

Material tailoring in finite torsional deformations of axially graded Mooney–Rivlin circular cylinder

Mathematics and Mechanics of Solids
2015, Vol. 20(2) 183–189
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DOI: 10.1177/1081286514544260
mms.sagepub.com



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Received 9 September 2013; accepted 21 October 2013

Abstract

We study direct and material tailoring (or inverse) problems for finite torsional deformations of a solid circular cylinder made of a Mooney–Rivlin material with the two elastic moduli continuously varying in the axial direction and deformed by applying equal and opposite torques on its two end faces. We note that a Mooney–Rivlin material is isotropic and incompressible. For the direct problem, we derive the condition that the two material moduli must satisfy in order for the problem to have a solution. In particular, it is shown that finite torsional deformations can be produced if one modulus equals the negative one-half of the other one. For the material tailoring problem, we find the axial variation of the two material moduli to produce finite torsional deformations with prescribed axial variation of the angle of twist per unit length.

Keywords

Inverse problems, inhomogeneous materials, nonlinear elasticity, Poynting effect

1. Introduction

In [1], Ericksen described five families of finite deformations that can be produced by applying surface tractions to a body made of a homogeneous and isotropic hyperelastic material. Pure torsional deformations that preserve the cylinder length and radius, keep plane sections plane, and simply rotate one cross-section with respect to its neighbor are members of this family of deformations. Here we consider an inhomogeneous circular cylinder made of a Mooney–Rivlin material with the two material parameters varying only in the axial direction and derive conditions on the two material moduli for the cylinder to admit pure torsional deformations under the action of surface tractions alone. In particular, this condition is satisfied when one material modulus equals the negative one-half the other modulus, as defined later in the paper. We also analyze the material tailoring (or the inverse) problem of finding the axial variation of the two material moduli to produce torsional deformations with the prescribed variation along the cylinder axis of the angle of twist per unit length. The inverse problem is usually more challenging than the direct problem and may not have a solution. Bodies comprised of

I am pleased to dedicate this work to Professor KR Rajagopal in honor of his contributions to continuum mechanics.

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materials with moduli varying continuously in one or more directions are usually called functionally graded (FG).

Mechanics problems for inhomogeneous elastic bodies have been studied by several authors; here we cite only two, arbitrarily chosen, earlier works [2, 3]. Rooney and Ferrari [4] have found lower and upper bounds on the elastic moduli of a homogeneous isotropic elastic cylinder in terms of those of an inhomogeneous cylinder for torsional, bending, and flexural deformations, and have cited several earlier works that have studied boundary-value problems for inhomogeneous linear elastic bodies. Dryden and Batra [5] have found lower and upper bounds on elastic moduli of a homogeneous hollow cylinder in terms of those of an inhomogeneous hollow cylinder so that the two cylinders have equal energies of deformation under identical pressures applied on their inner and outer surfaces. Saravanan and Rajagopal [6] have shown, amongst others, that values of elastic moduli found using the equivalent energy principle can lead to erroneous results in the following sense. Surface tractions required to produce the same average deformation in the inhomogeneous body and the equivalent homogeneous body can significantly differ from each other. Of course, stress concentrations present in either particulate reinforced or fiber reinforced or porous inhomogeneous body are usually absent in the equivalent homogeneous body. The problem studied below for an FG cylinder does not require homogenization of material properties.

Here we extend our earlier work on torsional deformations of a solid cylinder composed of an incompressible linear elastic material with the material moduli continuously varying in the axial direction. It was shown in [7] that for an isotropic and incompressible Hookean material with the shear modulus varying in the axial direction, the angle of twist per unit length of a cross-section is inversely proportional to the value of the shear modulus in that cross-section. We prove here that for a cylinder made of an inhomogeneous Mooney–Rivlin material with the two material parameters smoothly varying in the axial direction, pure torsional deformations can occur only if the two elastic parameters are related to each other. In particular, the torsional deformation can occur if the elastic parameter multiplying $\mathbf{B}^{-1}$ equals negative one-half of that multiplying the left Cauchy–Green tensor $\mathbf{B}$. If the cylinder is made of a neo-Hookean material with the elastic modulus continuously varying in the axial direction, pure torsional deformations are not possible.

