A Thermomechanical Theory for a Porous Anisotropic Elastic Solid with Inclusions

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This work is dedicated to Jerald L. Ericksen on the occasion of his 60^{th} birthday

Abstract

We construct a mixture theory which describes a porous elastic anisotropic solid with inclusions. Thermal effects are taken into account. The theory is in accord with classical thermodynamics. Fully nonlinear isotropic and anisotropic materials are considered, and field equations are also given for a nontrivial special case which, though nonlinear, is controlled by a few material functions. When properly specialized, the theory reduces to the P-α model, a model widely used to describe porous solids.

1. Introduction

A number of applications arise where a porous solid body with inclusions is subjected to large stresses and the influence of heat transfer, both by conduction and by radiation. Often such a body is anisotropic either because of its method of manufacture, or in the case of geological materials, because of its method of formation. Though *a priori*, empirical, and semi-empirical models and partial models for such materials are commonly used in engineering calculations, no three-dimensional model exists which takes all of these phenomena into account, and also has been shown to be in accord with all of the accepted principles for formulating constitutive equations. Here we formulate such a model.

The basic principles we use are straightforward applications of techniques of rational mechanics, most developed many years ago, but some settled only quite recently. Since the problem we consider is intrinsically complicated, it is no surprise that the equations are complicated also. However, we do attempt to formulate the model with the greatest economy of formalism consistent with our goals. Our approach is as follows. Since we are dealing with a material composed of more than one constituent, we use a mixture theory. The constituent of primary interest is a porous hyperelastic solid, which we allow to undergo finite deformations. The other constituent is a dissipative material which we envision as filling part of the pore space in the elastic material. The rest of the pore space is treated as vacuous, though we allow for pore crushing through a dissipative mechanism.

One matter needs special emphasis. Though the mechanics we use is quite widely accepted, the fact that we deal with thermal effects means that some sort of thermodynamics will be useful. Exactly what form "the" theory of thermodynamics takes for a single material is currently the subject of some debate. The corresponding question for mixtures is even more problematical. One approach which is in favor because it has had very many successful applications is that of COLEMAN & NOLL [1963]. We use a generalization of that approach here. If, as some individuals claim, its psychology is unacceptable and its results are "obvious" when the mixture is "sufficiently close" to equilibrium and meaningless otherwise, it is appropriate to apply to this work whatever statements or incantations of that sort one applies to other works. Realistically but on a very tentative basis, indications are that the standard restrictions derived from thermodynamics, of whatever origin, may force some systems to be somewhat more strongly stable than they are in nature. If a conclusion of that type is correct, thermodynamics should eliminate most phenomena which are physically unacceptable as well as, conceivably, some which could be of great interest. This may be good in that, since the results of this work are intended to be useful in engineering applications, such excess stability should force systems away from bifurcations or similar phenomena. These, in turn, could be the cause of considerable inconvenience or even a possible source of error in some digital computations.

In our effort to make this work compact and expository, we have laid aside the custom of interrupting the presentation with numerous references and footnotes. It is principally a rational extension of well-established theory to an interesting physical case. We choose to cite our major sources in the list of references, but not mention to all of them in the body of the paper. If we feel something needs to be said about a particular reference, we say it in braces { } along with the bibliographical information. We do note that the formal literature on mixtures has been pervaded, almost since its beginning, by a number of works which are substantially incorrect. We cite none of these. Since we make no attempt here to give a "complete" set of references, it is also true that we leave uncited many works which are entirely free of error.

2. Kinematics and Equations of Balance and Conservation

Consider a solid material containing both pores and inclusions. Conceptually, there are two obvious ways to treat such a material. First, we might choose some reference state and delineate in that state the exact position and configuration of each pore and inclusion. The ultimate purpose of solving a boundary-value problem would then be to describe the evolution of the solid as well as the voids and inclusions in time and space. Such an approach is impractical not only because it usually leads to impossibly complex boundary-value problems but also, even assuming the initial state of the body were known precisely, because it yields as its outcome far more information than can be used in most engineering situations. Second, we could "smear out" the pores and inclusions and replace them with appropriate variables which vary smoothly in space and time. A useful model of this type would, of course, require that the corresponding physical entities, in some sense, also vary smoothly. Moreover, the amount of physical detail it could handle would depend strongly on the complexity built into the theory. This is the type of theory commonly used to model composites. We choose to use a model of this type, in fact a very simple model of this type, here.*

