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Saint-Venant's Principle for a Helical Piezoelectric Body

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Abstract. For a helical body made of a linear piezoelectric material and loaded at one end only by a system of self-equilibrated forces, it is shown that the *strain energy* stored in the part of the body distant at least s from the loaded end decreases exponentially with the distance s.

1. Introduction

Toupin [8] gave the following mathematical formulation of Saint-Venant's principle in 1965. For a linear elastic homogeneous prismatic body of arbitrary length and cross-section loaded on one end only by an arbitrary system of self-equilibrated forces, the elastic energy $U(s)$ stored in the part of the body which is beyond a distance s from the loaded end satisfies the inequality

$$U(s) \leqslant U(0)\exp[-(s-l)/s_c(l)]. \tag{1}$$

The characteristic length $s_c(l)$ depends upon the maximum and minimum elastic moduli of the material and the smallest nonzero characteristic frequency of free vibration of a slice of the prismatic body of axial length l. By using an estimate due to Ericksen [4] for the norm of the stress tensor in terms of the strain-energy density, one can show that $s_c(l)$ depends on the maximum elastic modulus only. Inequalities similar to (1) have been obtained by, amongst others, Berglund [3], Batra [1] and Batra and Yang [2]. Developments of Saint-Venant's principle in various settings of linear elasticity have been summarized by Horgan and Knowles [5] and by Horgan [6].

As pointed out by Toupin [8], the decay rate in Saint-Venant's principle depends strongly on the shape of the body. Here we consider a helical piezoelectric body with arbitrary but constant cross section. Following Batra and Yang [2], we modify the standard governing equations for linear piezoelectricity. We also propose a variational principle for the new form of governing equations. We describe deformations of the helical body in suitable curvilinear coordinates so as to keep our analysis close to that of Toupin.

2. Equations for Linear Piezoelectricity

Let the finite spatial region occupied by a piezoelectric body be B, the boundary of B be ∂B, the unit outward normal be $\mathbf{n}$ and ∂B be partitioned as $\partial B_{\mathrm{u}} \cup \partial B_{\mathrm{t}} = \partial B_{\mathrm{E}} \cup \partial B_{\mathrm{D}} = \partial B$ and $\partial B_{\mathrm{u}} \cap \partial B_{\mathrm{t}} = \partial B_{\mathrm{E}} \cap \partial B_{\mathrm{D}} = \phi$, where ∂B_{u} and ∂B_{t} are parts of ∂B on which mechanical displacements and surface tractions are prescribed respectively. ∂B_{E} is the part of ∂B which is in contact with a metal or an electrode, hence the tangential electric field vanishes on it. ∂B_{D} is the part of ∂B on which surface electric charge is prescribed. The surface electric charge is usually zero for dielectrics. The governing equations and boundary conditions for static linear piezoelectricity [7] are

$$
\begin{aligned}
&\nabla \cdot \mathbf{T} = 0, \\
&\nabla \cdot \mathbf{D} = 0, \qquad \nabla \times \mathbf{E} = \mathbf{0}, \\
&\mathbf{T} = \frac{\partial H}{\partial \mathbf{s}} = \mathbf{Cs} - \mathbf{eE}, \\
&\mathbf{D} = -\frac{\partial H}{\partial \mathbf{E}} = \mathbf{es} + \chi\mathbf{E}, \\
&\mathbf{s} = \tfrac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T), \quad \text{in } B,
\end{aligned}
\tag{2}
$$

and the boundary conditions are

$$
\begin{aligned}
&\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \partial B_{\mathrm{u}}, \qquad \mathbf{Tn} = \bar{\mathbf{f}} \quad \text{on} \quad \partial B_{\mathrm{t}}, \\
&\mathbf{n} \times \mathbf{E} = \mathbf{0} \quad \text{on} \quad \partial B_{\mathrm{E}}, \qquad \mathbf{D} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial B_{\mathrm{D}},
\end{aligned}
\tag{3}
$$

where $\mathbf{T}$ is the stress tensor, $\mathbf{s}$ the infinitesimal strain tensor, $\mathbf{u}$ is the mechanical displacement vector; $\mathbf{E}$ is the electric field vector, $\mathbf{D}$ is the electric displacement vector; $\bar{\mathbf{u}}$ and $\bar{\mathbf{f}}$ are prescribed boundary displacements and tractions respectively. $H(\mathbf{s}, \mathbf{E})$ is the electric enthalpy function given by

