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Free vibrations of a piezoelectric body

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Abstract. We present a systematic analysis of the eigenvalue problem associated with free vibrations of a finite piezoelectric body. The analysis is based on an abstract formulation of the three-dimensional theory of piezoelectricity. A series of fundamental properties of free vibrations of a piezoelectric body are proved concisely. The problem of free vibrations of a piezoelectric plate governed by the two-dimensional plate equations due to Mindlin is treated in a similar manner.

1. Introduction

The free vibrations of a finite piezoelectric body has been of interest for a long time because of its applications in resonators. It has been studied either by using the three dimensional equations of piezoelectricity or the two-dimensional plate theory [1]. The two sets of equations are rather complicated. This has obscured insight into the mathematical structure of the equations and has made the mathematical manipulations tedious.

In this paper, an abstract formulation is employed. Based on the introduction of abstract vectors and operators and the construction of appropriate function spaces, several fundamental properties of free vibrations of a piezoelectric body are proved in a systematic and concise manner. Following the proof of the essential property that the operators involved are self-adjoint and positive on appropriate function spaces, the reality and positivity of the eigenvalues, the orthogonality of eigenvectors corresponding to distinct eigenvalues, and a variational principle in Rayleigh quotient form for the eigenvalues are established. The Rayleigh quotient is non-negative on an appropriate function space. This leads to a few properties of the smallest eigenvalue or the lowest resonant frequency. These properties generalize the corresponding results of classical elasticity. The problem of frequency shift due to small disturbances, a problem of great practical interest, is also studied

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within the abstract formulation by a perturbation method and a variational method. It is shown that the perturbation method gives equivalent results to those obtained by the Ritz method with a special choice of trial functions. The problem involving free vibrations of a piezoelectric plate governed by the two-dimensional plate equations of Mindlin is also formulated in an abstract form and then similar results follow automatically.

2. Free vibrations of a three-dimensional body

2.1. Governing equations

Let the finite spatial region occupied by the piezoelectric body be Ω, the boundary surface of Ω be S, the unit outward normal of S be n_i, and S be partitioned as

$$S_u \cup S_T = S_\phi \cup S_D = S,$$
$$S_u \cap S_T = S_\phi \cap S_D = \varnothing. \tag{1}$$

Physically, S_u, S_T, S_ϕ, and S_D are, respectively, parts of the boundary S on which mechanical displacement, traction vector, electric potential, and surface electric charge are prescribed.

For the time-harmonic free vibrations of a piezoelectric body with circular frequency ω, the governing equations and boundary conditions in rectangular Cartesian coordinates are [2]

$$-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj} = \rho\omega^2 u_i \quad \text{in } \Omega,$$
$$-e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki} = 0 \quad \text{in } \Omega,$$
$$u_i = 0 \quad \text{on } S_u,$$
$$T_{ji}(\mathbf{u}, \phi)n_j = (c_{jikl}u_{k,l} + e_{kji}\phi_{,k})n_j = 0 \quad \text{on } S_T,$$
$$\phi = 0 \quad \text{on } S_\phi,$$
$$D_i(\mathbf{u}, \phi)n_i = (e_{ikl}u_{k,l} - \varepsilon_{ik}\phi_{,k})n_i = 0 \quad \text{on } S_D, \tag{2}$$

where u_i is mechanical displacement, T_{ji} stress, ϕ electric potential, D_i electric displacement, ρ mass density, c_{ijkl} elastic moduli, ε_{ij} electric permittivity, and e_{ijk} piezoelectric constants. Throughout this paper, a repeated index implies summation over the range of the index, and a comma followed by an index j stands for partial differentiation with respect to x_j. The material constants have the following symmetry properties

$$c_{ijkl} = c_{jikl} = c_{klij},$$
$$e_{ijk} = e_{ikj}, \quad \varepsilon_{ij} = \varepsilon_{ji}, \tag{3}$$

and are positive definite, i.e. for any nonzero symmetric tensor a_{ij} and vector b_i

$$c_{ijkl}a_{ij}a_{kj} > 0, \quad \varepsilon_{ij}b_i b_j > 0 \tag{4}$$

In Eq. (2), values of ω^2 are sought corresponding to which nontrivial functions u_i and/or ϕ exist, hence we have an eigenvalue problem. We note that for the eigenfunctions u_i and ϕ corresponding to an eigenvalue ω^2, u_i alone must be nontrivial. This can be seen by setting $u_i = 0$ in Eq. (2), then ϕ must vanish identically.

2.2. *An abstract formulation*

For convenience, we denote ω^2 by λ and introduce vectors $\mathbf{U}$ and $\mathbf{V}$, and operators $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{U} = \{u_i, \phi\}, \quad \mathbf{V} = \{v_i, \psi\},$$
$$\mathbf{AU} = \{-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj}, -e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki}\},$$
$$\mathbf{BU} = \{\rho u_i, 0\}. \tag{5}$$

Then equation (2) can be written as

$$\mathbf{AU} = \lambda\mathbf{BU} \quad \text{in } \Omega,$$
$$u_i = 0 \quad \text{on } S_u,$$
$$T_{ji}(\mathbf{U})n_j = (c_{jikl}u_{k,l} + e_{kji}\phi_{,k})n_j = 0 \quad \text{on } S_T,$$
$$\phi = 0 \quad \text{on } S_\phi,$$
$$D_i(\mathbf{U})n_i = (e_{ikl}u_{k,l} - \varepsilon_{ik}\phi_{,k})n_i = 0 \quad \text{on } S_D. \tag{6}$$

We also introduce a function space $\Xi(\Omega)$:

$$\Xi(\Omega) = \{\mathbf{U}\,|\,\mathbf{U} \text{ satisfies boundary conditions } (6)_{2-5}\}. \tag{7}$$

We note that $\Xi(\Omega)$ contains all eigenvectors of Eq. (2). With the above definitions, the eigenvalue problem defined by Eq. (2) can be stated as: Find λ for which there exists a nontrivial $\mathbf{U} \in \Xi(\Omega)$ such that

$$\mathbf{AU} = \lambda\mathbf{BU}. \tag{8}$$

For later use, we also introduce a subspace $\Xi^*(\Omega) \subset \Xi(\Omega)$:

$$\Xi^*(\Omega) = \{\mathbf{U} \text{ real} \mid \mathbf{U} \in \Xi(\Omega),\ -e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki} = 0 \text{ in } \Omega\}. \tag{9}$$

