull overlaps that for the initially void region. As the hull penetrates into water regions occupied by the hull and the fluid change. On the hull/water interface γ_{12} continuity conditions (18) and (19) are satisfied using the penalty method (Aquelet and Souli, 2003; Aquelet et al., 2006) (see Appendix A) and selecting appropriate values of the penalty stiffness parameters k_d and c (see Eq. (A.4) in Appendix A). The free water surface is tracked in LS-DYNA using the Simple Linear Interface Calculation (SLIC) technique (Woodward and Colella, 1984).

Appropriate values of the penalty parameters depend on the speed of impact, elastic moduli of the hull material, hull shape, bulk modulus and viscosity of water, size of FEs, and the deadrise angle. Numerical studies with different values of parameters show that a low value of the penalty stiffness poorly satisfies Eqs. (18) and (19), thus water penetrates the hull–water interface. The water pressure p at the interface becomes oscillatory if a high value of the penalty stiffness parameter is used. Here, most problems are studied with at least three values of the penalty parameter to ascertain its appropriate value for the problem.

3. Results and discussion

3.1. Water slamming on rigid wedges moving with constant downward velocity

The problem studied is a rigid V-shaped wedge, with each arm 2 m long, entering calm water with a constant downward velocity of 10 m/s and having deadrise angle of 10°, 30°, 45°, or 81°. The fluid domain ($L_1 \times L_2$) with L_1 =2 m,

 $L_2=2 \text{ m}, L_3=1 \text{ m}, L_4=5 \text{ m}, L_5=6 \text{ m}$, and width in the x_3 -direction=0.01 m (Fig. 2(a)) near the rigid hull is discretized by $100 \times 100 \times 1$ 8-node brick elements. The mesh covering the entire domain (initially void and fluid domains, $L_4 \times L_5$) has $150 \times 150 \times 1$ 8-node brick elements, and the wedge is discretized into 100×1 4-node shell elements along the length and the width (the x_3 -) directions. All nodes of the fluid, the void and the solid structure are restrained from motion in the x_3 -direction. The water is modeled as an inviscid fluid with bulk modulus of 2.9 GPa and $\rho_0=1000 \text{ kg/m}^3$; thus the speed of a volumetric wave equals ~ 1700 m/s, and it reaches the right boundary at ~3 ms and the bottom surface at ~2.4 ms. The effect of waves reflected from these boundaries will be felt by the wedge at ~5 ms implying that pressures computed at the wedge surface approximate those acting on the wedge striking an infinite body of water only for times less than ~5 ms.

In order to find an appropriate value of k_d we have shown in Fig. 3 variations of the pressure coefficient $C_p = 2p/\rho V^2$ along the length of the water-wedge interface for various explicitly specified values of k_d with the damping coefficient c=0. It is evident from results shown in Fig. 3 for the wedge of deadrise angle 10° penetrating water at 10 m/s that for k_d =0.125, 1.25, 12.5, and 125 GPa/m the maximum amplitudes of oscillation of C_p are 60, 10, 5, and 20, respectively. For very low values of k_d (e.g., k_d =0.0125 GPa/m), a noticeable amount of water penetration through the fluid-structure interface occurs because continuity conditions at the interface are not well satisfied, and for k_d =1.25 and 125.0 GPa/m the maximum amplitude of oscillations in values of C_p divided by its mean value is large. Only for k_d =12.5 the variation of C_p is relatively smooth along the span. Similarly, for wedges with deadrise angles of 30°, 45°, and 81°, it is found that the variation of C_p is relatively smoother for k_d =1.25, 1.25, and 0.125 GPa/m, respectively, than those computed using higher values of k_d . Results shown in Fig. 4 suggest that with an increase in the value of the contact stiffness k_d , the mass of water penetrating through the wedge-water interface normalized by the mass of displaced water decreases. We note that for t close to zero the mass of displaced water is small therefore normalized water penetration is large. As t increases the normalized water penetration approaches a constant value. The computed results for deadrise angles of 30°, 45°, and 81° (not shown here for the sake of brevity but included in Das (2009)) reveal that the water penetration through the interface for t > 10 ms does not change when the value of k_d is increased from 1.25 to 125.0 GPa/m. However, for $k_d \ge 1.25$ GPa/m and deadrise angle=81°, the pressure profile along the length of the panel is oscillatory as compared to the pressure profile for $k_d \le 0.125$ GPa/m. For each one of the four values of the deadrise angle, Table 1 lists values of k_d for which the pressure profile is "smoother" than that for other values of k_d . Increasing k_d from the optimal value does not appreciably reduce the water penetration through the interface. One can find a better value for k_d by considering oscillations in the pressure profile and the normalized water penetrated through the water/solid interface. For the wedge with deadrise angle of 10°, oscillations in the pressure profile are observed even for a larger problem domain ($L_4=10$ m, $L_5=11$ m); thus the spatial oscillations in the pressure on the interface are not caused by waves reflected from the boundaries.

For the deadrise angle of 30° and the corresponding optimal value of k_d listed in Table 1, we have shown in Fig. 5 variations of the pressure coefficient C_p along the span of the wedge for three values of the contact damping factor c. It is evident that the value of c does not appreciably affect the value of C_p ; similar results were obtained for other values of the deadrise angle, e.g., see Das (2009).

The effect of the mesh size on values of the pressure coefficient C_p is shown in Fig. 6 where results for three FE meshes are plotted; FE mesh 2 is obtained from FE mesh 1 by subdividing each brick (shell) element in mesh 1 into 4 equal brick (2 equal shell) elements; sides of elements in the x_3 -direction are not subdivided into two parts. Similarly, FE mesh 3 is



Fig. 3. (Colour online) For deadrise angle of 10°, variations of the pressure coefficient at t=20 ms along the normalized span of the hull for different values of the contact stiffness k_d (GPa/m).



Fig. 4. (Colour online) For different values of the contact stiffness k_d (GPa/m), time histories of the mass of water penetration normalized by the mass of displaced water through the water–wedge interface for deadrise angle=10°.

Table 1Optimum values of k_d .

Deadrise angle (deg)	10	30	45	81
Optimum value of k_d (GPa/m)	12.5	1.25	1.25	0.125



Fig. 5. (Colour online) Variations of the pressure coefficient along the span of the hull for different values of the contact damping factor c for deadrise angle=30° and k_d =1.25 GPa/m.

obtained from FE mesh 2. For deadrise angles of 30° , 45° , and 81° and the corresponding optimal values of k_d listed in Table 1, the computed values of C_p along the wedge–water interface were found to be essentially independent of the mesh used. However, for the deadrise angle of 10° , the peak value of C_p computed using mesh 3 is ~20% higher than that obtained with mesh 2. For deadrise angle of 81° , meshes 2 and 3 produce smoother variation of C_p than that given by mesh 1. For the sake of brevity results only for deadrise angle= 10° are shown in Fig. 6; results for deadrise angles of 30° , 45° , and 81° are included in Das (2009).



