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Saint-Venant's principle in linear elasticity with microstructure

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Abstract

Toupin's version of the Saint-Venant principle in linear elasticity is generalized to the case of linear elasticity with microstructure. That is, it is shown that, for a straight prismatic bar made of an isotropic linear elastic material with microstructure and loaded by a self-equilibrated force system at one end only, the strain energy stored in the portion of the bar which is beyond a distance s from the loaded end decreases exponentially with the distance s.

Introduction

Mathematical versions of Saint-Venant's principle in linear elasticity due to Sternberg, Knowles, Zanaboni, Robinson and Toupin have been discussed by Gurtin [1] in his monograph. In this paper we prove an analogue of Toupin's version of the Saint-Venant principle for linear elastic materials with microstructure. For a linear elastic homogeneous prismatic body of arbitrary length and cross-section loaded on one end only by an arbitrary system of self-equilibrated forces, Toupin [2] showed that the elastic energy $U(s)$ stored in that part of the body which is beyond a distance s from the loaded end satisfies the inequality

$$U(s) \leqslant U(0) \exp(-(s-\ell)/s_c(\ell)). \tag{1}$$

The characteristic decay length $s_c(\ell)$ depends upon the maximum and the minimum elastic moduli of the material and the smallest nonzero characteristic frequency of free vibration of a slice of the cylinder of length ℓ. By using an estimate due to Ericksen ([7], p. 88) for the norm of the stress tensor in terms of the strain-energy density, one can show that $s_c(\ell)$ depends on the maximum elastic modulus and not one the minimum elastic modulus.

Inequalities similar to (1) have been obtained by Berglund [3] for linear elastic micropolar prismatic bodies and by Batra [4,5] for non-polar and micropolar linear elastic helical bodies. Herein we prove a similar result for a straight prismatic body made of a linear elastic material with microstructure. These materials were first studied by Mindlin [6]. In them a deformable unit cell is assumed to be attached to each point of the body. The deformation of this unit cell is independent of the deformation of material points of the body. The unit cell may be interpreted as a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material. Micropolar elastic materials are a special class of these materials for which the unit cell is taken to be rigid.

We assume that the cross-sections are materially uniform in the sense that one-cross-section can be obtained from the other by a rigid body motion. Thus the material properties are independent of the axial coordinate of the point. Ericksen [7] has discussed material uniformity in more general terms. We use equilibrium equations in the form of Euler-Lagrange equations derived by extremising a functional. The characteristic decay length is found to depend upon the maximum elastic modulus and the characteristic frequency of free vibration of a slice of cylindrical body of length ℓ. Thus a relatively large elasticity associated with the deformation of the microstructure will reduce the decay rate of the energy.

Formulation of the problem

Consider an unstressed straight prismatic bar with materially uniform cross-sections and made of a linear elastic material with microstructure. Introduce a fixed rectangular Cartesian Coordinate system so that in the unstressed reference configuration the X_3-axis coincides with the axis of the bar, one end is contained in the plane $X_3 = 0$ and for points in the bar $X_3 \geqslant 0$. We denote the coordinates of a material point in the reference configuration by X_i, the coordinates of the same material point after the deformation by x_i and the displacement of the material point by u_i.

For an elastic material with microstructure, embedded in each material particle there is assumed to be a microvolume V' in which X_i' and x_i' are the components of the referential and spatial position vectors, respectively, referred to axes parallel to those of the X_i or x_i and with origin always attached to the particle. A microdisplacement

$$u_i' = x_i' - X_i'$$

gives the displacement of the microvolume relative to the particle carrying the microvolume.