Axial variation of material moduli in a cylinder of rubberlike material can be achieved by continuously varying either the vulcanizing agent or volume fractions of reinforcements. For polymers the degree of cure that affects values of material moduli can be varied by exposing the polymer to ultraviolet light for different time durations. Watanabe et al. [8] have analyzed the centrifugal method to fabricate ceramic/metal FG materials (FGMs). The book [9] by Schwartz describes 14 different ways to manufacture FGMs.

It is challenging to experimentally find the axial variation of material properties. For a cylindrical body deformed quasi-statically in either uniaxial tension or compression one can find local strains on the surface of the cylinder by using either laser or optical (e.g., diffraction grating and polarized light [10]) strain gages. Since the average axial stress on each cross-section is the same, the variation of the axial strain is an indicator of the change in the axial modulus, and the change in cross-section area of the variation in Poisson's effect. Of course, it is not easy to determine the spatial variation of the moduli if material properties vary continuously in all three directions. One may be able to use dispersion of waves to characterize the spatial distribution of material properties.

Infinitesimal torsional deformations of an inhomogeneous and anisotropic circular cylinder have recently been studied by Escedi [13]. Horgan and Chan [14] and Horgan [15] have also studied torsion of inhomogeneous linear elastic cylinders. Material tailoring problems for cylinders and spheres made of Mooney–Rivlin materials have been analyzed by Batra [16], and those of linear elastic structures by Nie et al. [17–21].

2. Problem formulation

We use cylindrical coordinates with the origin at the centroidal axis and the z -axis aligned along the cylinder axis to study deformations of the solid cylinder induced by equal and opposite twisting moments applied at the two end faces; e.g., see Figure 1.

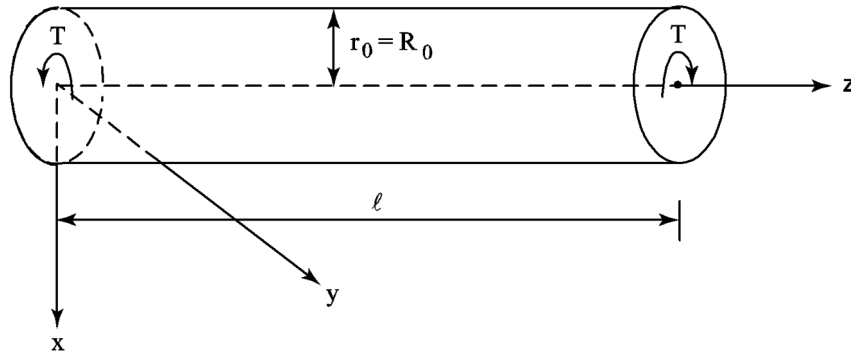


Figure 1. Schematic sketch of the problem studied.

Let (r, θ, z) denote the position of a point in the deformed (or the present) configuration of a material point that occupies the place (R, Θ, Z) in the unstressed reference configuration. In the absence of body forces, equations governing deformations of the cylinder are [10]

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0, \\ \frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} &= 0, \\ \frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} &= 0. \end{aligned} \quad (1)$$

The requirement that the resultant axial force and the resultant tangential force vanish on a cross-section and hence on the two end faces $z=0$ and ℓ can be expressed as

$$\int_0^{r_0} \int_0^{2\pi} \sigma_{zz} r dr d\theta = \int_0^{r_0} \int_0^{2\pi} \sigma_{zr} r dr d\theta = \int_0^{r_0} \int_0^{2\pi} \sigma_{z\theta} r dr d\theta = 0, \quad (2a)$$

and similarly, the equal and opposite torque, T , applied at the end faces of the cylinder results in boundary conditions listed in

$$\int_0^{r_0} \int_0^{2\pi} \sigma_{z\theta} r^2 dr d\theta = \pm T \text{ on } z=0, \ell. \quad (2b)$$

For the mantle of the cylinder to be traction free, the following equation must hold:

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0 \text{ on } r=r_0, \quad (2c)$$

where $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor, r_0 the outer radius of the undeformed and the deformed cylinder, and ℓ its initial and final length.