Fig. 6. (Colour online) Variation of the pressure coefficient along the span of the hull for three FE meshes for deadrise angle=10° and k_d =12.5 GPa/m.



Fig. 7. (Colour online) Variation of the pressure coefficient along the span of the wedge for deadrise angle=30°. For the sake of brevity similar results for deadrise angles of 10°, 45°, and 81° are not shown here but are included in Das (2009).

Table 2

Maximum difference in values of the pressure coefficient obtained from the present solution and that reported by Mei et al. (1999) with the consideration of the jet flow.

$\beta = 10^{\circ} (t = 20 \text{ ms})$	$\beta = 30^{\circ} (t = 40 \text{ ms})$	$\beta = 45^{\circ} (t = 48 \text{ ms})$	$\beta = 81^{\circ} (t = 150 \text{ ms})$
15% at $x_2/Vt = -0.5$	11% at $x_2/Vt = -0.7$	7% at $x_2/Vt = -0.5$	\sim 20% (for 0.2 > x_2/Vt > -0.075) at x_2/Vt =-0.15

Using the optimal values of k_d , we compare in Fig. 7 the presently computed values of the pressure coefficient with those reported by Mei et al. (1999). We note that the present solution obtained using LS-DYNA incorporates effects of the jet flow. The maximum percentage difference between the presently computed pressure coefficient and that reported by Mei et al. (1999) with the consideration of the jet flow is listed in Table 2. For deadrise angles of 10°, 30°, and 45°, the maximum percentage difference between the two sets of results does not occur at the peak pressure. For the deadrise angle of 81° the presently computed pressure coefficient near the keel does not compare well with that reported by Mei et al. (1999) with the consideration of the jet flow. It is also found that the pressure coefficient computed with LS-DYNA

does not vary significantly with time. The pressure coefficient reported by Mei et al. is constant with time. For 81° deadrise angle, the apex of the wedge penetrating water is very sharp and a finer mesh is needed to accurately compute the variation of the pressure coefficient near the keel which is the apex of the wedge. Due to limited computational resources, computations with a finer mesh were not performed. We note that in practice, hulls have relatively small deadrise angles. In Fig. 8, we have compared shapes of the deformed water regions computed using LS-DYNA with those reported by Mei et al. (1999) without the consideration of the jet flow. For deadrise angles of 30° and 45°, a significant jet flow is found in our numerical simulations. Except for the jet flow region, the water splash-up in the present numerical solution compares well with that reported by Mei et al. (1999). Our assumption that cavitation occurs when the tensile pressure at a fluid particle exceeds 10 GPa results in the formation of water bubbles at the tip of the jet flow shown in Fig. 8. The formation of water bubbles is affected by the limiting value of the tensile pressure; however, we did not compute results with other values of the limiting tensile pressure. Except for the bubbles formed, the consideration of the jet flow does not affect much the shape of the free surface of the deformed water region.



Fig. 8. (Colour online) Deformed shapes of the water region during the water entry of rigid wedges; black lines are water surfaces from Mei et al.'s (1999) solution without considering the jet flow.



Fig. 9. (Colour online) Comparison of the slamming pressure coefficient computed from forces in springs used in the contact algorithm and that at centroids of fluid elements adjoining the hull-water interface.

In results presented and discussed above, the interface pressure or the slamming pressure on the hull was calculated from forces in contact springs (see Appendix A) on the fluid–structure interface used to enforce the continuity of the normal component of velocity. Alternatively, we can find this from values of the pressure at centroids of fluid elements contacting the hull. Fig. 9 shows the interface pressures computed from two different techniques. These two sets of results agree well with each other except at points near the extremities of the wetted region where the pressure found from forces in springs used in the contact algorithm is close (at least qualitatively) to that reported by Mei et al. For subsequent analyses, the interface pressure is computed from forces in springs used in the contact algorithm since they are readily available.

Fig. 10(a) shows at t=4 ms the position of the rigid wedge of deadrise angle 30° moving vertically downwards at a velocity of 10 m/s, the deformed water region, and fringe plots of the speed of water particles. The normal and the tangential velocities of a point on the wedge are $-10 \cos(30^\circ) = -8.66$ m/s and $-10 \sin(30^\circ) = -5.0$ m/s, respectively. Fig. 10(b) exhibits variations of the normal and the tangential velocities of water particles at the wedge–water interface, and also of a point on the wedge–water interface which is confirmed by results plotted in Fig. 10(b) except at points in the region A of the jet flow. From results shown in Fig. 10(b), the percentage difference between the normal velocities of the wedge point and the corresponding water particle touching it is found to be $\sim 3\%$ except in region A of the jet flow. The tangential velocity of the are noticeable oscillations in the normal velocity of fluid particles in region A of the jet flow. The tangential velocity of the water at the interface increases from zero at the keel to ~ 8 m/s in the region A of the jet flow. Within a small portion of



Fig. 10. (Colour online) At t=4 ms, (a) rigid wedge and deformed water region, and (b) variation of the velocity of the wedge and the water particle at the wedge-water interface versus the x_1 -coordinate of the water particle.

the span of the wedge (\sim 0.2 m) in region A of the jet flow, the tangential velocity of the water increases from 8 to 50 m/s before decreasing to \sim 42 m/s in region B of the jet flow.

3.2. Water slamming on rigid wedges moving with variable downward velocity

In this section, we study the local slamming of a rigid wedge impacting at normal incidence the initially calm water and consider the deceleration of the wedge due to the hydrodynamic pressure; thus the downward velocity of the wedge need not stay constant in time. Three *V*-shaped wedges of mass 241, 94, and 153 kg are considered, and presently computed results are compared with those found experimentally by Zhao et al. (1996) and Yettou et al. (2007).