$$
H = \tfrac{1}{2}\mathbf{s} \cdot \mathbf{Cs} - \tfrac{1}{2}\mathbf{E} \cdot \chi\mathbf{E} - \mathbf{E} \cdot \mathbf{eS},
$$

where $\mathbf{C}$ is the fourth-order elasticity tensor, χ the second-order electric permittivity tensor, $\mathbf{e}$ is the third-order piezoelectric tensor. These material constants exhibit the following symmetries: $C_{ijkl} = C_{jikl} = C_{klij}$, $e_{ijk} = e_{ikj}$, $\chi_{ij} = \chi_{ji}$, and furthermore $\mathbf{C}$ and χ are positive definite. It is easy to verify that $H(\mathbf{s}, \mathbf{E})$ is indefinite. We use the internal energy density

$$
W = W(\mathbf{s}, \mathbf{D}) = \tfrac{1}{2}\mathbf{T} \cdot \mathbf{s} + \tfrac{1}{2}\mathbf{D} \cdot \mathbf{E} = H + \mathbf{E} \cdot \mathbf{D},
$$

and note that W is positive definite.

Following Batra and Yang [2], we introduce an electric displacement potential vector ψ and an anti-symmetric electric displacement (second-order) tensor $\mathcal{D}$ as follows:

$$\mathcal{D} = -\tfrac{1}{2}(\nabla\psi - (\nabla\psi)^T),$$
$$\mathbf{D} = \tfrac{1}{2}\nabla \times \psi.$$

Corresponding to $\mathcal{D}$, we also introduce an equivalent second-order anti- symmetric tensor

$$\mathcal{E} = \epsilon\mathbf{E},$$

where ϵ is the third-order alternating tensor.

Taking $\mathbf{s}$ and $\mathcal{D}$ as variables for W, and $\mathbf{u}$ and ψ as basic variables, then $\nabla\cdot\mathbf{D} = 0$ is satisfied identically, and the governing equations can be written as

$$\begin{aligned} &W = W(\mathbf{s}, \mathcal{D}),\\ &\nabla\cdot\mathbf{T} = 0, \qquad \nabla\cdot\mathcal{E} = 0,\\ &\mathbf{T} = \frac{\partial W}{\partial \mathbf{s}}, \qquad \mathcal{E} = \frac{\partial W}{\partial \mathcal{D}}, \end{aligned} \tag{4}$$

and boundary conditions $(3)_{3,4}$ are replaced by

$$\mathcal{E}\mathbf{n} = 0 \quad \text{on} \quad \partial B_{\mathbf{E}}, \qquad \psi = 0 \quad \text{on} \quad \partial B_{\mathbf{D}}. \tag{5}$$

Since the internal energy density W is a positive definite quadratic function of 9 variables, s_{ij} and $\mathcal{D}_{ij}$, we denote the ordered pair $(s_{ij}, \mathcal{D}_{ij})$ by $\mathbf{\Gamma}$. Thus we have

$$W = \tfrac{1}{2}\mathbf{\Gamma}\cdot\mathcal{C}\mathbf{\Gamma}, \tag{6}$$

where $\mathcal{C}$ is a linear transformation from a 9-dimensional linear space into a 9-dimensional linear space, and because of the positive definiteness of W,

$$\frac{\partial W}{\partial \mathbf{\Gamma}}\cdot\frac{\partial W}{\partial \mathbf{\Gamma}} = \mathcal{C}\mathbf{\Gamma}\cdot\mathcal{C}\mathbf{\Gamma} = \mathbf{\Gamma}\cdot\mathcal{C}^2\mathbf{\Gamma} \leqslant 2\alpha_M W, \tag{7}$$

where α_M is the largest eigenvalue of $\mathcal{C}$.