The constitutive relation for an isotropic, incompressible, and inhomogeneous Mooney–Rivlin material is [11]

$$\boldsymbol{\sigma} = -p\mathbf{1} + c_1(z)\mathbf{B} + c_2(z)\mathbf{B}^{-1}, \quad (3)$$

where $\mathbf{1}$ is the identity tensor, and the hydrostatic pressure, p , cannot be determined from the cylinder deformation. Material parameters c_1 and c_2 are assumed to continuously depend upon the axial coordinate z . Empirical inequalities [11] require that $c_1 > 0$ and $c_2 \leq 0$. These inequalities imply that a

principal axis of stress is also a principal axis of strain in an isotropic nonlinear elastic material [12]. However, there is no theoretical basis for these inequalities.

3. Problem solution

The problem defined by equations (1)–(3) is nonlinear and in general cannot be analytically solved for a cylinder made of an inhomogeneous material. We use one of Ericksen's [1] families of solutions, and seek its solution by a semi-inverse method. That is, we assume a solution of the form

$$r = R, \theta = \Theta + Q(z), z = Z, \quad (4)$$

and find the function $Q(z)$ so that equations (1) to (3) are satisfied. If one or more of the boundary conditions listed in equation (2) are not satisfied, we find the surface tractions needed to produce the deformation given by equation (4) in a cylinder made of an inhomogeneous Mooney–Rivlin material.

For the deformation described by equation (4), physical components of the deformation gradient, $\mathbf{F}$, are given by

$$[F] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & rQ' \\ 0 & 0 & 1 \end{bmatrix} \quad (5)$$

where $Q' = dQ/dz$. Thus

$$[B] = [F][F]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + r^2 Q'^2 & rQ' \\ 0 & rQ' & 1 \end{bmatrix}, \quad (6a)$$

$$[B]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -rQ' \\ 0 & -rQ' & 1 + r^2 Q'^2 \end{bmatrix}. \quad (6b)$$

We note that $\det[F] = 1$; thus the assumed deformation given by equation (4) is isochoric and is admissible in a body comprised of an incompressible material.

Substitution from equations (6a) and (6b) into equation (3) gives

$$\sigma_{rr} = -p + (c_1 + c_2), \quad (7a)$$

$$\sigma_{\theta\theta} = -p + (c_1 + c_2) + c_1 r^2 Q'^2, \quad (7b)$$

$$\sigma_{zz} = -p + (c_1 + c_2) + c_2 r^2 Q'^2, \quad (7c)$$

$$\sigma_{r\theta} = \sigma_{rz} = 0, \sigma_{\theta z} = (c_1 - c_2) r Q'. \quad (7d-f)$$

It follows from equation (7) that $\sigma_{rr} \neq \sigma_{\theta\theta} \neq \sigma_{zz}$ and the *universal relation* $\sigma_{\theta\theta} - \sigma_{zz} = \sigma_{\theta z}(rQ')$. Furthermore for $\sigma_{\theta z}$ to be non-zero, $c_1(z)$ must not equal $c_2(z)$. For infinitesimal deformations, the shear modulus equals $2(c_1 - c_2)$. Thus for the shear modulus to be positive, $c_1 > c_2$. With stresses given by equation (7), equilibrium equation (1) require that

$$-\frac{\partial p}{\partial r} - r(c_1 Q'^2) = 0, \quad (8a)$$

$$-\frac{1}{r} \frac{\partial p}{\partial \theta} + r((c_1 - c_2) Q')' = 0, \quad (8b)$$

$$-\frac{\partial p}{\partial z} + (c_1 + c_2)' + r^2 (c_2 Q'^2)' = 0. \quad (8c)$$