Next we introduce an inner product $\langle\, ; \rangle$ for vectors in $\Xi(\Omega)$:

$$\langle \mathbf{U}; \mathbf{V} \rangle = \int_\Omega (u_i v_i + \phi\psi)\, \mathrm{d}\Omega. \tag{10}$$

2.3. Self-adjointness and non-negativeness of the operators

For any $\mathbf{U}$ and $\mathbf{V} \in \Xi(\Omega)$, we have

$$\begin{aligned}
\langle \mathbf{AU}; \mathbf{V} \rangle &= \langle \{-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj},\ -e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki}\}; \{v_i, \psi\} \rangle \\
&= \int_\Omega [(-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj})v_i + (-e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki})\psi]\, \mathrm{d}\Omega \\
&= \int_S [-T_{ji}(\mathbf{U})n_j v_i - D_i(\mathbf{U})n_i\psi]\, \mathrm{d}S \\
&\quad + \int_\Omega [c_{jikl}u_{k,l}v_{i,j} + e_{kji}\phi_{,k}v_{i,j} + e_{ikl}u_{k,l}\psi_{,i} - \varepsilon_{ik}\phi_{,k}\psi_{,i}]\, \mathrm{d}\Omega \\
&= -\int_S [T_{ji}(\mathbf{U})n_j v_i + D_i(\mathbf{U})n_i\psi]\, \mathrm{d}S + \int_S [T_{kl}(\mathbf{V})n_l u_k + D_k(\mathbf{V})n_k\phi]\, \mathrm{d}S \\
&\quad + \int_\Omega [-c_{klij}v_{i,jl} - e_{ikl}\psi_{,il})u_k + (-e_{kji}v_{i,jk} + \varepsilon_{ik}\psi_{,ki})\phi]\, \mathrm{d}\Omega \\
&= -\int_S [T_{ji}(\mathbf{U})n_j v_i + D_i(\mathbf{U})n_i\psi]\, \mathrm{d}S \\
&\quad + \int_S [T_{kl}(\mathbf{V})n_l u_k + D_k(\mathbf{V})n_k\phi]\, \mathrm{d}S + \langle \mathbf{U}; \mathbf{AV} \rangle \\
&= \langle \mathbf{U}; \mathbf{AV} \rangle
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\langle \mathbf{BU}; \mathbf{V} \rangle &= \langle \{\rho u_i, 0\}; \{v_i, \psi\} \rangle \\
&= \int_\Omega \rho u_i v_i\, \mathrm{d}\Omega \\
&= \langle \mathbf{U}; \mathbf{BV} \rangle,
\end{aligned} \tag{12}$$

which show that the operators $\mathbf{A}$ and $\mathbf{B}$ are self-adjoint on $\Xi(\Omega)$. On $\Xi^*(\Omega)$, we have

$$
\begin{aligned}
\langle \mathbf{AU}; \mathbf{U}\rangle &= \langle\{-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj},\ -e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki}\};\ \{u_i,\ \phi\}\rangle \\
&= \int_{\Omega} [(-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj})u_i + (-e_{ikl}u_{k,li} - \varepsilon_{ik}\phi_{,ki})\phi]\, \mathrm{d}\Omega \\
&= \int_{S} [-T_{ji}(\mathbf{U})n_j u_i - D_i(\mathbf{U})n_i\phi]\, \mathrm{d}S \\
&\quad + \int_{\Omega} [c_{jikl}u_{k,l}u_{i,j} + e_{kji}\phi_{,k}u_{i,j} + e_{ikl}u_{k,l}\phi_{,i} - \varepsilon_{ik}\phi_{,k}\phi_{,i}]\, \mathrm{d}\Omega \\
&= \int_{\Omega} [c_{jikl}u_{k,l}u_{i,j} + \varepsilon_{ik}\phi_{,k}\phi_{,i} + 2(e_{ikl}u_{k,l}\phi_{,i} - \varepsilon_{ik}\phi_{,k}\phi_{,i})]\, \mathrm{d}\Omega \\
&= \int_{\Omega} [c_{jikl}u_{k,l}u_{i,j} + \varepsilon_{ik}\phi_{,k}\phi_{,i} - 2(e_{ikl}u_{k,li} - \varepsilon_{ik}\phi_{,ik})\phi]\, \mathrm{d}\Omega \\
&\quad + \int_{S} 2(e_{ikl}u_{k,l} - \varepsilon_{ik}\phi_{,k})n_i\phi\, \mathrm{d}S \\
&= \int_{\Omega} [(c_{jikl}u_{k,l}u_{i,j} + \varepsilon_{ik}\phi_{,k}\phi_{,i}]\, \mathrm{d}\Omega \geqslant 0
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
\langle \mathbf{BU}; \mathbf{U}\rangle &= \langle\{\rho u_i, 0\};\ \{u_i,\ \phi\}\rangle \\
&= \int_{\Omega} \rho u_i u_i\, \mathrm{d}\Omega \geqslant 0,
\end{aligned} \tag{14}
$$

which show that **A** and **B** are non-negative on $\Xi^*(\Omega)$.

2.4. Reality of eigenvalues

Having shown the self-adjointness of operators **A** and **B**, we prove that all eigenvalues of Eq. (2) are real. Let λ be an eigenvalue, and **U** the corresponding eigenvector. Then (λ, **U**) satisfies Eq. (8). Taking the inner product of both sides of Eq. (8) with $\bar{\mathbf{U}}$, the complex conjugate of **U**, we obtain

$$\langle \mathbf{AU}; \bar{\mathbf{U}}\rangle = \lambda\langle \mathbf{BU}; \bar{\mathbf{U}}\rangle \tag{15}$$

Subtracting the complex conjugate of both sides of equation (15) from it, we arrive at

$$\langle \mathbf{AU}; \bar{\mathbf{U}}\rangle - \langle \bar{\mathbf{A}}\bar{\mathbf{U}}; \mathbf{U}\rangle = \lambda\langle \mathbf{BU}; \bar{\mathbf{U}}\rangle - \bar{\lambda}\langle \bar{\mathbf{B}}\bar{\mathbf{U}}; \mathbf{U}\rangle. \tag{16}$$