Here we assume that deformations of the fluid and the wedge are plane strain in the x_1x_2 -plane and use two different FE meshes for the wedge of mass 241 kg. Each arm of the v-shaped wedge is 0.25 m long, which is the same as that in Zhao et al.'s (1996) experiments. For the first mesh, regions $L_1 \times L_2$ and $L_4 \times L_5$ are discretized, respectively, with $200 \times 100 \times 1$ and $220 \times 200 \times 1$ 8-node brick elements (see Fig. 2), and the rigid wedge, modeled as a shell, by 120×1 4-node shell elements, where $L_1=2$ m, $L_2=1$ m, $L_3=1$ m, $L_4=5$ m, and $L_5=2$ m. For the second mesh, regions $L_1 \times L_2$ and $L_4 \times L_5$ are discretized, respectively, with $75 \times 50 \times 1$ and $130 \times 205 \times 1$ 8-node brick elements, and the rigid wedge, by 240×1 4-node shell elements where $L_1=0.75$ m, $L_2=0.5$ m, $L_3=0.75$ m, $L_4=5$ m, and $L_5=5.75$ m. We note that for the second mesh, the depth of the water region is 5 times larger and the region $L_1 \times L_2$ is smaller than the respective regions of the first mesh. The elements in the second mesh are not of uniform size; they are smaller near the region where the wedge impacts water and the size of an element in the impact zone is one-half of that of the first mesh. It is found that results computed using the two FE meshes are not significantly different, and results computed with the second mesh are reported here. Optimum values, $k_d=0.01$ GPa/m and c=1, are chosen following the process described in the previous section.

Figs. 11 and 12 show, respectively, the time histories of the downward velocity of the wedge and the total upward hydrodynamic force acting on the wedge. The wedge downward velocity observed in experiments (Zhao et al., 1996) is also shown in Fig. 11. The presently computed absolute downward velocity is lower than that found experimentally and the maximum difference between the two sets of results is 6.5% at t=0.025 s.

The presently computed total upward force is in excellent agreement with that found experimentally up to t=0.014 s when the flashed up water reached the chine; however, in the experiment, the flashed up water reached the chine at 0.016 s. For t > 0.014 s, the computed and the experimental results agree qualitatively. The total forces computed numerically and analytically by Zhao et al. (1996) and Mei et al. (1999), respectively, are also shown in Fig. 12. They also computed this force by considering a correction factor to account for the finite width of the wedge, i.e., the 3-D effect. The total force found with the correction factor is in good agreement with the presently computed and the experimental results. We note that the presently computed force is for a 2-D domain. Thus, the difference between the simulated results of Zhao et al. (1996), Mei et al. (1999) and experimental results of Zhao et al. (1996) may not be due to the 3-D effect but due to other simplifying assumptions made in Zhao et al. (1996) and Mei et al. (1999).

Figs. 13–15 show variations of the pressure coefficient along the span of the wedge at an initial stage of slamming (t=0.0044 s), at an intermediate stage after the water reached the chine (t=0.0158 s) and at a late stage of slamming (t=0.0202 s). At the initial stage of slamming, the presently computed pressure coefficient agrees well with that found numerically and experimentally by Zhao et al. (1996) except near the peak pressure region. Both the presently computed pressure coefficient and that computed numerically by Zhao et al. (1996) are less than that found from the



Fig. 11. (Colour online) Time history of the downward velocity of the rigid wedge.



Fig. 12. (Colour online) The time history of the total upward force on the rigid wedge.



Fig. 13. (Colour online) At t=0.0044 s, the variation of the pressure coefficient along the span of the rigid wedge for deadrise angle=30° and mass 241 kg.

experimental data. We note that the peak pressure coefficient computed from LS-DYNA and that determined numerically by Zhao et al. (1996) agree well with each other. The variation of the pressure coefficient as reported by Mei et al. (1999) for a wedge moving with a constant velocity is also shown in Fig. 13 for comparison with that for the wedge moving with a variable velocity. It is clear that the pressure coefficient for the constant velocity wedge is higher than that for the wedge with variable velocity. At t=0.0158 s, the water just reached the chine in Zhao et al.'s experiment; however, the water reached the chine at t=0.0134 s in the present simulation. For comparison, the pressure coefficient for a 1.2 m long wedge with the same mass as that of the 0.25 m long wedge is computed and shown in Fig. 14. For the 1.2 m long wedge, the water does not reach the chine at t=0.0158 s. The presently computed pressure coefficient for the 0.25 m long wedge is in good agreement with that found experimentally except near the peak pressure region. In comparison, the analytically found pressure coefficient by Mei et al. (1999) for a wedge of constant velocity is greater than the presently computed pressure coefficient. The numerically computed pressure coefficients as reported by Zhao et al. (1996) and that found for the 1.2 m long wedge increase to a peak value and then decrease to zero at the extremity of the wetted region. The presently computed pressure coefficient for the 0.25 m long wedge is approximately constant along the length of the hull except within a small region near the chine, where the pressure coefficient increases to a peak value. It is possible that the sudden increase of pressure near the chine is due to the separation of water from the surface of the wedge at the chine. This was not observed in the experiments because no pressure sensor was present sufficiently close to the chine.



Fig. 14. (Colour online) At t=0.0158 s, the variation of the pressure coefficient along the span of the rigid wedge for deadrise angle=30° and mass 241 kg.



Fig. 15. (Colour online) At t=0.0202 s, the variation of the pressure coefficient along the span of the rigid wedge for deadrise angle=30° and mass 241 kg; h_w equals the wedge height.

For t=0.0202 s, the presently computed pressure coefficient is closer to that found experimentally and lower than that found numerically in Zhao et al. (1996). The presently computed pressure coefficient is qualitatively similar to that found numerically in Zhao et al. (1996) and is nearly constant from the apex to about 80% length of the wedge. Near the chine, the numerically computed pressure coefficient of Zhao et al. (1996) drops to zero. The presently computed pressure coefficient drops to ~1.5 near the chine from its value of 2.5 near the apex of the wedge before suddenly increasing to the peak value within a very small region at the chine, where the water separates from the wedge.

For the two v-shaped wedges of mass 94 and 153 kg with each arm of V=0.87 m long, we set $L_1=2$ m, $L_2=1$ m, $L_3=1$ m, $L_4=5$ m, and $L_5=2$ m (see Fig. 2); these dimensions are the same as those in the experimental setup of Yettou et al. (2007). We analyze the problem using three FE meshes; for the first mesh, regions $L_1 \times L_2$ and $L_4 \times L_5$ are discretized, respectively, with $100 \times 50 \times 1$ and $110 \times 100 \times 1$ 8-node brick elements, and the rigid wedge, modeled as a shell, by 60×1 elements. The second FE mesh is constructed by dividing each brick (shell) element of the first mesh into four (two) equal elements and the two meshes have only one element along the x_3 -direction. Similarly, mesh 3 is constructed from mesh 2. Two different values of the contact parameters are used to ascertain their influence on the solution of the problem. It has been found that results computed using FE meshes 2 and 3 with either $P_f=0.025$ (see Appendix A for the definition of P_f) or $k_d=0.01$ GPa/m are virtually the same. The FE mesh 2 with $P_f=0.025$ and c=1 is used to compute results shown in Figs. 16

and 17, and results shown in Figs. 18 and 19 are computed using the FE mesh 3 with k_d =0.01 GPa/m and c=1. At t=0 the downward velocity of the wedge equals 5.05 m/s and the apex of the wedge just touches the calm water surface.