We assume that the microdisplacement can be expressed as

$$u_j' = x_k' \psi_{kj}, \qquad \psi_{kj} = \psi_{kj}(\mathbf{x}), \tag{2}$$

and

$$\left|\frac{\partial u_i}{\partial X_j}\right| \ll 1, \qquad \left|\frac{\partial u_i'}{\partial X_j'}\right| \ll 1,$$

so that

$$\frac{\partial u_i}{\partial X_j} \simeq \frac{\partial u_i}{\partial x_j} \equiv h_{ij}, \tag{3}$$

$$\frac{\partial u_i'}{\partial X_j'} \simeq \frac{\partial u_i'}{\partial x_j'}.$$

For the linear theory being studied here the strain tensors (Cf. Mindlin [6])

$$e_{ij} = (h_{ij} + h_{ji})/2, \tag{4}$$

$$\gamma_{ij} = h_{ij} - \psi_{ij}, \tag{5}$$

$$\chi_{ijk} = \frac{\partial \psi_{jk}}{\partial x_i} \equiv \partial_i \psi_{jk}, \tag{6}$$

describe the deformation of the continuum. Also the strain energy density W is taken as a positive-definite, homogeneous quadratic function of the forty-two variables e_{ij}, γ_{ij} and χ_{ijk}. To save some writing we denote the ordered triplet $(\mathbf{e}, \boldsymbol{\gamma}, \boldsymbol{\chi})$ by Γ and write W as

$$W = \tfrac{1}{2}\Gamma \cdot \mathbf{E}\Gamma. \tag{7}$$

Thus $\mathbf{E}$, called the elasticity matrix, is a linear transformation from a 42-dimensional linear space into a 42-dimensional linear space. However, only 903 of the 1764 elasticities $\mathbf{E}$ are independent. Because of the positive definiteness of W,

$$\frac{\partial W}{\partial \Gamma} \cdot \frac{\partial W}{\partial \Gamma} = \mathbf{E}\Gamma \cdot \mathbf{E}\Gamma = \Gamma \cdot \mathbf{E}^2\Gamma \leqslant a_M \Gamma \cdot \mathbf{E}\Gamma = 2a_M W, \tag{8}$$

where

a_M = supremum of the eigen-values of $\mathbf{E}$.

We note that $\mathbf{E}$ depends only on X_1, X_2 since the cross-sections of the bar are assumed to be materially uniform. When the expressions (4)-(6) for strains are substituted into (7) we obtain

$$W = \overline{W}(h_{ij}, \psi_{ij}, \chi_{ijk}, X_A), \qquad A = 1, 2 \tag{9}$$

in which $\overline{W}$ is a homogeneous quadratic function of the indicated variables except X_A.

An infinitesimal rigid body displacement is described by a uniform translation c_i and a rotation $b_{ij} = -b_{ji}$ of the macromaterial and an equal rotation $\psi_{[ij]} \equiv (\psi_{ij} - \psi_{ji})/2$ of the micromaterial. The displacements associated with a rigid displacement are

$$w_j = c_j + b_{ij}x_i, \quad w_j' = x_i'\psi_{[ij]} = x_i'b_{ij}. \tag{10}$$

Thus if

$$v_i = u_i + w_i, \quad v_i' = u_i' + w_i', \quad \phi_{ij} = \psi_{ij} + b_{ij},$$

$$\bar{h}_{ij} = \frac{\partial v_i}{\partial x_j}, \quad \chi_{ijk}' = \frac{\partial \phi_{ij}}{\partial x_k}, \tag{11}$$

then

$$e_{ij}(\mathbf{u}) = e_{ij}(\mathbf{v}), \quad \gamma_{ij}(\mathbf{u}, \mathbf{u}') = \gamma_{ij}(\mathbf{v}, \mathbf{v}'),$$

$$\chi_{ijk}(\mathbf{u}') = \chi_{ijk}(\mathbf{v}'), \tag{12}$$

and

$$\overline{W}(\bar{h}_{ij}, \phi_{ij}, \bar{\chi}_{ijk}, X_A) = \overline{W}(h_{ij}, \psi_{ij}, \chi_{ijk}, X_A). \tag{13}$$

That is, the addition of a rigid displacement leaves $\overline{W}$ unchanged.