For p to be a single-valued function of θ , equation (8b) implies that p must be independent of θ . Thus, $\frac{\partial p}{\partial \theta} = 0$, and

$$((c_1 - c_2)Q')' = 0. \quad (9)$$

In order for p to be a continuous function of r and z , p must satisfy

$$\frac{\partial^2 p}{\partial z \partial r} = \frac{\partial^2 p}{\partial r \partial z}. \quad (10)$$

Thus

$$((c_1 + 2c_2)Q^2)' = 0. \quad (11)$$

Equations (9) and (11) imply that

$$\begin{aligned} (c_1 - c_2)Q' &= \alpha, \\ (c_1 + 2c_2)Q^2 &= \beta, \end{aligned} \quad (12a,b)$$

where α and β are constants. These equations imply that the shear stress, $\sigma_{\theta z}$, is independent of the axial coordinate, z .

Recalling that the hydrostatic pressure p is independent of θ , integrating equation (8c) we get

$$p = c_1 + c_2 + r^2 c_2 Q'^2 + f(r), \quad (13)$$

which when substituted into equation (8a) results in

$$f(r) = \beta(r_0^2 - r^2)/2, \quad (14)$$

where we have used the boundary condition $\sigma_{rr}|_{r=r_0} = 0$. Thus stresses found from equations (7) and (14) are given by

$$\begin{aligned} \sigma_{rr} &= c_1 Q'^2 (r^2 - r_0^2)/2, \\ \sigma_{\theta\theta} &= c_1 Q'^2 (3r^2 - r_0^2)/2, \\ \sigma_{zz} &= c_1 Q'^2 (r^2 - r_0^2)/2 + c_2 Q'^2 r^2, \\ \sigma_{\theta z} &= (c_1 - c_2)rQ', \sigma_{r\theta} = \sigma_{rz} = 0. \end{aligned} \quad (15a-f)$$

Equations (15d) and (2b) imply that the torque T required to deform the cylinder is given by

$$T = Q'(\ell)(c_1(\ell) - c_2(\ell))J = Q'(0)(c_1(0) - c_2(0))J = \alpha J, \quad (16)$$

where J equals the polar moment of inertia. We note that boundary conditions (2a) at the end faces of the circular cylinder require that the axial force

$$N = -\frac{\pi r_0^4}{4}(c_1(\ell) - 2c_2(\ell))Q'(\ell)^2 = -\frac{\pi r_0^4}{4}(c_1(0) - 2c_2(0))Q'(0)^2, \quad (17)$$

be applied at the end faces, $z = 0, \ell$. Furthermore, material parameters $c_1(z)$ and $c_2(z)$ and the twist/length $Q'(z)$ must satisfy equation (12).

We thus note that end surfaces will have null tractions if

$$(c_1(\ell) - 2c_2(\ell))Q'(\ell)^2 = (c_1(0) - 2c_2(0))Q'(0)^2 = 0, \quad (18)$$

or equivalently $c_1(\ell) = 2c_2(\ell)$ and $c_1(0) = 2c_2(0)$. When $c_1(\ell) \neq 2c_2(\ell)$ and $c_1(0) \neq 2c_2(0)$ and there is no axial force N applied, the cylinder length will change, as was pointed out by Poynting.

We discuss below various scenarios for satisfying equation (12).