For the wedge of mass 94 kg and deadrise angle 25°, we have compared in Fig. 16 the computed time histories of the downward velocity v_2 of the wedge with the experimental (Yettou et al., 2007) and the analytical (Zhao and Faltinsen, 1993) ones. It is clear that the presently computed velocity matches well with the experimental and the analytical ones, and the maximum percentage differences $100(v_2^{sim} - v_2^{m})/v_2^{sim}$ and $100(v_2^{sim} - v_2^{anl})/v_2^{nl}$ between the presently computed velocity v_2^{sim} , the experiment velocity v_2^{exp} and the analytical velocity v_2^{anl} are less than 6% and 3.5%, respectively. Fig. 17 shows the time history of the total upward force exerted by the water on the rigid wedge which is not reported in Zhao and Faltinsen (1993) and Yettou et al. (2007).

In order to demonstrate that results shown in Figs. 16 and 17 are also valid for other wedges, we analyzed the problem for another rigid wedge that was also studied analytically and experimentally in Yettou et al. (2007). Fig. 18 shows the time histories of the total upward force F_{up} exerted by the water on a rigid wedge of mass 153 kg and deadrise angle 30° obtained analytically in Yettou et al. (2007) and of that computed with LS-DYNA. In Yettou et al. (2007), two different analytical methods, namely pressure integration and Newton's 2nd law are used (for details see Yettou et al., 2007). The presently computed peak force of 15.9 kN is very close to the peak forces 15.7 and 15.0 kN found using the pressure integration and Newton's law, respectively. During the late stage of slamming, presently computed force differs from that



Fig. 16. (Colour online) Time history of the downward velocity of the rigid wedge.



Fig. 17. The time history of the total upward force on the rigid wedge.



Fig. 18. (Colour online) The time history of the total upward force on the rigid wedge.



Fig. 19. Variations of the pressure along the span of the hull at four different times. Values of time for different curves are as follows: red, 14.7 ms; blue, 23.7 ms; green, 35.5 ms; and purple, 48.5 ms. Solid lines are present results; dashed lines are taken from plots of the analytical results reported in Yettou et al. (2007); solid circles are experimental results from Yettou et al. (2007) (for interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).

found analytically. The maximum differences between the force computed using LS-DYNA and that found using the pressure integration method is 40% and between the present result and that computed using Newton's law is 15% at t=0.06 s. In comparison, the maximum difference between the forces computed with the pressure integration and Newton's 2nd law is ~30% at t=0.06 s. It is likely that a small difference in the hydrodynamic pressure at an early stage of the slamming event changes the velocity of the wedge and the difference between results from the two methods accumulates over time. We note that results shown in Figs. 16 and 17 are for different wedges, and the computed velocities are compared with the experimental values and the computed force with that derived from the analytical solution; this was necessitated by results given in Yettou et al. (2007).

The results reported in Yettou et al. (2007) are reproduced here from figures provided therein. Explicit expressions for the velocity of the wedge and the pressure on the wedge exerted by water are not given in Yettou et al. (2007), whereas it is possible to reproduce results given in Yettou et al. (2007) by solving the IBVP following the procedure explained in Yettou et al. (2007) but it has not been done here.

Fig. 19 shows variations of the analytically (Yettou et al., 2007), experimentally (Yettou et al., 2007), and the presently obtained pressures along the span of the hull at four different times for a wedge of mass 153 kg, deadrise angle 30°, the FE mesh 3, k_d =0.1 GPa/m and c=1; values for the analytical solution are taken from results plotted in Yettou et al. (2007) and not from the analytical solution of the problem provided by the authors. Although the analytical and the presently computed values of the total force shown in Fig. 19 agree well with each other at early stages of slamming before the peak total force is reached, i.e., t < 11 ms, the pressure variations are not close to each other for t > 11 ms. If we find areas under curves shown in Fig. 19 and multiply them with the width 1.2 m of the hull and $\cos(30^\circ)$ we should get the total upward force shown in Fig. 18. The total force at 14.7, 23.7, 35.5, and 48.5 ms computed by integrating numerical results of Fig. 19 (solid red, blue, green, and purple curves) differ by 1.9%, 2.8%, 3.2%, and 3.4%, respectively, from the values shown in Fig. 18 (red curve). However, the total forces computed by integrating results represented by dotted red, blue, green and purple curves in Fig. 19 differ from the analytical total force shown in Fig. 18 (purple curve) by ~25% for the four values of time considered. Thus the total upward force shown in Yettou et al. (2007) differs significantly from that obtained by integrating the pressure profile reported in Yettou et al. (2007) over the wedge span. The presently computed pressures differ from the experimental ones shown in Fig. 19 by solid circles, and we cannot explain reasons for these discrepancies.

3.3. Water slamming of sandwich hulls

We now study plane strain deformations in the x_1x_2 -plane of a 1 m long, 30 mm thick core sandwich composite plate with 12 mm thick face sheets and clamped at both ends. The dimensions of the fluid and the vacuum domains are L_1 =1.5 m, L_2 =1 m, L_3 =0.5 m, L_4 =2.5 m, and L_5 =2 m. The material of the face sheets is transversely isotropic with the axis of transverse isotropy along the length ($\overline{x_1}$ -axis, cf. Fig. 2(b)) of the hull, Young's modulus along the length E_1 =138 GPa, Young's modulus along the thickness E_2 =8.66 GPa, Poisson's ratio v_{12} =0.3, and the shear modulus G_{12} =7.1 GPa. The core material is isotropic with Young's modulus E=2.8 GPa and Poisson's ratio v=0.3. For results reported in this subsection, unless stated otherwise, the deadrise angle and the downward velocity of the plate equal 5° and 10 m/s, respectively. Mass densities of the core and the face sheets are 150 and 31,400 kg/m³, respectively; the mass density of the face sheets includes the non-structural dead weight. The problem studied is the same as that analyzed by Qin and Batra (2009) and presently computed results are compared with those reported in Qin and Batra (2009) using the {3, 2}-order plate theory for the core, the Kirchhoff plate theory for the face sheets, and the modified Wagner theory for finding the hydrodynamic pressure acting on the hull.