Henceforth, we often call the solid material the *matrix* and, when using index notation, assign it the index 1. We assign the inclusions the index 2, and use the terms *pore* and *void* interchangeably. At each material point $\boldsymbol{X}$ and time t, we let ϕ_1 be the portion of space occupied by the matrix, and ϕ_2 be the portion of space occupied by the inclusions. We will often call these quantities *volume fractions*. The total volume fraction ϕ is given by $\phi = \phi_1 + \phi_2$, and the material is called *saturated* if $\phi = 1$ and *unsaturated* if $\phi < 1$. The case $\phi > 1$ is of no physical interest. Each of the materials, of course, will have an intrinsic mass density γ_a ($a = 1, 2$), which is the mass of that material per unit volume *of that material*, and a mass density ϱ_a, which is the mass of that material per unit *total* volume. It then follows that $\varrho_a \leqq \gamma_a$ and

$$\varrho_a = \gamma_a \phi_a. \tag{2.1}$$

The mass density ϱ of the material is given by $\varrho = \varrho_1 + \varrho_2$. We abbreviate that equation by

$$\varrho = \Sigma \varrho_a. \tag{2.2}$$

We often sum quantities over both constituents of a mixture. When we do, we always use notation of the type set forth in (2.2).

The physical situation which is of primary interest here is that in which neither material diffuses through the other, so that each has the same deformation gradient $\boldsymbol{F}(\boldsymbol{X}, t)$ and velocity $\boldsymbol{v}(\boldsymbol{X}, t)$. We assume the deformation function $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ invertible so we may also write the deformation gradient and velocity as functions of the spatial co-ordinate $\boldsymbol{x}$ and time. We allow each material to have its own temperature θ_a. We expect to have balance laws for mass, momentum, moment of momentum, energy and entropy as in any continuum theory. In addition, the presence of volume fractions implies that there should be some equations which give information about them. Here we use an approach derived from that of GOODMAN & COWIN [1972]. We treat both the inclusions and the pores as *centers of dilatation* in much the same sense as does LOVE [1927], but we allow for local inertia associated with opening or closing of inclusions or pores. Thus, there are local equations of balance, called *equations of equilibrated force*, which describe these mechanisms, but which do not affect the equations of linear momentum for the body. Interactions among the effects arise, of course, in the equations of balance of energy.

To ensure that a mixture theory is formulated in an entirely self-consistent fashion, a very substantial axiomatic structure, much of which is explained by

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* Most of the material in this chapter duplicates, in terms of the referential description, the article of PASSMAN, NUNZIATO, & WALSH [1984].

TRUESDELL [1957, 1969] and by PASSMAN, NUNZIATO, & WALSH [1984], must be satisfied. Though we write out only those aspects which we use specifically, all of the equations we use here agree fully with that structure. For example, we postulate principles of balance for a material body. Then, if the appropriate fields are sufficiently smooth, we may derive their local forms. They are

$$\bar{\varrho}_a = \varrho_a \det \boldsymbol{F}, \qquad (2.3)_1 \ \textit{Mass}$$

$$\boldsymbol{m}_a^+ = \bar{\varrho}_a \dot{\boldsymbol{v}} - \operatorname{DIV} \boldsymbol{S}_a - \bar{\varrho}_a \boldsymbol{b}_a, \qquad (2.3)_2 \ \textit{Momentum}$$

$$\boldsymbol{M}_a^+ = \boldsymbol{F}\boldsymbol{S}_a^T - \boldsymbol{S}_a \boldsymbol{F}^T, \qquad (2.3)_3 \ \textit{Moment of Momentum}$$

$$g_a^+ = \bar{\varrho}_a (k_a \dot{\phi}_a)^{\cdot} - \operatorname{DIV} \boldsymbol{h}_a - \bar{\varrho}_a (l_a + f_a), \qquad (2.3)_4 \ \textit{Equilibrated Force}$$