Let

$$\pi = \int_B W(\mathbf{s}, \mathcal{D})\,\mathrm{d}v - \int_{\partial B_{\mathbf{t}}} \mathbf{t}\cdot\mathbf{u}\,\mathrm{d}a + \int_{\partial B_{\mathbf{D}}} \psi\cdot\phi\,\mathrm{d}a, \tag{8}$$

where $\phi = \mathcal{E}\mathbf{n}$. It is easy to check that $\delta\pi = 0$ is equivalent to equations (4), $(3)_{1,2}$ and (5). We note that the commonly used potential is of the form

$$\tilde{\Gamma} = \int_B W(\mathbf{s}, \mathbf{E})\,\mathrm{d}v - \int_{\partial B_\mathrm{t}} \mathbf{t} \cdot \mathbf{u}\,\mathrm{d}a + \int_{\partial B_\mathbf{D}} \phi \mathbf{D} \cdot \mathbf{n}\,\mathrm{d}a, \tag{9}$$

where ϕ is defined by $\mathbf{D} = -\nabla\phi$.

3. Formulation of the Problem

Consider a linear piezoelectric body B which is a clockwise helix of arbitrary but constant cross section in the unstressed state. Introduce a fixed rectangular Cartesian coordinate system $\mathbf{X}$ with the X^3-axis coinciding with the axis of the helix, the plane $X^3 = 0$ containing one end cross-section, and $X^3 \geqslant 0$ for points in B. The base vectors for $\mathbf{X}$-coordinates are $\mathbf{e}_i$ $(i = 1, 2, 3)$. Following Batra [1], we introduce the curvilinear coordinates $\mathbf{Y}$ by the transformation

$$\begin{pmatrix} Y^1 \\ Y^2 \\ Y^3 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}, \tag{10}$$

where $\theta = bX^3 = bY^3$, b is the angle of twist of the helix. In index notation,

$$X^i = R^i_j Y^j. \tag{11}$$

Through transformation (10), we transform the helix in the $\mathbf{X}$-coordinates into a straight prismatic body in the $\mathbf{Y}$-coordinates.

The covariant base vectors of the $\mathbf{Y}$-coordinates are given by

$$\mathbf{g}_i = \frac{\partial \mathbf{X}}{\partial Y^i} = \frac{\partial X^k}{\partial Y^i}\mathbf{e}_k,$$

i.e.

$$\begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ -bY^1\sin\theta + bY^2\cos\theta & -bY^1\cos\theta - bY^2\sin\theta & 1 \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \tag{12}$$

The metric tensor $\mathbf{G}$, defined by $G_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$, has the form

$$[G_{ij}] = \begin{bmatrix} 1 & 0 & bY^2 \\ 0 & 1 & -bY^1 \\ bY^2 & -bY^1 & 1 + b^2((Y^1)^2 + (Y^2)^2) \end{bmatrix}, \tag{13}$$

and the non-zero components of the Christoffel symbol are

$$\begin{aligned} \Gamma^1_{23} = \Gamma^1_{32} = b, \qquad \Gamma^1_{33} = -b^2 Y^1, \\ \Gamma^2_{13} = \Gamma^2_{31} = -b, \qquad \Gamma^2_{33} = -b^2 Y^2. \end{aligned} \tag{14}$$

We now write governing equations and boundary conditions in the **Y**-coordinates. For simplicity we derive the governing equations by the variational principle (8). The key issue is to relate components of $\mathbf{s}, \mathbf{u}, \mathcal{D}$ and ψ in the two coordinate systems. In order to simplify subsequent algebraic manipulations, we denote the components of **u** with respect to **X** coordinates by $\hat{u}^i$ and introduce the ordered triplet u^i by

$$\hat{u}^i = R^i_j u^j.$$

Thus

$$\begin{aligned} \mathbf{u} &= \hat{u}^i \mathbf{e}_i \\ &= (u^1 - bY^2 u^3)\mathbf{g}_1 + (u^2 + bY^1 u^3)\mathbf{g}_2 + u^3 \mathbf{g}_3 \\ &\equiv \tilde{v}^1 \mathbf{g}_1 + \tilde{v}^2 \mathbf{g}_2 + \tilde{v}^3 \mathbf{g}_3. \end{aligned} \tag{15}$$

So u^3 is the component of **u** in the $\mathbf{g}_3$ direction in the **Y** coordinates, but u^1 and u^2 are not components of **u** in the $\mathbf{g}_1$ and $\mathbf{g}_2$ directions.