Since **A** and **B** are real and self-adjoint, we have

$$0 = (\lambda - \bar{\lambda})\langle \mathbf{BU}; \bar{\mathbf{U}} \rangle. \tag{17}$$

For the eigenfunction **U**,

$$\langle \mathbf{BU}; \bar{\mathbf{U}} \rangle = \int_{\Omega} \rho u_i \bar{u}_i \, d\Omega > 0, \tag{18}$$

which implies that $\lambda - \bar{\lambda} = 0$, or the eigenvalue λ is real. For a real eigenvalue, we can always choose the corresponding eigenvector to be real. To see this, we take the complex conjugate of Eq. (8), and obtain, with the reality of **A**, **B**, and λ,

$$\mathbf{A}\bar{\mathbf{U}} = \lambda \mathbf{B}\bar{U}. \tag{19}$$

Thus $\bar{\mathbf{U}}$ is also an eigenvector corresponding to λ. The linearity of the problem implies that $\frac{1}{2}(\mathbf{U} + \bar{\mathbf{U}})$ and $1/(2i)(\mathbf{U} - \bar{\mathbf{U}})$ are also eigenvectors corresponding to λ. In the following, we will assume that eigenvectors have been chosen real so that $\Xi^*(\Omega)$ contains all eigenvectors and $\Xi(\Omega)$ has real vectors only.

2.5. *Positivity of eigenvalues*

On $\Xi^*(\Omega)$, for an eigenpair $(\lambda, \mathbf{U})$, we take the inner product of both sides of Eq. (8) with **U** and obtain

$$\langle \mathbf{AU}; \mathbf{U} \rangle = \lambda \langle \mathbf{BU}; \mathbf{U} \rangle. \tag{20}$$

Since both **A** and **B** are non-negative on $\Xi^*(\Omega)$, and for eigenvectors they are strictly positive, Eq. (20) shows that all eigenvalues λ must be positive.

2.6. *Orthogonality of eigenvectors*

Let $\lambda^{(m)}$ and $\lambda^{(n)}$ be two distinct eigenvalues of Eq. (2) and the corresponding eigenvectors be $\mathbf{U}^{(m)}$ and $\mathbf{U}^{(n)}$. Thus

$$\begin{aligned} \mathbf{AU}^{(m)} &= \lambda^{(m)}\mathbf{BU}^{(m)}, \\ \mathbf{AU}^{(n)} &= \lambda^{(n)}\mathbf{BU}^{(n)}. \end{aligned} \tag{21}$$

Then taking the inner product of both sides of Eqs $(21)_1$ and $(21)_2$ with $\mathbf{U}^{(n)}$ and $\mathbf{U}^{(m)}$ respectively, and subtracting one from the other, we obtain

$$0 = (\lambda^{(m)} - \lambda^{(n)})\langle \mathbf{BU}^{(m)}; \mathbf{U}^{(n)} \rangle, \tag{22}$$

where we have used the fact that **A** and **B** are self-adjoint. Since $\lambda^{(m)} \neq \lambda^{(n)}$,

$$\langle \mathbf{BU}^{(m)}; \mathbf{U}^{(n)} \rangle = \int_\Omega \rho u_i^{(m)} u_i^{(n)}\, \mathrm{d}\Omega = 0, \tag{23}$$

which implies that eigenvectors associated with distinct eigenvalues are mutually orthogonal (Tiersten [3]). We note that Eqs (21) and (23) also imply another form of the orthogonality condition, viz.,

$$\begin{aligned}\langle \mathbf{AU}^{(m)}; \mathbf{U}^{(n)} \rangle &= \int_\Omega [(-c_{jikl} u_{k,lj}^{(m)} - e_{kji}\phi_{,kj}^{(m)}) u_i^{(n)} + (-e_{ikl} u_{k,li}^{(m)} + \varepsilon_{ik}\phi_{,ki}^{(m)})\phi^{(n)}]\, \mathrm{d}\Omega \\ &= \int_\Omega [c_{jikl} u_{k,l}^{(m)} u_{i,j}^{(n)} + e_{kji}\phi_{,k}^{(m)} u_{i,j}^{(n)} + e_{ikl} u_{k,l}^{(m)}\phi_{,i}^{(n)} - \varepsilon_{ik}\phi_{,k}^{(m)}\phi_{,i}^{(n)}]\, \mathrm{d}\Omega = 0.\end{aligned} \tag{24}$$

2.7. Variational formulation

We now give a variational formulation for the eigenvalue problem defined by Eq. (2). For a fractional functional

$$\Pi = \frac{\Lambda}{\Gamma}, \tag{25}$$

$\delta\Pi = 0$ implies that

$$\delta\Lambda - \Pi\delta\Gamma = 0. \tag{26}$$

We consider the following functional of $\mathbf{U} \in \Xi(\Omega)$:

$$\Pi(\mathbf{U}) = \frac{\langle \mathbf{AU}; \mathbf{U} \rangle}{\langle \mathbf{BU}; \mathbf{U} \rangle} \tag{27}$$

By using the self-adjointness of **A** and **B**, the stationary condition of Π is seen to be

$$\begin{aligned}\delta\Lambda - \Pi\delta\Gamma &= \langle \mathbf{AU}; \delta\mathbf{U} \rangle + \langle \delta\mathbf{AU}; \mathbf{U} \rangle - \Pi(\langle \mathbf{BU}; \delta\mathbf{U} \rangle + \langle \delta\mathbf{BU}; \mathbf{U} \rangle) \\ &= \langle \mathbf{AU}; \delta\mathbf{U} \rangle + \langle \mathbf{A}\delta\mathbf{U}; \mathbf{U} \rangle - \Pi(\langle \mathbf{BU}; \delta\mathbf{U} \rangle + \langle \mathbf{B}\delta\mathbf{U}; \mathbf{U} \rangle) \\ &= \langle \mathbf{AU}; \delta\mathbf{U} \rangle + \langle \delta\mathbf{U}; \mathbf{AU} \rangle - \Pi(\langle \mathbf{BU}; \delta\mathbf{U} \rangle + \langle \delta\mathbf{U}; \mathbf{BU} \rangle) \\ &= 2\langle \mathbf{AU} - \Pi\mathbf{BU}; \delta\mathbf{U} \rangle = 0,\end{aligned} \tag{28}$$