Results reported in this subsection are computed using two FE meshes. The coarse mesh has 100 uniform elements along the length of the hull, 6 along the thickness (2 in each face sheet and 2 in the core), and regions $L_1 \times L_2$ (see Fig. 2), $L_1 \times L_3$, and $L_4 \times L_5$ of the water, void and combined domains, respectively, have 150×150 , 150×65 , and 165×230 elements, respectively. The fine mesh is obtained from the coarse mesh by dividing each brick element of the coarse mesh into four elements. In order to obtain optimum values of contact parameters k_d and c, we employ the procedure of Subsection 3.1 and found that for $k_d=1.25$ GPa/m and c=1, the time histories of the slamming pressure are relatively "smoother" than those for $k_d=62.5$ and 312.5 GPa/m with either c=0 or 1. A detailed account of obtaining optimal values of k_d and c for this problem can be found in Das (2009).

Fig. 20 exhibits the presently computed deflection of the centroid of the hull for three different deadrise angles and that shown in Qin and Batra (2009). Values assigned to penalty parameters are k_d =1.25 GPa/m, c=1, and we use two different



Fig. 20. (Colour online) Time histories of the downward deflection of the centroid of the hull for three different deadrise angles; fine and coarse in the inset correspond to results computed with the fine and the coarse FE meshes.

FE meshes, namely the fine mesh and the coarse mesh. For deadrise angles of 5° and 10°, the presently computed deflections with both FE meshes are close to those reported by Qin and Batra (2009); however, for the deadrise angle of 14°, the maximum deviation of the deflection found using LS-DYNA and the fine FE mesh from that given in Qin and Batra (2009) is 50%. Qin and Batra's analysis (Qin and Batra, 2009) of the slamming pressure on the hull by using Wagner's theory modified to consider hull's infinitesimal elastic deformations is valid for small deadrise angles. Using the fine FE mesh, and for the hull deadrise angle of 5°, we have compared in Fig. 21(a) variation along the span of the hull of the presently computed deflection, $w_0^{LS-DYNA}(x_1)$, of the centerline of the hull with that, $w_0^{Qin}(x_1)$, found by Qin and Batra (2009). The relative L_2 norm $(\int_0^{1m} (w_0^{LS-DYNA}(x_1)-w_0^{Qin}(x_1))^2)^{0.5} dx_1/(\int_0^{1m} (w_0^{Qin}(x_1))^2)^{0.5} dx$ of the difference between these two sets of results equals 0.26, 0.19, 0.17, 0.15, and 0.11 for t=2.735, 3.247, 4.026, 5.471, and 6.018 ms, respectively. We note that the L_2 norm does not compare local deflections which may differ noticeably over a small length even though the value of the L_2 norm is small. The local deadrise angle of the hull at different times shown in Fig. 21(a) and adding to the local slope the initial deadrise angle of 5°.

In Fig. 22(a) and (b), we have compared the presently computed time histories of the slamming pressure at three different locations on the span of the sandwich hull by taking k_d =1.25 GPa/m and either c=0 (Fig. 22(a)) or c=1 (Fig. 22(b))



Fig. 21. (a) Deflections of the centerline of the hull; red: t=2.735 ms, blue: t=3.247 ms, green: t=4.026 ms, brown: t=5.471 ms, purple: t=6.018 ms; dashed line: results from Qin and Batra (2009), solid line: present solution using LS-DYNA. (b) Local slopes (deadrise angle) of the centerline of the hull for an initial deadrise angle of 5° (for interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).



Fig. 22. (Colour online) Time histories of the interface pressure at three locations on the hull-water interface for k_d=1.25 GPa/m: (a) c=0 and (b) c=1.

with those reported in Qin and Batra (2009). As expected, the non-zero value of *c* suppresses oscillations in the time history of the slamming pressure; however, it also reduces the peak pressure at a point from ~ 10 to ~ 6 MPa which is undesirable. Whereas the present solution gives finite values of the hydrodynamic pressure, Qin and Batra's solution, because of the singularity in the expression for the pressure, provides an unrealistically high value of the pressure when water just reaches the point under consideration. Except for these large initial differences, the two sets of pressures are close to each other.

Results shown in Figs. 23–28 are computed using k_d =1.25 GPa/m and c=1. The water level reaches locations x_1 =0.24, 0.35, and 0.57 m, respectively, at 1.4, 2.0 and 3.2 ms; the decay with time of the slamming pressures at x_1 =0.24, 0.35, and 0.57 m compare well with those reported by Qin and Batra (2009). At different times, the distributions of the pressure on the hull wetted surface from the two approaches shown in Fig. 23 reveal that the two sets of results agree only till t=3.2 ms; at subsequent times the two approaches give significantly different pressure distributions on the wetted surface. It is possible that differences in the pressure at time t₁ noticeably affect the hydroelastic pressure and the deformations of the sandwich beam at later times. One can quantify this by solving several water slamming problems with the hydroelastic pressure perturbed at different times. However, this has not been attempted here.

For t=5.741 ms, we have shown in Fig. 24(a) and (b) deformed position of the hull, the water, fringe plots of the speed, and variations of the normal and the tangential velocities of the water and the hull particles on the interface versus their x_1 -coordinates. Since water has been assumed to be inviscid, therefore, only the normal component of velocity should be continuous at the hull–water interface which is verified by results shown in Fig. 24(b). The difference in the normal



Fig. 23. (Colour online) Distribution of the slamming pressure along the hull; solid curves: solution using LS-DYNA; dashed curves: results from Qin and Batra (2009).

velocities of the hull and water particles on the hull-water interface is found to be less than 10% except in the jet flow. Oscillations in the normal component of the water velocity within the jet flow are possibly due to errors in estimating the local slope of the hull-water interface. The tangential velocity of the water at the interface increases from zero at the keel to \sim 30 m/s at a point near the jet flow; in the jet flow the tangential velocity of water rapidly increases to \sim 100 m/s. We note that the maximum tangential velocity of water in the jet flow of a rigid wedge of deadrise angle 30° impacting water at 10 m/s was \sim 50 m/s (cf. Subsection 3.1). The presently computed time history of the wetted length agrees well with that reported in Qin and Batra (2009), cf. Fig. 25. For this problem, the wetted length increases at an average rate of 170 m/s.