Recall the definition of the gradient in curvilinear coordinates (e.g. see Malvern [9])

$$\begin{aligned} \mathbf{s} &= \tfrac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \\ &= \frac{1}{2}\left(G_{li}\frac{\partial \tilde{v}^l}{\partial Y^j} + G_{lj}\frac{\partial \tilde{v}^l}{\partial Y^i} + (G_{li}\Gamma^l_{jk} + G_{lj}\Gamma^l_{ik})\tilde{v}^k \right) \mathbf{g}^i \mathbf{g}^j \\ &= S_{ij}\mathbf{g}^i \mathbf{g}^j. \end{aligned} \tag{16}$$

Written explicitly,

$$
\begin{aligned}
S_{11} &= \frac{\partial u^1}{\partial Y^1}, \qquad S_{12} = \frac{1}{2}\left(\frac{\partial u^1}{\partial Y^2} + \frac{\partial u^2}{\partial Y^1}\right), \qquad S_{22} = \frac{\partial u^2}{\partial Y^2},\\
S_{13} &= \frac{1}{2}\left(\frac{\partial u^3}{\partial Y^1} + \frac{\partial u^1}{\partial Y^3} + bY^2\frac{\partial u^1}{\partial Y^1} - bY^1\frac{\partial u^2}{\partial Y^1} + bu^2\right),\\
S_{23} &= \frac{1}{2}\left(\frac{\partial u^2}{\partial Y^3} + \frac{\partial u^3}{\partial Y^2} + bY^2\frac{\partial u^1}{\partial Y^2} - bY^1\frac{\partial u^2}{\partial Y^2} - bu^1\right),\\
S_{33} &= \frac{\partial u^3}{\partial Y^3} + bY^2\frac{\partial u^1}{\partial Y^3} - bY^1\frac{\partial u^2}{\partial Y^3} + b^2(Y^1u^1 + Y^2u^2).
\end{aligned}
\tag{17}
$$

Similarly, if we define $\psi = \hat{\psi}^i \mathbf{e}_i$ and $\hat{\psi}^i = R^i_j \psi^j$, we have

$$
\begin{aligned}
\mathcal{D}_{11} &= \mathcal{D}_{22} = \mathcal{D}_{33} = 0,\\
\mathcal{D}_{12} &= -\mathcal{D}_{21} = -\frac{1}{2}\left(\frac{\partial \psi^1}{\partial Y^2} - \frac{\partial \psi^2}{\partial Y^1}\right),\\
\mathcal{D}_{13} &= -\mathcal{D}_{31} = -\frac{1}{2}\left(\frac{\partial \psi^1}{\partial Y^3} - \frac{\partial \psi^3}{\partial Y^1} + bY^1\frac{\partial \psi^2}{\partial Y^1} - bY^2\frac{\partial \psi^1}{\partial Y^1} + b\psi^2\right),\\
\mathcal{D}_{23} &= -\mathcal{D}_{32} = -\frac{1}{2}\left(\frac{\partial \psi^2}{\partial Y^3} - \frac{\partial \psi^3}{\partial Y^2} + bY^1\frac{\partial \psi^2}{\partial Y^2} - bY^2\frac{\partial \psi^1}{\partial Y^2} - b\psi^1\right).
\end{aligned}
\tag{18}
$$

Thus

$$
W = W(\mathbf{S}, \mathcal{D}) = \hat{W}(u^i_{,j}, \psi^i_{,j}, u^i, \psi^i),
$$

where a comma followed by the index j implies partial differentiation with respect to Y^j. The potential function (8) in the **Y**-coordinates can be written as

$$
\pi = \int_B \hat{W}\,\mathrm{d}v - \int_{\partial B_{\mathbf{t}}} t_i u^i\,\mathrm{d}a + \int_{\partial B_{\mathbf{D}}} \psi^i \phi_i\,\mathrm{d}a, \tag{19}
$$

where $\mathrm{d}v = \mathrm{d}Y^1\,\mathrm{d}Y^2\,\mathrm{d}Y^3, \mathrm{d}a = \mathrm{d}U^1\,\mathrm{d}U^2, Y^i = Y^i(U^1, U^2)$ being the parametric representation of the lateral surface of the helical body. $\delta\pi = 0$ gives

$$
\left(\frac{\partial \hat{W}}{\partial u^i_{,j}}\right)_{,j} - \frac{\partial \hat{W}}{\partial u^i} = 0, \qquad \left(\frac{\partial \hat{W}}{\partial \psi^i_{,j}}\right)_{,j} - \frac{\partial \hat{W}}{\partial \psi^i} = 0, \tag{20}
$$

and

$$
\frac{\partial \hat{W}}{\partial u^i_{,j}} n_j = f_i, \qquad \frac{\partial \hat{W}}{\partial \psi^i_{,j}} n_j = \phi_i. \tag{21}
$$