$$\begin{aligned} e_a^+ = {} & \bar{\varrho}_a \dot{e}_a + \boldsymbol{m}_a^+ \cdot \boldsymbol{v} - \boldsymbol{S}_a \cdot \dot{\boldsymbol{F}}^T + \operatorname{DIV} \boldsymbol{q}_a - \bar{\varrho}_a r_a \\ & + \bar{\varrho}_a \dot{\phi}_a \left(f_a + \frac{g_a^+}{\varrho_a} - \tfrac{1}{2} \dot{k}_a \dot{\phi}_a \right) \\ & - \boldsymbol{h}_a \cdot \operatorname{GRAD} \dot{\phi}_a. \end{aligned} \qquad (2.3)_5 \ \textit{Energy}$$

Here $\bar{\varrho}_a$ is the mass density in the reference configuration, $\boldsymbol{S}_a$ is the first Piola-Kirchhoff stress tensor, DIV denotes divergence with respect to $\boldsymbol{X}$, *etc.* We assume there are no interchanges of mass among constituents. The quantities $\boldsymbol{m}_a^+, \boldsymbol{M}_a^+, \ldots$, allow for interchange of momentum, moment of momentum, ..., among constituents. We allow momentum, moment of momentum, ..., to change form, but do not allow the total mixture to produce these quantities. This requires

$$\Sigma \boldsymbol{m}_a^+ = 0, \qquad (2.4)_1$$

$$\Sigma \boldsymbol{M}_a^+ = 0, \qquad (2.4)_2$$

$$\Sigma g_a^+ = 0, \qquad (2.4)_3$$

$$\Sigma e_a^+ = 0. \qquad (2.4)_4$$

These equations are statements that, if one looks at the body as a whole, one cannot tell it is a mixture in the sense that it satisfies the same *balance equations* (but not necessarily constitutive equations) as a single body. To see this, define

$$\varrho = \Sigma \varrho_a, \quad \bar{\varrho} = \Sigma \bar{\varrho}_a, \qquad (2.5)_1$$

$$\boldsymbol{S} = \Sigma \boldsymbol{S}_a, \qquad (2.5)_2$$

$$\bar{\varrho} \boldsymbol{b} = \Sigma \bar{\varrho}_a \boldsymbol{b}_a, \qquad (2.5)_3$$

$$\bar{\varrho}(l + f) = \Sigma \bar{\varrho}_a (l_a + f_a), \qquad (2.5)_4$$

$$\bar{\varrho}\left(e + \frac{k\dot{\phi}^2}{2}\right) = \Sigma \bar{\varrho}_a \left(e_a + \frac{k_a \dot{\phi}_a^2}{2}\right), \qquad (2.5)_5$$

$$\boldsymbol{q} - \boldsymbol{h}\dot{\phi} = \Sigma(\boldsymbol{q}_a - \boldsymbol{h}_a \dot{\phi}_a), \qquad (2.5)_6$$

$$\boldsymbol{h} = \Sigma \boldsymbol{h}_a, \qquad (2.5)_7$$

$$\bar{\varrho}(r + l\dot{\phi}) = \Sigma \bar{\varrho}_a (r_a + l_a \dot{\phi}_a). \qquad (2.5)_8$$

Most of these definitions are motivated either by elementary measure-theoretic considerations, or by classical kinetic theory. It then follows that

$$\bar{\varrho} = \varrho \det \boldsymbol{F}, \tag{2.6}$$

$$\bar{\varrho}\dot{\boldsymbol{v}} = \text{DIV}\, \boldsymbol{S} + \bar{\varrho}\boldsymbol{b}, \tag{2.7}$$

$\boldsymbol{F}\boldsymbol{S}^T$ is symmetric (and thus the Cauchy stress $\boldsymbol{T} = \boldsymbol{S}\boldsymbol{F}^T/\det \boldsymbol{F}$ and the second Piola-Kirchhoff stress tensor $\tilde{\boldsymbol{T}} = \boldsymbol{F}^{-1}\boldsymbol{S}$ are symmetric), and we have

$$\bar{\varrho}(k\dot{\phi})^{\cdot} = \text{DIV}\, \boldsymbol{h} + \bar{\varrho}(l + f), \tag{2.8}$$

$$\bar{\varrho}\dot{e} = \boldsymbol{S} \cdot \dot{\boldsymbol{F}}^T - \text{DIV}\, \boldsymbol{q} + \bar{\varrho}r + \bar{\varrho}\frac{\dot{k}\dot{\phi}^2}{2} + \boldsymbol{h} \cdot \text{GRAD}\, \dot{\phi} - \bar{\varrho}f\dot{\phi}. \tag{2.9}$$