Fig. 26(a) and (b) exhibits comparisons of the distributions of the strain energy density stored in the two face sheets and the core at an early and at a terminal stage of the slamming impact event computed using the fine FE mesh with those obtained by Qin and Batra (2009). We compute the strain energy density of the face sheets using the expression $U_{total}=0.5\varepsilon_{ij}T_{ij}$, where U_{total} is the strain energy density and a repeated index implies summation over the range 1, 2 of the index. Noting that strains induced are infinitesimal, the strain energy densities due to the transverse shear and the transverse normal strains in the core are given, respectively, by $U_{shear} = \varepsilon_{12}T_{12}$ and $U_{normal} = 0.5\varepsilon_{22}T_{22}$. The strain ε_{ij} is obtained at the centroid (Gauss point) of each FE, T_{ii} is computed using the constitutive relation (8) from ε_{ii} and U_{total} U_{shear} , and U_{normal} are integrated through the thickness of either the face sheets or the core using the trapezoidal rule to obtain the strain energy densities, which are shown in Fig. 26. Note that the damage in the sandwich composite panel is not considered. Results from the solution of the problem with the fine mesh differ from those from the solution of the problem with the coarse mesh only at points near the fixed supports (Das, 2009). Whereas the variation of the strain energy stored in the face sheets at 2.735 ms computed from the present solution agrees well with that found by Qin and Batra (2009), values from the two approaches differ noticeably at 6.028 ms. Qin and Batra found that strain energy in the core is mainly due to transverse shear deformations, present results suggest that it is primarily due to the transverse normal strains. This difference could be due to the {3, 2}-order plate theory employed by Qin and Batra whereas we have modeled face sheets and the core as, respectively, transversely isotropic and isotropic homogeneous linear elastic materials, and the significant differences in the slamming pressure distributions shown in Fig. 24. With either approach, deformations of the core account for a significant portion of the work done by the slamming pressure.

Figs. 27 and 28 exhibit fringe plots of ε_{12} and ε_{22} in the hull at t=2.735 and 6.028 ms, respectively. At t=2.735 (6.028) ms, the magnitude of ε_{22} in the core varies between 4.52×10^{-3} and -3.66×10^{-2} (6.80×10^{-3} and -1.90×10^{-2}), whereas the magnitude of ε_{12} varies between 1.14×10^{-3} and -8.97×10^{-6} (6.34×10^{-4} and -1.47×10^{-3}). This corroborates results shown in Fig. 26, i.e., the strain energy in the core is mainly due to the transverse normal strains. Note that fringe plots display a range of values of the strain rather than its exact value.

3.4. Delamination of the foam core from face sheets in a sandwich composite plate

For the sandwich hull studied in Subsection 3.3, we examine if delamination occurs at the core/face sheet interface. The FE mesh used in the simulation is similar to the fine mesh described in Subsection 3.3 except that at the interface between the core and the face sheets the FE mesh for the core and the face sheets have separate nodes which are tied using the contact algorithm CONTACT_TIED_SURFACE_TO_SURFACE_FAILURE in LS-DYNA and the tie is released when the following



Fig. 24. (Colour online) At *t*=5.471 ms: (a) deformable hull and water region and (b) variations of the velocity components of the hull and the adjoining water particle.

condition is satisfied at a node:

$$\left(\frac{\operatorname{Max}(0.0, T_{nomal})}{F_{normal}}\right)^{2} + \left(\frac{T_{shear}}{F_{shear}}\right)^{2} - 1 \ge 0.$$
(21)

Here T_{normal} and T_{shear} are the normal and the tangential tractions at a point on the interface between the core and the face sheet, F_{normal} and F_{shear} are corresponding strengths of the interface, and we have set F_{normal} =1.0 MPa and F_{shear} =1.0 MPa. After the tie is released between two overlapping nodes, the core and the face sheet nodes can slide over each other as if the two contacting surfaces were smooth but not interpenetrating; the contact algorithm CONTACT_AUTOMATIC_SURFACE_TO_SURFACE in LS-DYNA is used to accomplish this. The contact algorithm is checked by studying forced vibrations of the hull described in detail in Das (2009). For a very high value assigned to F_{normal} and F_{shear} to prevent delamination, the lowest natural frequency of the plate is found to be 107 Hz, which is the same as that computed without using the tie and the contact algorithms. With F_{normal} =1.0 MPa and F_{shear} =1.0 MPa, delamination occurred all along the interface during the forced vibration analysis, and the lowest natural frequency of the delaminated plate decreased to 30 Hz.

Results shown in Fig. 29 illustrate that time histories of the deflection of the centroid of the hull with and without the consideration of delamination between the core and the face sheets are quite different. At 6 ms, when the top most point



Fig. 25. Time history of the length $c_f(t)$ of the wetted surface.



Fig. 26. Strain energy density in the core and face sheets at (a) an early stage of slamming when t=2.735 ms and (b) an ending stage at t=6.018 ms. Present results computed with LS-DYNA are compared with those reported by Qin and Batra (2009).

of the jet flow reaches the chine, the difference in the two sets of results is about 40%. Fig. 30 shows the deformed hull at three times, and arrows point to the delaminated regions. A portion of the hull with delaminated face sheets at t=4.7 ms is shown in Fig. 31.



Fig. 27. (Colour online) Fringe plots of (a) ε_{22} and (b) ε_{12} at *t*=2.745 ms.



Fig. 28. (Colour online) Fringe plots of (a) ε_{22} and (b) ε_{12} at *t*=6.018 ms.

Fig. 32(a) shows local slopes ζ and η of the top face sheet and the core at points (x_1, x_2) and (y_1, y_2) . The points (x_1, x_2) and (y_1, y_2) on the top face sheet and the core, respectively, are initially coincident $(X_1=Y_1 \text{ and } X_2=Y_2)$ and are at the midspan of the panel, therefore, $X_1=Y_1=0.5$ m. At ~ 1.5 ms the delamination occurs and two points separate from each other. Until about 1.5 ms the two slopes are the same and equal the initial slope (deadrise angle) of 5°. After ~ 1.5 ms, the local slopes of the face sheet and the core deviate from each other until about 3.0 ms, at which time, the two slopes again become equal. At ~ 5.0 ms, the slopes again deviate from each other. Fig. 32(b) shows the components of relative displacement u_n and u_t , which are, respectively, normal and tangential to the top surface of the core at (y_1, y_2) and are given by

$$u_n = (x_2 - y_2)\cos\eta - (x_1 - y_1)\sin\eta, \tag{22}$$

and

$$u_t = (x_2 - y_2) \sin \eta + (x_1 - y_1) \cos \eta.$$

(23)



Fig. 29. (Colour online) Time histories of the deflection of the hull centroid with and without the consideration of delamination.