or

$$\mathbf{AU} - \Pi\mathbf{BU} = 0 \tag{29}$$

since $\delta\mathbf{U}$ is arbitrary. Therefore, we have the following variational principle: In order that the functional $\Pi(\mathbf{U})$ defined by (27) be stationary in $\Xi(\Omega)$ Eq. $(2)_{1,2}$ must be satisfied, and the stationary value of Π equals ω^2. Here the boundary conditions $(2)_{3-6}$ are regarded as constraints which must be satisfied by all admissible vectors $\mathbf{U}$ for Π. Written explicitly, the Rayleigh quotient Π is given by

$$\Pi(u_i, \phi) = \frac{\int_\Omega (c_{ijkl}u_{i,j}u_{k,l} - \varepsilon_{ij}\phi_{,i}\phi_{,j} + 2e_{ikl}u_{k,l}\phi_{,i})\,\mathrm{d}\Omega}{\int_\Omega \rho u_i u_i\,\mathrm{d}\Omega}, \tag{30}$$

which was obtained in [4]. Here it is a direct consequence of the self-adjointness of $\mathbf{A}$ and $\mathbf{B}$.

We further have, on $\Xi^*(\Omega)$

$$\Pi(u_i, \phi) = \frac{\int_\Omega (c_{ijkl}u_{i,j}u_{k,l} + \varepsilon_{ij}\phi_{,i}\phi_{,j})\,\mathrm{d}\Omega}{\int_\Omega \rho u_i u_i\,\mathrm{d}\Omega}. \tag{31}$$

Since Π is non-negative on $\Xi^*(\Omega)$, it is bounded from below. Therefore the smallest eigenvalue must be a minimum. Following standard arguments in variational analysis [5], we obtain the following results.

The smallest eigenvalue will increase if (i) ρ decreases; and (ii) c_{ijkl} increases to c'_{ijkl} such that $(c'_{ijkl} - c_{ijkl})a_{ij}a_{kl} > 0$ for any nonzero symmetric a_{ij}.

2.8. Frequency shift by a perturbation method

In applications, it is often of interest to study small change in the eigenvalues caused by small variations in the operators which may be due to small variations in the physical and/or geometrical parameters of the system. The small change in an eigenvalue is usually called a frequency shift. For the frequency shift problem associated with the eigenvalue problem (8), we consider the following problem:

$$(\mathbf{A} + \varepsilon\mathbf{A}^*)\mathbf{U} = \lambda(\mathbf{B} + \varepsilon\mathbf{B}^*)\mathbf{U}, \tag{32}$$

where ε is a small parameter, $\varepsilon\mathbf{A}^*$ and $\varepsilon\mathbf{B}^*$ are the small changes in operators $\mathbf{A}$ and $\mathbf{B}$, and λ and $\mathbf{U}$ are unknowns. We make the following perturbation expansions:

$$\lambda = \lambda^o + \varepsilon\lambda^* + O(\varepsilon^2),$$
$$\mathbf{U} = \mathbf{U}^o + \varepsilon\mathbf{U}^* + O(\varepsilon^2), \tag{33}$$

where λ^* is the first-order frequency shift which is to be determined. Substituting (33) into (32), and collecting terms of equal powers of ε, we have the following perturbation problems of successive orders.

Zeroth order:

$$\mathbf{AU}^o = \lambda^o\mathbf{BU}^o \tag{34}$$

First order:

$$\mathbf{AU}^* + \mathbf{A}^*\mathbf{U}^o = \lambda^*\mathbf{BU}^o + \lambda^o\mathbf{B}^*\mathbf{U}^o + \lambda^o\mathbf{BU}^*. \tag{35}$$

The solution to the zeroth order problem (34) is assumed known. In order to solve the first-order problem (35) for λ^*, we take the inner product of both sides of (35) with $\mathbf{U}^o$, and use the self-adjointness of $\mathbf{A}$ and $\mathbf{B}$ to obtain

$$\lambda^* = \frac{\langle \mathbf{U}^o;\ \mathbf{A}^*\mathbf{U}^o\rangle - \lambda^o\langle \mathbf{U}^o;\ \mathbf{B}^*\mathbf{U}^o\rangle}{\langle \mathbf{U}^o;\ \mathbf{BU}^o\rangle}. \tag{36}$$

Since all terms on the right-hand side of (36) are known, we can compute the frequency shift. The above perturbation method was used in [8] to find the frequency shift caused by a small change in the thickness of a quartz plate. The derivation, leading to an expression equivalent to (36), was very lengthy. Here it follows immediately from the self-adjointness of $\mathbf{A}$ and $\mathbf{B}$. Equation (36) delineates clearly how small changes in operators contribute to the frequency shift.

2.9. Frequency shift by a variational method

If the small changes in $\mathbf{A}$ and $\mathbf{B}$ are also self-adjoint, then the Rayleigh quotient for (32) is

$$\Pi(\mathbf{U}) = \frac{\langle(\mathbf{A} + \varepsilon\mathbf{A}^*)\mathbf{U}; \mathbf{U}\rangle}{\langle(\mathbf{B} + \varepsilon\mathbf{B}^*)\mathbf{U}; \mathbf{U}\rangle}. \tag{37}$$