For the problem under consideration, the loads are applied at $Y^3 = 0$, and the remainder of the boundary is free of mechanical and electrical loads. Let

$$C_s = \{\mathbf{Y}\colon \mathbf{Y} \in B, Y^3 = X^3 = s\}, \tag{22}$$

then we require that

$$\int_{C_0} \frac{\partial \hat{W}}{\partial u^i_{,3}} \,\mathrm{d}Y^1\,\mathrm{d}Y^2 = \int_{C_0} f_i \,\mathrm{d}Y^1\,\mathrm{d}Y^2 = 0,$$

$$\int_{C_0} \delta^{kl}\epsilon_{ijk} Y^j f_l \,\mathrm{d}Y^1\,\mathrm{d}Y^2 = 0. \tag{23}$$

Here moments are taken with respect to the origin and C_0 is defined by (22).

Before we state our theorem, we note that for a rigid body motion $\boldsymbol{\omega}$ and for any given $\mathbf{u}$, if

$$\mathbf{v} = \mathbf{u} + \boldsymbol{\omega}, \tag{24a}$$

then

$$S_{ij}(\mathbf{v}) = S_{ij}(\mathbf{u}), \qquad \hat{W}(v^i_{,j}, \psi^i_{,j}, v^i, \psi^i) = \hat{W}(u^i_{,j}, \psi^i_{,j}, u^i, \psi^i). \tag{24b}$$

With the definitions,

$$a \equiv b \sup_{\mathbf{Y}\in C_0} (|Y^1|, |Y^2|),$$

$$C_{s,l} \equiv \{\mathbf{Y}\colon \mathbf{Y} \in B, s \leqslant Y^3 \leqslant s + l\}, \tag{25}$$

$$U(s) = \int_{Y^3 \geqslant s} W \,\mathrm{d}v,$$

we have the following

THEOREM. *If for an unstressed helical body made of a linear piezoelectric material, the loads applied at* $Y^3 = 0$ *satisfy* (23) *and*

$$\frac{\partial \hat{W}}{\partial u^i_{,j}} n_j = 0 \quad on \quad \partial B - C_0,$$

$$\frac{\partial \hat{W}}{\partial \psi^i_{,j}} n_j = 0 \quad on \quad \partial B,$$

then

$$U(s) \leqslant U(0) \exp(-(s - l)/s_c(l)), \tag{26}$$

where

$$s_c(l) = \sqrt{\frac{\alpha_M}{\lambda_0(l)}}(1+2a).$$

Here $\lambda_0(l)$ is the smallest non-zero characteristic value of free vibration of a slice of the helical body of axial length l, with the mass density per unit volume equal to one, and with the inertia density associated with ψ equal to one.

Proof. Since $\hat{W}$ is a homogeneous quadratic form of $u^i_{,j}$, $\psi^i_{,j}$, u^i, ψ^i, by Euler's theorem,

$$\begin{aligned} U(s) &= \int_{Y^3 \geqslant s} \hat{W}\,\mathrm{d}v \\ &= \frac{1}{2}\int_{Y^3 \geqslant s}\left(\frac{\partial \hat{W}}{\partial u^i_{,j}}u^i_{,j} + \frac{\partial \hat{W}}{\partial \psi^i_{,j}}\psi^i_{,j} + \frac{\partial \hat{W}}{\partial u^i}u^i + \frac{\partial \hat{W}}{\partial \psi^i}\psi^i\right)\mathrm{d}v \\ &= \frac{1}{2}\int_{C_s}\left(\frac{\partial \hat{W}}{\partial u^i_{,j}}u^i n_j + \frac{\partial \hat{W}}{\partial \psi^i_{,j}}\psi^i n_j\right)\mathrm{d}a \\ &= -\frac{1}{2}\int_{C_s}\left(\frac{\partial \hat{W}}{\partial u^i_{,3}}u^i + \frac{\partial \hat{W}}{\partial \psi^i_{,3}}\psi^i\right)\mathrm{d}a. \end{aligned} \tag{27}$$

To obtain (27), we used the divergence theorem and the equilibrium equations (20).