These are the accepted balance laws for a single porous material. Under moderate additional conditions, the saturation constraint $\phi = 1$ forces the equation of equilibrated force to be satisfied identically, and the energy equation to reduce to

$$\bar{\varrho}\dot{e} = \boldsymbol{S} \cdot \dot{\boldsymbol{F}}^T - \text{DIV}\, \boldsymbol{q} + \bar{\varrho}r, \tag{2.10}$$

the accepted form for a single nonporous continuum. We do not impose the saturation constraint in this work.

The preceding presentation of balance principles is quite standard. Though the method of exposition is open to considerable adjustment, the final results do not leave cause for substantive debate. It is useful to have one final type of balance law—one for entropy. The form of that principle allows room for discussion, and the current literature contains much of it. Here we choose a treatment deriving from that of COLEMAN & NOLL and of BOWEN [1967]. PASSMAN, NUNZIATO, & WALSH [1984] have discussed this particular treatment in some detail.

Consider a body[*] $\mathscr{B}_a$ of the a^{th} constituent. No material of the a^{th} constituent may cross the boundary $\partial\mathscr{B}_a$ of $\mathscr{B}_a$, which itself may move in time. Material of other constituents may cross the boundary of $\mathscr{B}_a$. Let the entropy equation for the a^{th} constituent be

$$\int_{\mathscr{B}_a} \eta_a^+ \, dV = \frac{d}{dt} \int_{\mathscr{B}_a} \bar{\varrho}_a \eta_a \, dV + \int_{\partial\mathscr{B}_a} \frac{\boldsymbol{q}_a}{\theta_a} \cdot \bar{\boldsymbol{n}} \, dA - \int_{\mathscr{B}_a} \frac{\bar{\varrho}_a r_a}{\theta_a} dV. \tag{2.11}$$

Here η_a is the entropy per unit mass of $\mathscr{B}_a$, and η_a^+ is the entropy exchanged between $\mathscr{B}_a$ and other bodies which, since we are dealing with a mixture, may occupy the same region of space as $\mathscr{B}_a$ at the same time. The local form of this equation is

$$\eta_a^+ = \bar{\varrho}_a \dot{\eta}_a + \text{DIV}\left(\frac{\boldsymbol{q}_a}{\theta_a}\right) - \frac{\bar{\varrho}_a r_a}{\theta_a}. \tag{2.12}$$

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[*] In order to maintain maximum clarity in this exposition, we set forth the balance principle for entropy in somewhat more detail than we have done for the other balance principles.

Define the Helmholtz free energy ψ_a by

$$\psi_a = e_a - \theta_a \eta_a. \tag{2.13}$$

Then the entropy equation for each constituent is

$$\eta_a^+ = \frac{\bar{\varrho}_a}{\theta_a}(\dot{e}_a - \dot{\theta}_a \eta_a - \dot{\psi}_a) + \mathrm{DIV}\left(\frac{\boldsymbol{q}_a}{\theta_a}\right) - \frac{\bar{\varrho}_a r_a}{\theta_a}. \tag{2.14}$$

We postulate as the entropy inequality

$$\Sigma \eta_a^+ \theta_a \geqq 0, \tag{2.15}$$

and obtain

$$\Sigma\left[\bar{\varrho}_a\{-\dot{\theta}_a \eta_a - \dot{\psi}_a\} + \left\{\boldsymbol{S}_a \cdot \dot{\boldsymbol{F}}^T - \bar{\varrho}_a \dot{\phi}_a\left(f_a + \frac{g_a^+}{\varrho_a} - \tfrac{1}{2} k_a \dot{\phi}_a\right)\right.\right.$$
$$\left.\left. + \boldsymbol{h}_a \cdot \mathrm{GRAD}\, \dot{\phi}_a\right\} - \frac{\boldsymbol{q}_a \cdot \mathrm{GRAD}\, \theta_a}{\theta_a}\right] \geqq 0, \tag{2.16}$$

as our reduced entropy inequality. This completes the theory of balance equations.