Fig. 30. Deformed shapes of the hull and the water at three different times showing the delamination (indicated by arrows). The boxed area in Fig. 30(c) is zoomed in Fig. 31.



Fig. 31. Deformed shape of a part of the hull near the chine (see the boxed area in Fig. 30(c)) showing the separation of the core from face sheets due to the delamination at t=4.7 ms.

Displacements u_n and u_t remain zero till 1.5 ms indicating no delamination. From 1.5 ms till about 3.0 ms, u_n and u_t increase and then decrease to zero. This implies that the gap between the two points increases initially from 1.5 to 2.5 ms before closing at ~3.0 ms. From ~3.0 to ~5.5 ms u_n remains almost zero while the absolute value of u_t increases indicating sliding between the core and the face sheet at the interface after the closure of the gap between them.



Fig. 32. (Colour online) (a) Local slopes ζ and η of the face sheet and the core at two initially coincident points (x_{1t}, x_2) and (y_1, y_2) at the midspan of the panel, (b) the normal (tangential) relative displacement $u_n(u_t)$, and (c) tangential (normal) velocities v_t and $v_t'(v_n$ and $v_n)$ of points (x_1, x_2) and (y_1, y_2) , respectively.



Fig. 33. (Colour online) Strain energy density in the core and face sheets at t=2.735 ms.

Fig. 32(c) shows tangential (normal) velocities $v_t^1(v_n^1)$ and $v_t^2(v_n^2)$ of points (x_1, x_2) and (y_1, y_2) , respectively. These velocities are given by

$$v_t^1 = v_1^1 \cos\zeta + v_2^1 \sin\zeta,\tag{24}$$

$$v_n^1 = -v_1^1 \sin\zeta + v_2^1 \cos\zeta,$$
(25)

$$v_t^1 = v_1^2 \cos \eta + v_2^2 \sin \eta,$$
 (26)

$$v_n^1 = -v_1^2 \sin\eta + v_2^2 \cos\eta.$$
(27)

Here, v_1^1 (v_2^1) and v_1^2 (v_2^2) are the x_1 - (x_2 -) velocities of points (x_1, x_2) and (y_1, y_2), respectively. After delamination at ~1.5 ms tangential velocities of the two points deviate from each other. Normal velocities of the two points deviate from each other between ~1.5 and ~3.0 ms, at which point, they become almost same before deviating again at about ~5.5 ms.

We have shown in Fig. 33 variations of the strain energy density in the core and the two face sheets at t=2.735 ms both with and without the consideration of delamination; results for the latter case are from Subsection 3.3. The comparison of these two sets of results shows that the energy due to core deformations is insignificant after delamination, and the work done by external forces is used to deform the face sheets. After delamination the core is either separated from the face sheets or slides between them and there is no load transfer between the core and the face sheets except when the two are in contact and only the normal traction can be transferred from a face sheet to the core. The strain energy due to the transverse shear and the transverse normal strains in the core is negligible, whereas, without the consideration of the delamination, only the strain energy in the core due to the transverse shear strain is negligible but that due to transverse normal strain is significant.



Fig. 34. (Colour online) Fringe plots of the transverse normal strain in the core and the face sheets at t=2.735 ms.



Fig. 35. (Colour online) Fringe plots of the strain in the core and the face sheets at t=4.7 ms with the consideration of the delamination.

Fig. 34 shows fringe plots of the transverse normal strain in the core and face sheets at t=2.735 ms with consideration of delamination, and Fig. 27(a) shows the similar result without considering delamination. When the face sheets are allowed to separate from the core the transverse normal strain in the core is significantly lower than that in the core when delamination is not allowed.

Figs. 35 and 36 show fringe plots of strains at t=4.7 ms with and without the consideration of the delamination, respectively. Table 3 lists the maximum and the minimum values of strains in the core and the two face sheets at t=4.7 ms. It is observed that the transverse normal strain in the core is significantly higher when delamination is not allowed than that when delamination is allowed.



Fig. 36. Fringe plots of the strain in the core and the face sheets at t=4.7 ms without the consideration of the delamination.

Fable 3
Maximum and minimum strains in the core and the two face sheets of the sandwich composite panel at $t=4.7$ ms.

Strain	Delamination allowed		Delamination not allowed	
	Face sheets	Core	Face sheets	Core
Longitudinal normal s	train			
Min	-0.003	0.0000	-0.004	-0.0004
Max	+0.004	0.0000	+0.005	+0.0005
Transverse normal stra	ain			
Min	0.0000	-0.0025	0.000	-0.007
Max	0.0000	0.0023	0.0001	+0.007
Transverse shear strain	n			
Min	-0.005	0.0000	-0.004	0.0000
Max	0.004	0.0000	+0.005	0.0000

4. Conclusions

We have simulated the slamming impact of rigid and deformable hulls by using a coupled Lagrangian and Eulerian formulation available in the commercial finite element (FE) software LS-DYNA, and approximating deformations of the hull and the water as two dimensional (plane strain). The Lagrangian (Eulerian) description of motion is used to describe deformations of the wedge (water), and the penalty method is used to satisfy continuity conditions between the wedge and the water surface. Effects of different values of parameters in the penalty method on the pressure distribution at the interface and the penetration of water into the hull are delineated. It is found that values of these parameters to minimize oscillations in the pressure and reduce water penetration depend upon the deadrise angle, the FE mesh, the initial velocity of the wedge, and whether or not deformations of the hull material are considered. Whenever possible, computed results are compared with those available in the literature. Effects of damage induced in a sandwich composite panel due to the hydroelastic pressure are studied.

Other conclusions are summarized below:

- 1. Fluid pressure at the solid–fluid interface and the velocity profile near the interface depend significantly on values of parameters in the penalty method (see Appendix A), the FE mesh used, materials of the hull, and the hull's deadrise angle. We used at least two different FE meshes to ascertain the likely error in the numerical solution.
- 2. For v-shaped rigid wedges entering initially calm water with a constant downward velocity, and deformations assumed to be symmetric about the vertical axis passing through the apex of the v, computed pressures at the fluid–hull interface are found to agree reasonably well with those reported in the literature (e.g., see Fig. 7). For deadrise angles of 30° and 45°, the interface pressure variation along the span of the wedge is approximately uniform. For small deadrise angles (e.g., 10°) the peak pressure occurs at the jet flow, and for relatively large deadrise angles (e.g., 81°), the peak pressure occurs at the keel (for details see Das, 2009).
- 3. For rigid v-shaped hulls of deadrise angles of 30° and 45°, the length of jet flow near the chine is more than that for other deadrise angles studied here (cf. Figs. 8 and 10).
- 4. For a rigid v-shaped wedge the tangential component of the water velocity at the wedge-water interface increases from zero at the keel to the maximum value in region A of the jet flow (cf. Fig. 10(a)).
- 5. For v-shaped rigid wedges falling freely through initially calm water with deformations symmetric about the vertical axis passing through the apex of the v, the time history of the total upward force acting on the hull agreed well with that reported in Yettou et al. (2007) but the computed pressure distribution on the fluid–wedge interface differed noticeably from that given in Yettou et al. (2007).
- 6. The core material in a sandwich composite panel is effective in absorbing the impact energy due to water slamming. However, delamination between the core and the face sheets significantly reduces the effectiveness of the core material.