Because of (24), we can write (27) as

$$U(s) = -\frac{1}{2}\int_{C_s}\left(\frac{\partial \hat{W}}{\partial v^i_{,3}}v^i + \frac{\partial \hat{W}}{\partial \psi^i_{,3}}\psi^i\right)\mathrm{d}a. \tag{28}$$

Using expressions (17) and (18) for S_{ij} and $\mathcal{D}_{ij}$, we have

$$\begin{aligned} \frac{\partial \hat{W}}{\partial v^1_{,3}} &= \frac{1}{2}\frac{\partial W}{\partial S_{m3}}\delta_{1m} + \frac{\partial W}{\partial S_{33}}bY^2, \\ \frac{\partial \hat{W}}{\partial v^2_{,3}} &= \frac{1}{2}\frac{\partial W}{\partial S_{m3}}\delta_{2m} - \frac{\partial W}{\partial S_{33}}bY^1, \\ \frac{\partial \hat{W}}{\partial v^3_{,3}} &= \frac{\partial W}{\partial S_{33}}, \end{aligned} \tag{29}$$

and similar expressions for $(\partial\hat{W}/\partial\psi^i_{,3})$ can be obtained. Substituting (29) and similar expressions for $(\partial\hat{W}/\partial\psi^i_{,3})$ into (28), using the Schwarz and geometric-arithmetic mean inequalities (see [8]), i.e.

$$2\int_v fh\,\mathrm{d}v \leqslant \gamma\int_v f^2\,\mathrm{d}v + \frac{1}{\gamma}\int_v h^2\,\mathrm{d}v \tag{30}$$

for every $\gamma > 0$, we have

$$\begin{aligned} U(s) \leqslant\ & \frac{\Gamma}{4}\delta_{ij}\left[\gamma\int_{C_s}\left(\frac{\partial W}{\partial S_{i3}}\frac{\partial W}{\partial S_{j3}} + \frac{\partial W}{\partial \mathcal{D}_{i3}}\frac{\partial W}{\partial \mathcal{D}_{j3}}\right)\mathrm{d}a \right.\\ & \left. + \frac{1}{\gamma}\int_{C_s}(v^i v^j + \psi^i\psi^j)\,\mathrm{d}a\right] \\ \leqslant\ & \frac{\Gamma}{4}\left[\gamma\int_{C_s}\delta_{ij}\delta_{rs}\left(\frac{\partial W}{\partial S_{ir}}\frac{\partial W}{\partial S_{js}} + \frac{\partial W}{\partial \mathcal{D}_{ir}}\frac{\partial W}{\partial \mathcal{D}_{js}}\right)\mathrm{d}a \right.\\ & \left. + \frac{1}{\gamma}\int_{C_s}\delta_{ij}(v^i v^j + \psi^i\psi^j)\,\mathrm{d}a\right] \\ \leqslant\ & \frac{\Gamma}{4}\left[2\alpha_M\gamma\int_{C_s}W\,\mathrm{d}a + \frac{1}{\gamma}\int_{C_s}\delta_{ij}(v^i v^j + \psi^i\psi^j)\mathrm{d}a\right], \end{aligned} \tag{31}$$

where $\Gamma = (1 + 2a)$ and we have also used (7). Integrate both sides of (31) with respect to Y^3 from $Y^3 = s$ to $Y^3 = s + l$ for some $l > 0$ and set

$$\frac{1}{l}\int_s^{s+l} U(t)\,\mathrm{d}t = Q(s,l). \tag{32}$$

The result is

$$Q(s,l) \leqslant \frac{\Gamma}{4l}\left[2\gamma\alpha_M\int_{C_{s,l}} W\,\mathrm{d}v + \frac{\delta_{ij}}{\gamma}\int_{C_{s,l}}(v^i v^j + \psi^i\psi^j)\,\mathrm{d}v\right]. \tag{33}$$

To bound the second integral on the right-hand side of (33) by an integral of W, we consider the free vibration problem of the helical spring and define the following eigenvalue problem:

$$\begin{aligned} &\left(\frac{\partial\hat{W}}{\partial u^i_{,j}}\right)_{,j} - \frac{\partial\hat{W}}{\partial u^i} + \lambda u^i = 0, \\ &\left(\frac{\partial\hat{W}}{\partial \psi^i_{,j}}\right)_{,j} - \frac{\partial\hat{W}}{\partial \psi^i} + \lambda \psi^i = 0. \end{aligned} \tag{34}$$