3. Constitutive Equations—General Considerations

3.1. Formulation of Constitutive Equations. Our constitutive equations model a porous elastic body with inclusions. They are somewhat similar to those of Nunziato & Cowin [1979] for a single porous elastic body. However, they do include the concept that we have a mixture. In addition, we treat anisotropy in some detail.

Let

$$\mathscr{S}_a = \{\boldsymbol{F}, \phi_a, \dot{\phi}_a, \mathrm{GRAD}\, \phi_a, \theta_a, \mathrm{GRAD}\, \theta_a\}. \tag{3.1}$$

We postulate as constitutive equations*

$$\psi_a = \hat{\psi}_a(\mathscr{S}_a), \tag{3.2}$$

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\mathscr{S}_a), \tag{3.3}$$

$$\eta_a = \hat{\eta}_a(\mathscr{S}_a), \tag{3.4}$$

$$f_a = \hat{f}_a(\mathscr{S}_a), \tag{3.5}$$

$$\boldsymbol{h}_a = \hat{\boldsymbol{h}}_a(\mathscr{S}_a), \tag{3.6}$$

$$\boldsymbol{q}_a = \hat{\boldsymbol{q}}_a(\mathscr{S}_a), \tag{3.7}$$

$$g_a^+ = \hat{g}_a^+(\mathscr{S}_a, \mathscr{S}_b), \tag{3.8}$$

$$\boldsymbol{m}_a^+ = \hat{\boldsymbol{m}}_a^+(\mathscr{S}_a, \mathscr{S}_b), \tag{3.9}$$

$$k_a = \hat{k}_a(\phi_a). \tag{3.10}$$

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* Almost always, our constitutive equations will depend on an initial state $\{\boldsymbol{F}, \phi_a, \theta_a\} = \{\boldsymbol{1}, \bar{\phi}_a, \bar{\theta}_a\}$ of the body. We usually suppress that dependence in our notation for general considerations. When we consider specific applications, we sometimes use that dependence explicitly.

Then arguments deriving from those of COLEMAN & NOLL [1963] give

$$\psi_a = \hat{\psi}_a(\boldsymbol{F}, \phi_a, \text{GRAD}\,\phi_a, \theta_a), \tag{3.11}$$

$$\eta_a = -\frac{\partial \psi_a}{\partial \theta_a}, \tag{3.12}$$

$$\boldsymbol{S} = \bar{\varrho}\frac{\partial \psi}{\partial \boldsymbol{F}}, \tag{3.13}$$

$$\boldsymbol{h}_a = \bar{\varrho}_a \frac{\partial \psi_a}{\partial\,(\text{GRAD}\,\phi_a)}; \tag{3.14}$$

in $(3.13)_3$

$$\bar{\varrho}\left(\psi + \frac{k\dot{\phi}^2}{2}\right) = \Sigma\,\bar{\varrho}_a\left(\psi_a + \frac{k_a\dot{\phi}_a^2}{2}\right). \tag{3.15}$$

It is significant that all of these results may be written in terms of ψ only, rather than ψ_a. They are

$$\eta_a = -\frac{\bar{\varrho}}{\bar{\varrho}_a}\frac{\partial \tilde{\psi}}{\partial \theta_a}, \tag{3.16}$$

$$\boldsymbol{S} = \bar{\varrho}\frac{\partial \tilde{\psi}}{\partial \boldsymbol{F}}, \tag{3.17}$$

$$\boldsymbol{h}_a = \bar{\varrho}\frac{\partial \tilde{\psi}}{\partial\,(\text{GRAD}\,\phi_a)}, \tag{3.18}$$

with

$$\psi = \tilde{\psi}(\boldsymbol{F}, \phi_a, \text{GRAD}\,\phi_a, \theta_a), \tag{3.19}$$

given by*

$$\begin{aligned}\psi &= \frac{1}{\bar{\varrho}}\sum \bar{\varrho}_a\left(\psi_a + \frac{k_a\dot{\phi}_a^2}{2}\right) - \frac{k\dot{\phi}^2}{2},\\ &= \sum\frac{\bar{\varrho}_a\psi_a}{\bar{\varrho}} + \left(\sum\left(\frac{\bar{\varrho}_a}{\bar{\varrho}}\right)\frac{k_a\dot{\phi}_a^2}{2} - \frac{k\dot{\phi}^2}{2}\right),\\ &= \tilde{\psi}(\boldsymbol{F}, \phi_a, \text{GRAD}\,\phi_a, \theta_a) + \bar{K}(\phi_a, \dot{\phi}_a).\end{aligned} \tag{3.20}$$