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Appendix A. The penalty based contact algorithm

On the interface γ_{12} between the solid and the fluid regions, Eqs. (18) and (19) are satisfied using the penalty method (Aquelet and Souli, 2003; Aquelet et al., 2006). The contact algorithm computes the pressure at a point on the interface based upon the relative displacement **d**, where $\mathbf{d} = \mathbf{u}^s - \mathbf{u}^f$, and the material time derivative of **d**. These forces are applied to the contacting fluid and the structure nodes to prevent penetration of water into the solid region.

In the contact algorithm, the surface of the solid body is designated as "slave" and that of fluid contacting the solid body as "master". Forces due to the coupling affect nodes that are on the fluid–structure interface. For each node on the slave surface, the increment in **d** is computed at each time step, using the relative velocity $\dot{\mathbf{d}} = (\mathbf{v}^s - \mathbf{v}^f)$, where \mathbf{v}^s is the velocity of the slave node, and \mathbf{v}^f is the velocity of the master particle of the fluid body initially coincident with the slave node on the fluid/structure interface (Fig. A.1). Note that the master particle is not a FE node, but a particle of the fluid body, which should remain coincident with the slave node, which is a FE node of the solid body. \mathbf{v}^f is interpolated from the velocity at FE nodes of the fluid region at the current time.

At time $t = t^n$, **d**^{*n*} is updated incrementally by using

$$\mathbf{d}^{n+1} = \mathbf{d}^n + (\mathbf{v}^s_{n+1/2} - \mathbf{v}^f_{n+1/2})\Delta t$$
,

where Δt is the time increment. The coupling force acts only if penetration occurs, i.e., $|\mathbf{d}^{n+1}| > 0$. For clarity, the superscript on **d** is omitted, and we use **d** for the relative displacement. Penalty coupling behaves like a spring dashpot

(A.1)



Fig. A.1. Penalty coupling between the mesh for the solid region and that for the fluid region.

system and penalty forces are calculated proportionally to **d** and **d**. Fig. A.1 illustrates the spring and the dashpot attached to the slave node and the master particle within a fluid element that is intercepted by the structure.

The coupling force **F** is given by

$$\mathbf{F} = k\mathbf{d} + c\dot{\mathbf{d}},\tag{A.2}$$

where *k* and *c* represent the spring stiffness and the damping coefficient, respectively. For an inviscid fluid, **d** is the displacement between the slave node and the master particle in the direction normal to the surface of the solid body, so that **F=0** when fluid moves along the tangent to the surface of the solid body. The force **F** is applied to both the master particle and the slave node in opposite directions to satisfy force equilibrium at the interface. At the slave node, force $\mathbf{F}_s = \mathbf{F}$ is applied; whereas for the fluid, the coupling force \mathbf{F}_f is divided among the FE nodes based on the shape functions N_i (*i*=1,...,8) of the element on which the master particle is situated:

$$\mathbf{F}_{i}^{i}=N_{i}\mathbf{F}.$$

As the penalty stiffness k approaches infinity, **d** approaches zero, satisfying the interface condition. However, the penalty method increases the overall stiffness of the system affecting its dynamic behavior. The optimum value of k should be such that it does not alter the dynamics of the problem significantly and prevents relative displacements between the two contacting bodies.

In LS-DYNA the contact stiffness is defined per unit area as

$$k_d = \frac{\kappa}{A},\tag{A.4}$$

where *A* is the area of the solid–fluid interface on a structural element. The value of k_d can either be specified through the LS-DYNA input file or can be given in terms of the bulk modulus C_1 of the fluid, and the volume V_{fe} of the fluid element that contains the master fluid particle:

$$k_d = P_f \frac{C_1 A}{V_{fe}}.$$
(A.5)

The value of P_f , which is a scalar factor, can be specified through the LS-DYNA input file. Its suggested value is between 0.0 and 0.1. The optimum value of k_d or P_f is problem specific. Stenius et al. (2006) have proposed a method to find the optimum values of k_d and P_f for water slamming on a rigid wedge. With an increase in the value of k_d , the interface pressure becomes oscillatory. In this paper, we compute the amount of fluid penetration through the interface and the pressure profile over the interface. For the optimal value of k_d the pressure variation should be smooth over the interface and the amount of fluid penetration should be negligible, e.g., the average value of the component of **d** normal to the interface over the span of the hull should be at least an order of magnitude less than the thickness of the hull.

The software LS-DYNA does not have the option of solving problems for incompressible fluids. We have taken water to be compressible and shown that the computed slamming pressures acting on rigid panels entering calm water at ~ 10 m/s agree well with those found considering water to be incompressible. Therefore, the compressibility of water does not affect the pressure distribution on the panel. We note that if the bulk modulus C_1 of water is large then k_d will be large if Eq. (A.5) is used to find k_d and if P_f is not small enough. In the present study, k_d is computed using Eq. (A.4).

The viscous dashpot with damping coefficient *c* damps out high frequency oscillations due to the contact stiffness. Again, to prevent the damping force from altering the dynamics of the problem an optimal value of *c* should be used. The equation of motion for a spring-dashpot system can be written as

$$M\frac{d^2}{dt^2}d + c\frac{d}{dt}d + kd = 0,$$
(A.6)

where $M = \sqrt{k((m_s + m_f)/m_s m_f)}$, m_s and m_f equal lumped mass at the slave node and the master particle, respectively. Eq. (A.6) can be rewritten in terms of the damping factor $\xi = c/\sqrt{kM}$ and the frequency $\omega = \sqrt{k/M}$ as

$$\frac{d^2}{dt^2}d + \zeta\omega\frac{d}{dt}d + \omega^2 d = 0.$$
(A.7)

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The damping factor ξ can be specified through the LS-DYNA input file.

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