The use of unperturbed modes $\mathbf{U}^o$ as trial functions in the Ritz approximation method gives

$$\begin{aligned}
\lambda &\approx \frac{\langle(\mathbf{A}+\varepsilon\mathbf{A}^*)\mathbf{U}^o;\mathbf{U}^o\rangle}{\langle(\mathbf{B}+\varepsilon\mathbf{B}^*)\mathbf{U}^o;\mathbf{U}^o\rangle} \\
&= \frac{\langle\mathbf{A}\mathbf{U}^o;\mathbf{U}^o\rangle+\varepsilon\langle\mathbf{A}^*\mathbf{U}^o;\mathbf{U}^o\rangle}{\langle\mathbf{B}\mathbf{U}^o;\mathbf{U}^o\rangle+\varepsilon\langle\mathbf{B}^*\mathbf{U}^o;\mathbf{U}^o\rangle} \\
&= \frac{\langle\mathbf{A}\mathbf{U}^o;\mathbf{U}^o\rangle}{\langle\mathbf{B}\mathbf{U}^o;\mathbf{U}^o\rangle}\,\frac{1+\varepsilon\langle\mathbf{A}^*\mathbf{U}^o;\mathbf{U}^o\rangle/\langle\mathbf{A}\mathbf{U}^o;\mathbf{U}^o\rangle}{1+\varepsilon\langle\mathbf{B}^*\mathbf{U}^o;\mathbf{U}^o\rangle/\langle\mathbf{B}\mathbf{U}^o;\mathbf{U}^o\rangle} \\
&\approx \lambda^o+\varepsilon\,\frac{\langle\mathbf{U}^o;\mathbf{A}^*\mathbf{U}^o\rangle-\lambda^o\langle\mathbf{U}^o;\mathbf{B}^*\mathbf{U}^o\rangle}{\langle\mathbf{U}^o;\mathbf{B}\mathbf{U}^o\rangle}.
\end{aligned} \tag{38}$$

A comparison of (38) with (36) shows that the frequency shift computed by the Ritz method is asymptotically equal to that given by the first order perturbation method. This equivalence of results given by the two methods was also discussed in [8].

3. Free vibrations of a plate

3.1. Governing equations

We consider a piezoelectric plate of thickness $2b$, with Cartesian coordinate axes x_1 and x_3 in the middle plane, and x_2 normal to the plate. Let the two-dimensional region in the x_1-x_3 plane occupied by the piezoelectric plate be A, the boundary curve of A be C, the unit outward normal of C be n_i (with $n_2 = 0$), and C be partitioned as

$$\begin{aligned}
&C_u \cup C_T = C_\phi \cup C_D = C, \\
&C_u \cap C_T = C_\phi \cap C_D = \varnothing.
\end{aligned} \tag{39}$$

The eigenvalue problem for free vibrations of a linear piezoelectric plate is [9]

$$\begin{aligned}
&-T_{ji,j}^{(0)} = \omega^2 2b\rho u_i^{(0)}, \quad -T_{\beta\alpha,\beta}^{(1)} + T_{2\alpha}^{(0)} = \omega^2 \tfrac{2}{3} b^3 \rho u_\alpha^{(1)} \quad \text{in } A, \\
&-D_{i,i}^{(0)} = 0, \quad -D_{\alpha,\alpha}^{(1)} + D_2^{(0)} = 0 \quad \text{in } A, \\
&-S_{ij}^{(0)} + \tfrac{1}{2}(u_{i,j}^{(0)} + u_{j,i}^{(0)} + \delta_{2i} u_j^{(1)} + \delta_{2j} u_i^{(1)}) = 0 \quad \text{in } A, \\
&-S_{\alpha\beta}^{(1)} + \tfrac{1}{2}(u_{\alpha,\beta}^{(1)} + u_{\beta,\alpha}^{(1)}) = 0 \quad \text{in } A, \\
&E_i^{(0)} + \phi_{,i}^{(0)} + \delta_{2i}\phi^{(1)} = 0, \quad E_\alpha^{(1)} + \phi_{,\alpha}^{(1)} = 0 \quad \text{in } A, \\
&-T_{ij}^{(0)} + \frac{\partial H}{\partial S_{ij}^{(0)}} = 0, \quad -T_{\alpha\beta}^{(1)} + \frac{\partial H}{\partial S_{\alpha\beta}^{(1)}} = 0 \quad \text{in } A,
\end{aligned}$$

$$
\begin{aligned}
&D_i^{(0)} + \frac{\partial H}{\partial E_i^{(0)}} = 0, \quad D_\alpha^{(1)} + \frac{\partial H}{\partial E_\alpha^{(1)}} = 0 \quad \text{in } A,\\
&-u_i^{(0)} = 0, \quad -u_\alpha^{(1)} = 0 \quad \text{on } C_u,\\
&n_j T_{ji}^{(0)} = 0, \quad n_\beta T_{\beta\alpha}^{(1)} = 0 \quad \text{on } C_T,\\
&-\phi^{(0)} = 0, \quad -\phi^{(1)} = 0 \quad \text{on } C_\phi,\\
&n_i D_i^{(0)} = 0, \quad n_\alpha D_\alpha^{(1)} = 0 \quad \text{on } C_D,
\end{aligned} \tag{40}
$$

where $u_i^{(0)}$ and $u_\alpha^{(1)}$ are zeroth and first order displacements, $S_{ij}^{(0)}$ and $S_{\alpha\beta}^{(1)}$ the zeroth and first order strains, $T_{ij}^{(0)}$ and $T_{\alpha\beta}^{(1)}$ the zeroth and first order stresses, $\phi^{(0)}$ and $\phi^{(1)}$ the zeroth and first order electric potentials, $E_i^{(0)}$ and $E_\alpha^{(1)}$ the zeroth and first order electric fields, $D_i^{(0)}$ and $D_\alpha^{(1)}$ the zeroth and first order electric displacements, $2b$ the thickness of the plate, ρ the mass density, and ω the resonant frequency. We note that indices i, j, k range from 1 to 3, Greek indices α, β assume values 1 and 3 only, and $(\,)_{,2} = 0$. $H = H(\mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)})$ is the electric enthalpy function. For a linear piezoelectric plate, H and the corresponding linear constitutive relations [9] are

$$
\begin{aligned}
H = {} & b(c_{ijkl}^{(0)} S_{ij}^{(0)} S_{kl}^{(0)} - \varepsilon_{ij} E_i^{(0)} E_j^{(0)} - 2e_{ijk}^{(0)} E_i^{(0)} S_{jk}^{(0)})\\
& + \tfrac{1}{3} b^3 (c_{\alpha\beta\gamma\delta}^{(1)} S_{\alpha\beta}^{(1)} S_{\gamma\delta}^{(1)} - \varepsilon_{\alpha\beta} E_\alpha^{(1)} E_\beta^{(1)} - 2e_{\alpha\beta\gamma}^{(1)} E_\alpha^{(1)} S_{\beta\gamma}^{(1)}),
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
&T_{ij}^{(0)} = 2b(c_{ijkl}^{(0)} S_{kl}^{(0)} - e_{kij}^{(0)} E_k^{(0)}), \quad T_{\alpha\beta}^{(1)} = \tfrac{2}{3} b^3 (c_{\alpha\beta\gamma\delta}^{(1)} S_{\gamma\delta}^{(1)} - e_{\gamma\alpha\beta}^{(1)} E_\gamma^{(1)}),\\
&D_i^{(0)} = 2b(\varepsilon_{ij} E_j^{(0)} + e_{ijk}^{(0)} S_{jk}^{(0)}), \quad D_\alpha^{(1)} = \tfrac{2}{3} b^3 (\varepsilon_{\alpha\beta} E_\beta^{(1)} + e_{\alpha\beta\gamma}^{(1)} S_{\beta\gamma}^{(1)}),
\end{aligned} \tag{42}
$$