4. Discussion of results

For a *linear elastic (or Hookean)* material, terms involving Q'^2 are neglected; thus equation (12b) gives $\beta = 0$, and equation (12a) gives $(c_1 - c_2)Q' = \text{constant}$, a result derived in [1] with $[2(c_1 - c_2)]$ interpreted as the shear modulus. Thus the angle of twist/unit length, Q' , is inversely proportional to $(c_1 - c_2)$. In the solution of the direct (material tailoring) problem, $(c_1 - c_2)$ is a given function of z and we solve for $Q'((c_1 - c_2))$ as a function of z . For this case no axial force needs to be applied at the end faces.

For a *neo-Hookean* material, $c_2 = 0$. For the direct (material tailoring) problem, $c_1(z)$ ($Q'(z)$) is prescribed. Then we have two equations, i.e. equations (12a) and (12b), to determine either $Q'(z)$ or $c_1(z)$. These two equations have a solution only when Q' and c_1 are constants or equivalently the cylinder material is homogeneous and the angle of twist/length is a constant. This type of deformation has been studied in many books, e.g., see [11, 22, 33]. The circular cylinder remains circular in the absence of surface tractions on its mantle.

The situation for a *Mooney–Rivlin* material is more interesting. For the direct problem, $c_1(z)$ and $c_2(z)$ are prescribed, and we have equations (12a) and (12b) for the determination of $Q'(z)$. These two equations (or equivalently equations (9) and (11)) have a solution if and only if

$$2(c_1 + 2c_2)(c'_1 - c'_2) = (c_1 - c_2)(c'_1 + 2c'_2) \quad (19)$$

Equation (15f) implies that $c_1(z)$ must not equal $c_2(z)$. Assuming that equation (19) holds, then it follows from equation (12a) that $Q'(z)$ is inversely proportional to $(c_1(z) - c_2(z))$, similar to that for a Hookean material. However, for the Mooney–Rivlin material, the axial force given by equation (17) needs to be applied to keep the cylinder length constant.

One way to satisfy equation (19) is to set

$$c_2(z) = -c_1(z)/2. \quad (20)$$

Of course, there may be other solutions to equation (19). We note that equation (20) requires that $c_1(z)$ and $c_2(z)$ be of opposite signs, which is consistent with the empirical inequalities. When equation (20) holds, $N = -\frac{\pi r_0^4}{2}(c_1(\ell))Q'(\ell)^2 = -\frac{\pi r_0^4}{2}(c_1(0))Q'(0)^2 \neq 0$.

For the material tailoring problem, Q' is prescribed. Material parameters $c_1(z)$ and $c_2(z)$ found by solving equations (12a) and (12b) are given by

$$c_1 = (2\alpha/Q' + \beta/Q'^2)/3, c_2 = (-\alpha/Q' + \beta/Q'^2)/3. \quad (21)$$

Constants α and β are determined from values of $c_1(0)$ and $c_2(0)$ or equivalently from values of the torque T , the axial force N , $Q'(0)$, and the polar moment of inertia J .

5. Conclusions

We have studied finite torsional deformations of a circular cylinder made of a Mooney–Rivlin material with the two material moduli c_1 and c_2 varying in the axial direction z . On the assumption that the shear modulus, $2(c_1(z) - c_2(z))$, is positive everywhere, we have derived a condition that the moduli must satisfy for the torsional deformation to be producible. In particular, if $c_2(z) = -c_1(z)/2$, then a pure torsional deformation can be produced in an inhomogeneous circular cylinder and the angle of twist/length is inversely proportional to $c_1(z)$. As for the torsion of a homogeneous Mooney–Rivlin cylinder, a cylinder made of an inhomogeneous Mooney–Rivlin material exhibits the Poynting effect. For the material tailoring problem, we have found values of the two material moduli in terms of the angle of twist/length, the torque and the axial forces applied at the end faces, and the polar moment of inertia.

Acknowledgements

This work was carried out at Virginia Polytechnic Institute and State University. Views expressed in the paper are those of the author and neither those of the funding agency nor of the author's institution.

Funding

This work was supported by the Office of Naval Research (grant number N00014-11-1-0594) with Dr YDS Rajapakse as the Program Manager.

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