We also have the residual entropy inequality

$$\sum\left[\bar{\varrho}_a\left\{\frac{\partial \psi_a}{\partial \phi_a} + f_a + \frac{g_a^{+}}{\bar{\varrho}_a} - \tfrac{1}{2}\,\dot{k}_a\dot{\phi}_a\right\}\dot{\phi}_a + \frac{\boldsymbol{q}_a\cdot\text{GRAD}\,\theta_a}{\theta_a}\right] \leqq 0. \tag{3.21}$$

3.2. Material Frame-Indifference. The two invariance principles of central interest in the theory of constitutive equations are *frame-indifference* and *material symmetry*. For a theory with as many variables as the one we consider here, a general discussion of such principles would be so complicated as to be opaque. However, we have seen that, for the type of mixture we deal with, the free energy

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* Here we have used $(2.3)_1$ to eliminate γ_a in favor of ϕ_a in the last of equations (3.20).

ψ determines much of the behavior of the mixture. Thus we concentrate on dealing with it in detail. It is then easy to write out the corresponding results for other quantities, and we do so without proof.

Material frame-indifference requires that the response of the material be independent of the motion of the observer. Its only formal consequence in terms of (3.11) is

$$\tilde{\psi}(F, \phi_a, \operatorname{GRAD} \phi_a, \theta_a) = \check{\psi}(E, \phi_a, \operatorname{GRAD} \phi_a, \theta_a), \tag{3.22}$$

where

$$E = \tfrac{1}{2}(U^2 - 1) \tag{3.23}$$

is the finite strain tensor, and

$$U^2 = F^T F. \tag{3.24}$$

Under a change of frame, we have

$$E^* = E, \qquad (\operatorname{GRAD} \phi_a)^* = \operatorname{GRAD} \phi_a, \tag{3.25}$$

where E^* is the value of E under the change of frame, *etc.* As is obvious from (3.25),

$$\check{\psi}(E, \phi_a, \operatorname{GRAD} \phi_a, \theta_a) = \tilde{\psi}(E^*, \phi_a, (\operatorname{GRAD} \phi_a)^*, \theta_a). \tag{3.26}$$

3.3. Equilibrium. Consider the residual entropy inequality (3.21). Let

$$\Phi_a = \bar{\varrho}_a \left\{ \frac{\partial \psi_a}{\partial \phi_a} + f_a + \frac{\overset{+}{g_a}}{\bar{\varrho}_a} - \tfrac{1}{2} \dot{k}_a \dot{\phi}_a \right\}. \tag{3.27}$$

Note that Φ_a depends on $\{E, \operatorname{GRAD} \phi_a, \phi_a, \theta_a\}$ and (through $\overset{+}{g_a}$) $\{\operatorname{GRAD} \phi_b, \dot{\phi}_b, \phi_b, \theta_b, \operatorname{GRAD} \theta_b\}$, as well as the fact that k_a depends on ϕ_a. The inequality (3.21) then has the form

$$R \equiv \sum_a \Phi_a \dot{\phi}_a + \frac{q_a \cdot \operatorname{GRAD} \theta_a}{\theta_a} \leqq 0. \tag{3.28}$$

We define $\mathscr{B}_a$ to be in *equilibrium* if $\dot{\phi}_a \equiv 0$ and $\operatorname{GRAD} \theta_a = 0$. We use the notation

$$(\ )_E \equiv \text{value of } (\ ) \text{ in equilibrium}. \tag{3.29}$$

As a consequence of (3.28), R is a maximum with respect to $\dot{\phi}_a$ and $\operatorname{GRAD} \theta_a$ in equilibrium. It then follows that

$$\left(\frac{\partial R}{\partial \dot{\phi}_a}\right)_E = 0, \qquad \left(\frac{\partial R}{\partial (\operatorname{GRAD} \theta_a)}\right)_E = 0. \tag{3.30}$$