The equilibrium equations and boundary conditions obtained by seeking the extremum of the functional

$$\int_B \overline{W} dV - \int_{\partial B} f_i u_i \mathrm{d}S - \int_{\partial B} T_{jk}\psi_{jk} \mathrm{d}S$$

are

$$\partial_i\left(\frac{\partial \overline{W}}{\partial h_{ij}}\right) = 0 \qquad \text{in} \quad B, \tag{14}$$

$$\partial_i\left(\frac{\partial \overline{W}}{\partial \chi_{ijk}}\right)-\frac{\partial \overline{W}}{\partial \psi_{jk}}=0 \quad \text{in} \quad B, \tag{15}$$

and

$$\frac{\partial \overline{W}}{\partial h_{ij}}n_i=f_j \quad \text{on} \quad \partial B, \tag{16}$$

$$\frac{\partial \overline{W}}{\partial \chi_{ijk}}n_i=T_{jk} \quad \text{on} \quad \partial B, \tag{17}$$

where

$$\mathrm{d}S_i=n_i\mathrm{d}U^1\mathrm{d}U^2, \tag{18}$$

$$n_i=\pm\epsilon_{ijk}\frac{\partial X_j}{\partial U^1}\frac{\partial X_k}{\partial U^2}. \tag{19}$$

Here the boundary ∂B of the body B is assumed to have the local parametric representation $\mathbf{X}=\mathbf{X}(U^1, U^2)$ and the sign in (19) is selected so that the vector $\mathbf{n}$ points out of the body. The permutation symbol ϵ_{ijk} equals 1 or -1 accordingly as i, j and k form an even or an odd permutation of 1, 2 and 3 and is zero otherwise. The surface traction $\mathbf{f}$ and the double force T_{jk} are measured per unit coordinate area $\mathrm{d}U^1\mathrm{d}U^2$. The diagonal terms of T_{jk} are double forces without moments and the off-diagonal terms represent double forces with moments. The antisymmetric part $T_{[jk]}$ of the double traction T_{jk} is the Cosserat couple stress vector. In T_{jk}, the first subscript gives the orientation of the lever arm between forces and the second subscript gives the orientation of the forces.

Equations (14) and (15) are the equilibrium equations. Were we to write equations of motion in the absence of external body forces, the right-hand sides of (14) and (15) will be replaced by $\rho\ddot{u}_j$ and $I_{jk\ell m}\ddot{\psi}_{\ell m}$ wherein a superimposed dot indicates material time differentiation, ρ is the mass density and $I_{jk\ell m}$ is the inertia tensor associated with the microdeformation. Both ρ and $\mathbf{I}$ are evaluated in the reference configuration. We will need ρ and $\mathbf{I}$ subsequently when we consider the problem of free vibration of a slice of the prismatic body.

We are interested in the case when the part $X_3=0$ of the boundary is loaded and the remainder of the boundary is traction free. In order that there exist a solution of (14) and (15) under these conditions, the applied loads must be self-equilibrated and must satisfy

$$\int_{C_0}\frac{\partial \overline{W}}{\partial h_{3j}}\mathrm{d}S_3=0=\int_{C_0}\left(\frac{\partial \overline{W}}{\partial h_{3[j}}X_{k]}+\frac{\partial \overline{W}}{\partial \chi_{3[jk]}}\right)\mathrm{d}S_3. \tag{20}$$

Here moments are taken with respect to the origin and C_s is the cross-section of the body contained in the plane $X_3=s$. With the definition

$$U(s)=\int_{X_3\geqslant s}W\mathrm{d}V, \tag{21}$$

we state and prove below the

Theorem. If a prismatic body made of a linear elastic material with microstructure and with materially uniform cross-sections is loaded on C_0 by a self-equilibrated force system and if

$$f_i = 0 = T_{jk} \quad \text{on} \quad \partial B - C_0, \tag{22}$$

then

$$U(s) \leqslant U(0)\exp(-(s-\ell)/s_c(\ell)), \tag{23}$$

where

$$s_c(\ell) = 2(a_M/\lambda_0(\ell))^{1/2}, \tag{24}$$

$\lambda_0(\ell)$ is the smallest nonzero characteristic frequency of free vibration of a slice of the body of axial length ℓ, unit mass density and the inertia tensor associated with the microdeformation equal to the identity tensor.