where $c_{ijkl}^{(0)}$, $c_{\alpha\beta\gamma\delta}^{(1)}$, ε_{ij}, $e_{ijk}^{(0)}$ and $e_{\alpha\beta\gamma}^{(1)}$ are material properties. Given ρ, b, and H, values of ω^2 are sought corresponding to which nontrivial solutions $u_i^{(0)}$, $u_\alpha^{(1)}$, $S_{ij}^{(0)}$, $S_{\alpha\beta}^{(1)}$, $T_{ij}^{(0)}$, $T_{\alpha\beta}^{(1)}$, $\phi^{(0)}$, $\phi^{(1)}$, $E_i^{(0)}$, $E_\alpha^{(1)}$, $D_i^{(0)}$ and $D_\alpha^{(1)}$ exist. With (42), Eq. (40) can be written as a group of seven equations for the seven unknowns $u_i^{(0)}$, $u_\alpha^{(1)}$, $\phi^{(0)}$, and $\phi^{(1)}$, which are the counterpart of Eq. (2).

3.2. The abstract formulation

We introduce the following vector $\mathbf{U}$:

$$
\mathbf{U} = \xi\{u_i^{(0)}, u_\alpha^{(1)}, \phi^{(0)}, \phi^{(1)}, T_{ji}^{(0)}, T_{\beta\alpha}^{(1)}, D_i^{(0)}, D_\alpha^{(1)}, S_{ij}^{(0)}, S_{\beta\alpha}^{(1)}, E_i^{(0)}, E_\alpha^{(1)}\} \tag{43}
$$

and operators $\mathbf{A}$ and $\mathbf{B}$:

$$
\begin{aligned}
\mathbf{AU} = \Big\{ & -T_{ji,j}^{(0)}, \ -T_{\beta\alpha,\beta}^{(1)} + T_{2\alpha}^{(0)}, \ -D_{i,i}^{(0)}, \ -D_{\alpha,\alpha}^{(1)} + D_2^{(0)},\\
& -S_{ij}^{(0)} + \tfrac{1}{2}(u_{i,j}^{(0)} + u_{j,i}^{(0)} + \delta_{2i} u_j^{(1)} + \delta_{2j} u_i^{(1)}, \ -S_{\alpha\beta}^{(1)} + \tfrac{1}{2}(u_{\alpha,\beta}^{(1)} + u_{\beta,\alpha}^{(1)}),
\end{aligned}
$$

$$
\begin{aligned}
&E_i^{(0)} + \phi_{,i}^{(0)} + \delta_{2i}\phi^{(1)},\ E_\alpha^{(1)} + \phi_{,\alpha}^{(1)},\\
&-T_{ij}^{(0)} + \frac{\partial H}{\partial S_{ij}^{(0)}},\quad -T_{\alpha\beta}^{(1)} + \frac{\partial H}{\partial S_{\alpha\beta}^{(1)}},\\
&D_i^{(0)} + \frac{\partial H}{\partial E_i^{(0)}},\quad D_\alpha^{(1)} + \frac{\partial H}{\partial E_\alpha^{(1)}}\bigg\},
\end{aligned}
\tag{44}
$$

$$
\mathbf{BU} = \{2b\rho u_i^{(0)}, \tfrac{2}{3}b^3\rho u_\alpha^{(1)}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}.
$$

We also define

$$
\Xi(A) = \{\mathbf{U}\,|\,\text{boundary conditions in (40) are satisfied}\}. \tag{45}
$$

With the above definitions, the eigenvalue problem (40) can be stated as: Find $\lambda = \omega^2$ for which there exists a nontrivial $\mathbf{U} \in \Xi(\Omega)$ such that

$$
\mathbf{AU} = \lambda\mathbf{BU}. \tag{46}
$$

For two abstract vectors $\mathbf{U}$ and $\mathbf{V} \in \Xi(A)$, we define the inner product

$$
\begin{aligned}
\langle \mathbf{U}; \mathbf{V} \rangle = \int_A &(u_i^{(0)}v_i^{(0)} + u_\alpha^{(1)}v_\alpha^{(1)} + \phi^{(0)}\psi^{(0)} + \phi^{(1)}\psi^{(1)}\\
&+ T_{ji}^{(0)}\mathscr{T}_{ji}^{(0)} + T_{\beta\alpha}^{(1)}\mathscr{T}_{\beta\alpha}^{(1)} + D_i^{(0)}\mathscr{D}_i^{(0)} + D_\alpha^{(1)}\mathscr{D}_\alpha^{(1)}\\
&+ S_{ji}^{(0)}\mathscr{S}_{ji}^{(0)} + S_{\beta\alpha}^{(1)}\mathscr{S}_{\beta\alpha}^{(1)} + E_i^{(0)}\mathscr{E}_i^{(0)} + E_\alpha^{(1)}\mathscr{E}_\alpha^{(1)})\,\mathrm{d}A,
\end{aligned}
\tag{47}
$$

where $\mathbf{U}$ is given by (43) and

$$
\mathbf{V} = \{v_i^{(0)}, v_\alpha^{(1)}, \psi^{(0)}, \psi^{(1)}, \mathscr{T}_{ji}^{(0)}, \mathscr{T}_{\beta\alpha}^{(1)}, \mathscr{D}_i^{(0)}, \mathscr{D}_\alpha^{(1)}, \mathscr{S}_{ij}^{(0)}, \mathscr{S}_{\beta\alpha}^{(1)}, \mathscr{E}_i^{(0)}, \mathscr{E}_\alpha^{(1)}\}. \tag{48}
$$