Taking the inner product of $(34)_1$ with u_i, of $(34)_2$ with ψ_i, integrating the resulting equations over $C_{s,l}$ and adding the respective sides gives

$$\lambda(l) = \frac{2\int_{C_{s,l}} \hat{W}\,\mathrm{d}v}{\delta_{ij}\int_{C_{s,l}} (u^i u^j + \psi^i \psi^j)\,\mathrm{d}v}. \tag{35}$$

$\lambda(l) = 0$ if and only if $\hat{W} = W = 0$, and $W(\mathrm{s}, \mathcal{D}) = 0$ only when s, $\mathcal{D}$ both are equal to zero. We exclude this physically uninteresting case by requiring non-trivial u_j and ψ_j to be orthogonal to the eigenvector $(\boldsymbol{\omega}, \boldsymbol{\phi})$ corresponding to the zero eigenvalue:

$$\delta^{ij}\int_{C_{s,l}} (u_i\omega_j + \psi_i\phi_j)\,\mathrm{d}v = 0, \qquad \delta^{ij}\int_{C_{s,l}} (u_i u_j + \psi_i\psi_j)\,\mathrm{d}v \neq 0.$$

Thus the lowest eigenvalue $\lambda_0(l)$ will be positive and will satisfy

$$\lambda_0(l) \leqslant \frac{2\int_{C_{s,l}} W\,\mathrm{d}v}{\delta_{ij}\int_{C_{s,l}} (v^i v^j + \psi^i \psi^j)\,\mathrm{d}v}, \tag{36}$$

so from (33) and (36) we have

$$Q(s,l) \leqslant \frac{\Gamma}{2l}\left(\alpha_M\gamma + \frac{1}{\gamma\lambda_0(l)}\right)\int_{C_{s,l}} W\,\mathrm{d}v \tag{37}$$

for all $\gamma > 0$. We choose $\gamma = 1/(\alpha_M\lambda_0)^{1/2}$ so that $s_c(l) = \frac{\Gamma}{2}\left(\alpha_M\gamma + \frac{1}{\gamma\lambda_0(l)}\right)$ takes the minimum value $s_c(l) = \Gamma\left(\frac{\alpha_M}{\lambda_0(l)}\right)^{1/2}$. Thus

$$Q(s,l) \leqslant \frac{s_c(l)}{l}\int_{C_{s,l}} W\,\mathrm{d}v. \tag{38}$$

Differentiation of (32) with respect to s yields

$$\frac{\mathrm{d}Q}{\mathrm{d}s} = \frac{1}{l}[U(s+l) - U(s)] = -\frac{1}{l}\int_{C_{s,l}} W\,\mathrm{d}v,$$

which when combined with (38) gives

$$Q + s_c(l)\frac{\mathrm{d}Q}{\mathrm{d}s} \leqslant 0, \tag{39}$$

where we have used $W \geqslant 0$. Integrating (39) and using the property that $U(s)$ is nonincreasing, $U(s+l) \leqslant Q(s,l) \leqslant U(s)$, we obtain

$$\frac{U(s_2 + l)}{U(s_1)} \leqslant \exp[-(s_2 - s_1)/s_c(l)]. \tag{40}$$

The choice $s_1 = 0$, $s_2 = s - l$ in (40) yields the desired inequality (26).

4. Remarks

For $b = 0$ the helical body becomes straight and our result reduces to that of Batra and Yang [2]. The helical nature of the body affects the decay rate through the appearance of a (cf. (25)) in the decay rate. Also the smallest non-zero frequency $\lambda_0(l)$ of free vibration will be affected by the helix angle and the cross-section of the body. As pointed out by Batra and Yang [2], it is difficult to delineate the effect of the electric field on the decay rate. Since the presence of electric fields increases W, it may lower $\lambda_0(l)$. However, its effect on the maximum eigenvalue of the linear symmetric transformation C (cf. (6)) is unclear unless one considers a specific material. Thus it is hard to quantify how the helix angle and the piezoelectricity affect the decay rate.

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