Proof of the Theorem

Recalling that in (13) $\overline{W}$ is a homogeneous quadratic function of the indicated variables except X_A, we have by Euler's Theorem

$$U(s) = \int_{X_3 \geqslant s} \overline{W} \mathrm{d}V,$$

$$= \frac{1}{2}\int_{X_3 \geqslant s}\left[\frac{\partial \overline{W}}{\partial h_{ij}} h_{ij} + \frac{\partial \overline{W}}{\partial \psi_{jk}} \psi_{jk} + \frac{\partial \overline{W}}{\partial \chi_{ijk}} \chi_{ijk}\right] \mathrm{d}V, \tag{25}$$

$$= -\frac{1}{2}\int_{C_s}\left[\frac{\partial \overline{W}}{\partial h_{3j}} u_j + \frac{\partial \overline{W}}{\partial \chi_{3jk}} \psi_{jk}\right] \mathrm{d}S_3. \tag{26}$$

In order to obtain (26) from (25) we have used the divergence theorem, equilibrium equations (14) and (15), boundary conditions (22) and

$$n_k = -\delta_{3k} \quad \text{on} \quad C_s.$$

Because of (13) we can replace $\mathbf{u}$ and ψ in (26) by $\mathbf{v}$ and ϕ where $\mathbf{v}$ and ϕ are given by (11). Thus

$$U(s) = -\frac{1}{2}\int_{C_s}\left[\frac{1}{2}\frac{\partial \overline{W}}{\partial \bar{h}_{3j}} v_j + \frac{\partial \overline{W}}{\partial \bar{\chi}_{3jk}} \phi_{jk}\right] \mathrm{d}S_3. \tag{27}$$

Physically this expresses the requirement that a self-equilibrated force system does no work during a rigid displacement of the body. From Eqns. (4)–(6) and (11), we conclude that

$$\frac{\partial \overline{W}}{\partial \bar{h}_{3j}} = \frac{1}{2}\frac{\partial \overline{W}}{\partial e_{3j}} + \frac{\partial \overline{W}}{\partial \gamma_{3j}}, \tag{28}$$

and hence

$$U(s) = -\frac{1}{2}\int_{C_s}\left[\frac{1}{2}\frac{\partial \overline{W}}{\partial e_{3j}} v_j + \frac{\partial \overline{W}}{\partial \gamma_{3j}} v_j + \frac{\partial \overline{W}}{\partial \bar{\chi}_{3jk}} \phi_{jk}\right] \mathrm{d}S_3. \tag{29}$$

Using the inequality

$$2\int_B fh\,\mathrm{d}V \leqslant \left[\alpha\int_B f^2\mathrm{d}V + \frac{1}{\alpha}\int_B h^2\mathrm{d}V\right] \tag{30}$$

which holds for $\alpha > 0$ and is consequence of the Schwarz and geometric-arithmetic mean inequalities (e.g. see [2] p. 93), we obtain

$$-\int_{C_s}\frac{1}{2}\frac{\partial\overline{W}}{\partial e_{3j}}v_j\mathrm{d}S_3 \leqslant \frac{1}{2}\left[\alpha_1\int_{C_s}\frac{1}{4}\frac{\partial\overline{W}}{\partial e_{3j}}\frac{\partial\overline{W}}{\partial e_{3j}}\mathrm{d}S_3 + \frac{1}{\alpha_1}\int_{C_s}v_jv_j\mathrm{d}S_3\right],$$

$$\leqslant \frac{1}{2}\left[\alpha_1\int_{C_s}\frac{\partial\overline{W}}{\partial e_{ij}}\frac{\partial\overline{W}}{\partial e_{ij}}\mathrm{d}S_3 + \frac{1}{\alpha_1}\int_{C_s}v_jv_j\mathrm{d}S_3\right]. \tag{31}$$

Similarly

$$-\int\frac{\partial\overline{W}}{\partial\partial\gamma_{3j}}v_j\mathrm{d}S_3 \leqslant \frac{1}{2}\left[\alpha_2\int_{C_s}\frac{\partial\overline{W}}{\partial\gamma_{ij}}\frac{\partial\overline{W}}{\partial\gamma_{ij}}\mathrm{d}S_3 + \frac{1}{\alpha_2}\int_{C_s}v_jv_j\mathrm{d}S_3\right], \tag{32}$$