3.3. Self-adjointness of the operators

For vectors $\mathbf{U}$ and $\mathbf{V} \in \Xi(A)$, we obtain the following after integration by parts and using the boundary conditions given in equations (40):

$$
\begin{aligned}
\langle \mathbf{AU}; \mathbf{V} \rangle = \int_A \bigg\{ &-T_{ji,j}^{(0)}v_i^{(0)} + (-T_{\beta\alpha,\beta}^{(1)} + T_{2\alpha}^{(0)})v_\alpha^{(1)}\\
&-D_{i,i}^{(0)}\psi^{(0)} + (-D_{\alpha,\alpha}^{(1)} + D_2^{(0)})\psi^{(1)}\\
&+ [-S_{ij}^{(0)} + \tfrac{1}{2}(u_{i,j}^{(0)} + u_{j,i}^{(0)} + \delta_{2j}u_i^{(1)} + \delta_{2i}u_j^{(1)})]\mathscr{T}_{ji}^{(0)}
\end{aligned}
$$

$$
\begin{aligned}
&+ [-S^{(1)}_{\alpha\beta} + \tfrac{1}{2}(u^{(1)}_{\alpha,\beta} + u^{(1)}_{\beta,\alpha})]\mathscr{T}^{(1)}_{\beta\alpha} \\
&+ (E^{(0)}_i + \phi^{(0)}_{,i} + \delta_{2i}\phi^{(1)})\mathscr{D}^{(0)}_i + (E^{(1)}_\alpha + \phi^{(1)}_{,\alpha})\mathscr{D}^{(1)}_\alpha \\
&+ \left(-T^{(0)}_{ij} + \frac{\partial H}{\partial S^{(0)}_{ij}}\right)\mathscr{S}^{(0)}_{ij} + \left(-T^{(1)}_{\alpha\beta} + \frac{\partial H}{\partial S^{(1)}_{\alpha\beta}}\right)\mathscr{S}^{(1)}_{\beta\alpha} \\
&+ \left(D^{(0)}_i + \frac{\partial H}{\partial E^{(0)}_i}\right)\mathscr{E}^{(0)}_i + \left(D^{(1)}_\alpha + \frac{\partial H}{\partial E^{(1)}_\alpha}\right)\mathscr{E}^{(1)}_\alpha\Bigg\}\,\mathrm{d}A \\
= &\int_A \Bigg\{-\mathscr{T}^{(0)}_{ji,j}u^{(0)}_i + (-\mathscr{T}^{(1)}_{\beta\alpha,\beta} + \mathscr{T}^{(0)}_{2\alpha})u^{(1)}_\alpha \\
&- \mathscr{D}^{(0)}_{i,i}\phi^{(0)} + (-\mathscr{D}^{(1)}_{\alpha,\alpha} + \mathscr{D}^{(0)}_2)\phi^{(1)} \\
&+ [-\mathscr{S}^{(0)}_{ij} + \tfrac{1}{2}(v^{(0)}_{i,j} + v^{(0)}_{j,i} + \delta_{2j}v^{(1)}_i + \delta_{2i}v^{(1)}_j)]T^{(0)}_{ji} \\
&+ [-\mathscr{S}^{(1)}_{\alpha\beta} + \tfrac{1}{2}(v^{(1)}_{\alpha,\beta} + v^{(1)}_{\beta,\alpha})]T^{(1)}_{\beta\alpha} \\
&+ (\mathscr{E}^{(0)}_i + \psi^{(0)}_{,i} + \delta_{2i}\psi^{(1)})D^{(0)}_i + (\mathscr{E}^{(1)}_\alpha + \psi^{(1)}_{,\alpha})D^{(1)}_\alpha \\
&+ \left(-\mathscr{T}^{(0)}_{ij} + \frac{\partial \mathscr{H}}{\partial \mathscr{S}^{(0)}_{ij}}\right)S^{(0)}_{ij} + \left(-\mathscr{T}^{(1)}_{\alpha\beta} + \frac{\partial \mathscr{H}}{\partial \mathscr{S}^{(1)}_{\alpha\beta}}\right)S^{(1)}_{\beta\alpha} \\
&+ \left(\mathscr{D}^{(0)}_i + \frac{\partial \mathscr{H}}{\partial \mathscr{E}^{(0)}_i}\right)E^{(0)}_i + \left(\mathscr{D}^{(1)}_\alpha + \frac{\partial \mathscr{H}}{\partial \mathscr{E}^{(1)}_\alpha}\right)E^{(1)}_\alpha\Bigg\}\,\mathrm{d}A \\
= &\langle \mathbf{U}; \mathbf{AV}\rangle,
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
\langle \mathbf{BU}; \mathbf{V}\rangle &= \int_A (2b\rho u^{(0)}_i v^{(0)}_i + \tfrac{2}{3}b^3\rho u^{(1)}_\alpha v^{(1)}_\alpha)\,\mathrm{d}A \\
&= \langle \mathbf{U}; \mathbf{BV}\rangle,
\end{aligned}
\tag{50}
$$

where

$$
\begin{aligned}
\mathscr{H} = {} & b(c^{(0)}_{ijkl}\mathscr{S}^{(0)}_{ij}\mathscr{S}^{(0)}_{kl} - \varepsilon_{ij}\mathscr{E}^{(0)}_i\mathscr{E}^{(0)}_j - 2e^{(0)}_{ijk}\mathscr{E}^{(0)}_i\mathscr{S}^{(0)}_{jk}) \\
& + \tfrac{1}{3}b^3(c^{(1)}_{\alpha\beta\gamma\delta}\mathscr{S}^{(1)}_{\alpha\beta}\mathscr{S}^{(1)}_{\gamma\delta} - \varepsilon_{\alpha\beta}\mathscr{E}^{(1)}_\alpha\mathscr{E}^{(1)}_\beta - 2e^{(1)}_{\alpha\beta\gamma}\mathscr{E}^{(1)}_\alpha\mathscr{S}^{(1)}_{\beta\gamma}).
\end{aligned}
\tag{51}
$$

Relations (49) and (50) show that operators **A** and **B** are self-adjoint. It is important in deriving the two-dimensional plate theory from the three-dimensional theory that the self-adjointness be kept. With the self-adjointness, we can prove the reality of the eigenvalues in the same way as was done in Section 2.4. We assume below that the eigenvectors have also been chosen as real.