$$-\int_{C_s}\frac{\partial\overline{W}}{\partial\overline{\chi}_{3jk}}\phi_{jk}\mathrm{d}S_3 \leqslant \frac{1}{2}\left[\alpha_3\int_{C_s}\frac{\partial\overline{W}}{\partial\overline{\chi}_{ijk}}\frac{\partial\overline{W}}{\partial\overline{\chi}_{ijk}}\mathrm{d}S_3 + \frac{1}{\alpha_3}\int_{C_s}\phi_{jk}\phi_{jk}\mathrm{d}S_3\right], \tag{33}$$

and hence (29) can be written as

$$U(s) \leqslant \frac{1}{4}\left[\beta\int_{C_s}\left(\frac{\partial\overline{W}}{\partial e_{ij}}\frac{\partial\overline{W}}{\partial e_{ij}} + \frac{\partial\overline{W}}{\partial\gamma_{ij}}\frac{\partial\overline{W}}{\partial\gamma_{ij}} + \frac{\partial\overline{W}}{\partial\overline{\chi}_{ijk}}\frac{\partial\overline{W}}{\partial\overline{\chi}_{ijk}}\right)\mathrm{d}S_3 \right.$$

$$\left. + \frac{1}{\beta}\int_{C_s}(2v_jv_j + \phi_{jk}\phi_{jk})\mathrm{d}S_3\right], \tag{34}$$

where we have set $\alpha_1 = \alpha_2 = \alpha_3 = \beta$. Substitution from (8) into (34) results in

$$U(s) \leqslant \frac{1}{4}\left[\beta\int_{C_s}2a_M W\mathrm{d}S_3 + \frac{2}{\beta}\int_{C_s}(v_jv_j + \phi_{jk}\phi_{jk})\mathrm{d}S_3\right]. \tag{35}$$

Integration of both sides of (35) with respect to X_3 from $X_3 = s$ to $X_3 = s + \ell$ for some $\ell > 0$ and setting

$$\frac{1}{\ell}\int_s^{s+\ell}U(y)\mathrm{d}y = Q(s,\ell) \tag{36}$$

gives

$$Q(s,\ell) \leqslant \frac{\beta a_M}{2\ell}\int_{C_{s,\ell}}W\mathrm{d}V + \frac{1}{2\beta\ell}\int_{C_{s,\ell}}(v_jv_j + \phi_{jk}\phi_{jk})\mathrm{d}V, \tag{37}$$

in which

$$C_{s,\ell} \equiv \{\mathbf{X} : \mathbf{X} \in B,\ s \leqslant X_3 \leqslant s + \ell\}$$

= portion of the prismatic body between the planes $X_3 = s$ and $X_3 = s + \ell$.

In order to bound the last integral on the right-hand side of (37) by an integral of W, we consider the problem of free vibration of a prismatic body of length ℓ, unit mass density and the microdeformation inertia tensor equal to the unit tensor. Define a characteristic solution (e.g. see [1] Section 75) as the ordered triplet $[\lambda, \mathbf{u}, \boldsymbol{\psi}]$ such that λ is a scalar, $\mathbf{u}$ and $\boldsymbol{\psi}$ are fields on $C_{s,\ell}$, and

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \overline{W}}{\partial h_{ij}}\right) + \lambda u_j = 0 \qquad \text{in} \quad C_{s,\ell}, \tag{38}$$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \overline{W}}{\partial \chi_{ijk}}\right) - \frac{\partial \overline{W}}{\partial \psi_{jk}} + \lambda \psi_{jk} = 0 \quad \text{in} \quad C_{s,\ell}, \tag{39}$$

$$\int_{C_{s,\ell}} (u_i u_i + \psi_{jk}\psi_{jk})\mathrm{d}V = 1, \tag{40}$$

$$f_i = 0 = T_{jk} \qquad \text{on} \quad \partial C_{s,\ell}. \tag{41}$$

By taking the inner product of (38) with $\mathbf{u}$, of (39) with $\boldsymbol{\psi}$, adding the respective sides of these two equations and integrating the resulting equation over $C_{s,\ell}$, using the divergence theorem, equilibrium equations (14) and (15), and the boundary condition (41), we obtain (for details, see [1])

$$\lambda = \frac{2\int_{C_{s,\ell}} \overline{W}\mathrm{d}V}{\int_{C_{s,\ell}} (u_i u_i + \psi_{jk}\psi_{jk})\mathrm{d}V} = 2\int_{C_{s,\ell}} \overline{W}\mathrm{d}V. \tag{42}$$