3.4. Orthogonality of eigenvectors

Following the steps outlined in Section 2.6, we can prove that

$$\langle \mathbf{B}\mathbf{U}^{(m)}; \mathbf{U}^{(n)} \rangle = 0 \tag{52}$$

or

$$\int_A (2b\rho u_i^{(0)(m)} u_i^{(0)(n)} + \tfrac{2}{3} b^3 \rho u_\alpha^{(1)(m)} u_\alpha^{(1)(n)})\, \mathrm{d}A = 0. \tag{53}$$

Hence eigenvectors $\mathbf{U}^{(m)}$ and $\mathbf{U}^{(n)}$ associated with distinct eigenvalues are mutually orthogonal. Relation (53) was proved in [10] "after much tedious manipulation". Similarly, the other orthogonality condition

$$\langle \mathbf{A}\mathbf{U}^{(m)}; \mathbf{U}^{(n)} \rangle = 0 \tag{54}$$

yields

$$\begin{aligned}
\int_A \Big\{ & - T_{ji,j}^{(0)(m)} u_i^{(0)(n)} + (-T_{\beta\alpha,\beta}^{(1)(m)} + T_{2\alpha}^{(0)(m)}) u_\alpha^{(1)(n)} \\
& - D_{i,i}^{(0)(m)} \phi^{(0)(n)} + (-D_{\alpha,\alpha}^{(1)(m)} + D_2^{(0)(m)}) \phi^{(1)(n)} \\
& + [-S_{ij}^{(0)(m)} + \tfrac{1}{2}(u_{i,j}^{(0)(m)} + u_{j,i}^{(0)(m)} + \delta_{2j} u_i^{(1)(m)} + \delta_{2i} u_j^{(1)(m)})] T_{ji}^{(0)(n)} \\
& + [-S_{\alpha\beta}^{(1)(m)} + \tfrac{1}{2}(u_{\alpha,\beta}^{(1)(m)} + u_{\beta,\alpha}^{(1)(m)})] T_{\beta\alpha}^{(1)(n)} \\
& + (E_i^{(0)(m)} + \phi_{,i}^{(0)(m)} + \delta_{2i} \phi^{(1)(m)}) D_i^{(0)(n)} + (E_\alpha^{(1)(m)} + \phi_{,\alpha}^{(1)(m)}) D_\alpha^{(1)(n)} \\
& + \left(-T_{ij}^{(0)(m)} + \frac{\partial H^{(m)}}{\partial S_{ij}^{(0)(m)}} \right) S_{ij}^{(0)(n)} + \left(-T_{\alpha\beta}^{(1)(m)} + \frac{\partial H^{(m)}}{\partial S_{\alpha\beta}^{(1)(m)}} \right) S_{\beta\alpha}^{(1)(n)} \\
& + \left(D_i^{(0)(m)} + \frac{\partial H^{(m)}}{\partial E_i^{(0)(m)}} \right) E_i^{(0)(n)} + \left(D_\alpha^{(1)(m)} + \frac{\partial H^{(m)}}{\partial E_\alpha^{(1)(m)}} \right) E_\alpha^{(1)(n)} \Big\}\, \mathrm{d}A = 0,
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
H^{(m)} = {} & b(c_{ijkl}^{(0)} S_{ij}^{(0)(m)} S_{kl}^{(0)(m)} - \varepsilon_{ij} E_i^{(0)(m)} E_j^{(0)(m)} - 2 e_{ijk}^{(0)} E_i^{(0)(m)} S_{jk}^{(0)(m)}) \\
& + \tfrac{1}{3} b^3 (c_{\alpha\beta\gamma\delta}^{(1)} S_{\alpha\beta}^{(1)(m)} S_{\gamma\delta}^{(1)(m)} - \varepsilon_{\alpha\beta} E_\alpha^{(1)(m)} E_\beta^{(1)(m)} - 2 e_{\alpha\beta\gamma}^{(1)} E_\alpha^{(1)(m)} S_{\beta\gamma}^{(1)(m)}).
\end{aligned} \tag{56}$$

We note that (55) can be simplified to

$$\int_A [-T_{ji,j}^{(0)(m)} u_i^{(0)(n)} + (-T_{\beta\alpha,\beta}^{(1)(m)} + T_{2\alpha}^{(0)(m)}) u_\alpha^{(1)(n)}]\, \mathrm{d}A = 0, \tag{57}$$

which is equivalent to (53).

3.5. Variational formulation

The Rayleigh quotient for the eigenvalue problem (46) is

$$\Pi(\mathbf{U}) = \frac{\langle \mathbf{AU}; \mathbf{U} \rangle}{\langle \mathbf{BU}; \mathbf{U} \rangle}$$

$$= \left\{ \iint_A [T_{ji}^{(0)} u_{i,j}^{(0)} + T_{\beta\alpha}^{(1)} u_{\alpha,\beta}^{(1)} + T_{2\alpha}^{(0)} u_\alpha^{(1)} + D_i^{(0)} \phi_{,i}^{(0)} + D_\alpha^{(1)} \phi_{,\alpha}^{(1)} + D_2^{(0)} \phi^{(1)}\right.$$

$$+ H(\mathbf{S}^{(0)}, \mathbf{S}^{(1)}, \mathbf{E}^{(0)}, \mathbf{E}^{(1)}) + E_i^{(0)} D_i^{(0)} + E_\alpha^{(1)} D_\alpha^{(1)}$$

$$\left. - T_{ij}^{(0)} S_{ij}^{(0)} - T_{\alpha\beta}^{(1)} S_{\alpha\beta}^{(1)}] \,\mathrm{d}S \right\} \Big/ \left\{ \iint_A \frac{1}{2} (2b\rho u_i^{(0)} u_i^{(0)} + \tfrac{2}{3} b^3 \rho u_\alpha^{(1)} u_\alpha^{(1)}) \,\mathrm{d}S \right\}, \tag{58}$$

where some of the terms have been integrated by parts. We note that the variational principle is of mixed type with all of the field variables taken as independent variables, and similar variational principles were given in [11].

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