In order to conclude $(42)_2$ from $(42)_1$ we have used (40). Since $\overline{W} = 0$ for a rigid body displacement, the smallest characteristic frequency of free vibration of $C_{s,\ell}$ is zero. In order to eliminate the rigid body displacement and thereby the possibility of zero characteristic frequency we consider smooth fields $\mathbf{v}$ and $\boldsymbol{\phi}$ that satisfy

$$\int_{C_{s,\ell}} (v_i v_i + \phi_{ij}\phi_{ij})\mathrm{d}V \neq 0, \qquad \int_{C_{s,\ell}} v_i \mathrm{d}V = 0, \qquad 0 = \int_{C_{s,\ell}} \epsilon_{ijk} x_j v_k \mathrm{d}V. \tag{43}$$

As shown by Toupin [2], for a given $\mathbf{u}$ one can choose $\mathbf{w}$ in (11) such that $\mathbf{v}$ satisfies (43); the corresponding $\boldsymbol{\phi}$ is related to $\boldsymbol{\psi}$ by $(11)_3$. Thus the lowest non-zero characteristic frequency $\lambda_0(\ell)$ of free vibration of $C_{s,\ell}$ will satisfy the inequality

$$\lambda_0(\ell) \leqslant \frac{2\int_{C_{s,\ell}} \overline{W}\mathrm{d}V}{\int_{C_{s,\ell}} (v_i v_i + \phi_{ij}\phi_{ij})\mathrm{d}V}. \tag{44}$$

Substitution from (44) into (37) results in the following:

$$Q(s,\ell) \leqslant \frac{\overline{s_c(\ell)}}{\ell} \int_{C_{s,\ell}} \overline{W}\mathrm{d}V, \tag{45}$$

in which

$$\overline{s_c(\ell)} = \tfrac{1}{2}\beta a_M + \frac{2}{\lambda_0 \beta}.$$

We choose $\beta = 2/(a_M \lambda_0)^{1/2}$ so that $\overline{s_c(\ell)}$ takes on the minimum value

$$\overline{s_c(\ell)} = 2(a_M/\lambda_0)^{1/2}. \tag{46}$$

Differentiation of (36) with respect to s yields

$$\frac{\mathrm{d}Q}{\mathrm{d}s} = \frac{1}{\ell}[U(s+\ell) - U(s)] = -\frac{1}{\ell}\int_{C_{s,\ell}} W \mathrm{d}V.$$

This when combined with (45) results in

$$\overline{s_c(\ell)} \frac{\mathrm{d}Q}{\mathrm{d}s} + Q \leqslant 0. \tag{47}$$

Integrating (47) and using

$$U(s+\ell) \leqslant Q(s,\ell) \leqslant U(s)$$

which follows from the observation that $U(s)$ is a nonincreasing function of s, we arrive at

$$\frac{U(s_2+\ell)}{U(s_2)} \leqslant \exp\left(-(s_2 - s_1)/\overline{s_c(\ell)}\right).$$

The choice $s_1 = 0$ and $s_2 = s - \ell$ gives the desired inequality (23).

Remarks

For a linear elastic body without any microstructure, i.e. $\psi_{ij} \equiv 0$, the deformation is completely described by the strain tensor e_{ij} and the characteristic decay length $s_c(\ell)$ reduces to essentially that given by Toupin, the remaining difference being due to the sharper estimate (8) used herein. For a micropolar linear elastic body $\psi_{kj} + \psi_{jk} = 0$, e_{ij} and γ_{ij} describe the deformation of the body and again our results reduce to those obtained by Berglund except that we study anisotropic materials whereas Berglund assumed the bar to be made of an isotropic material.

As is apparent from (24), the characteristic decay length depends upon the smallest non-zero characteristic frequency of free vibration of a slice of the prismatic body of axial length ℓ. Methods exist [8] for estimating these characteristic frequencies, or these frequencies might even be obtained experimentally. Needless to say our estimate of the decay rate is not the optimum one since we have been arbitrarily strengthening the various inequalities.

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