Furthermore, consider the vector $\mathscr{U}$ with components

$$\mathscr{U} \equiv (\dot{\phi}_a, \operatorname{GRAD} \theta_a), \tag{3.31}$$

and compute the matrix

$$R_{\mathscr{U}\mathscr{U}} \equiv \frac{\partial^2 R}{\partial \mathscr{U}^2}. \tag{3.32}$$

Then that matrix must be nonpositive in equilibrium

$$(R_{\mathscr{U}\mathscr{U}})_E \leqq 0, \tag{3.33}$$

in the sense that its determinant and all of its subdeterminants must be nonpositive.

It is easiest to apply these conditions if we assume g_a^+ is independent of GRAD θ_b and q_a is independent of $\dot{\phi}_b$. We thus make these assumptions. We then conclude from (3.30) that

$$\left(\frac{\partial \psi_a}{\partial \phi_a} + f_a + \frac{g_a^+}{\bar{\varrho}_a}\right)_E = 0, \qquad (q_a)_E = 0. \tag{3.34}$$

A condition *sufficient* to satisfy $(3.30)_1$ is

$$g_a^+ + \bar{\varrho}_a f_a = -\bar{\varrho}_a \frac{\partial \psi_a}{\partial \phi_a} - \delta_a(\boldsymbol{E}, \phi_b, \text{GRAD}\, \phi_b, \theta_b)\, \dot{\phi}_a. \tag{3.35}$$

The condition

$$\delta_a(\boldsymbol{E}, \phi_a, \text{GRAD}\, \phi_a, \theta_a) \geqq 0, \tag{3.36}$$

suffices to render (3.35) consistent with (3.33).

3.4. Material Symmetry. The other invariance principle of interest is *material symmetry.* We let

$$\boldsymbol{w}_a \equiv \text{GRAD}\, \phi_a. \tag{3.37}$$

The function $\check{\psi}$ given by (3.22) depends on a number of scalar-valued variables which appear only parametrically in subsequent arguments. We suppress them in our notation by setting

$$\check{\psi}(\boldsymbol{E}, \boldsymbol{w}_a) \equiv \check{\psi}(\boldsymbol{E}, \phi_a, \text{GRAD}\, \phi_a, \theta_a). \tag{3.38}$$

The symmetry group $\mathscr{G}$ (isotropy group, peer group) of $\check{\psi}$ is the set of all unimodular transformations $\boldsymbol{H}$ that leave the value of $\check{\psi}$ unchanged:

$$\check{\psi}(\boldsymbol{E}, \boldsymbol{w}_a) = \check{\psi}(\boldsymbol{H}\boldsymbol{E}\boldsymbol{H}^T, \boldsymbol{H}\boldsymbol{w}_a). \tag{3.39}$$

In mechanics attention is usually directed towards thirteen groups $\mathscr{G}$: the eleven groups derived by certain invariance principles inherent in elasticity theory from the thirty-two crystal classes usually studied by crystallographers, and the groups corresponding to isotropic and transversely isotropic materials. The former are all finite groups, the latter are both infinite groups. Under each of these groups, (3.39) is restricted. The results of the restrictions yield special forms of $\check{\psi}$, and the theorems which yield these forms are *representation theorems.* Generally, the larger the group (and thus the more symmetry the material has), the greater will be the restriction and the simpler will be the representation for $\check{\psi}$. In particular, for isotropic materials, explicit representation theorems are known and clearly set out by SMITH [1969, 1971] and WANG [1969, 1970]. Representation theorems for other groups are less easily accessible, though many are known. {We note the treatise of SPENCER [1971], and the papers of PIPKIN & RIVLIN [1959–1960], and SMITH, SMITH, & RIVLIN [1963].} Here we give explicit results for isotropic and transversely isotropic materials.

4. Constitutive Equations for an Isotropic Material

For an isotropic material, the symmetry group $\mathscr{G}$ is $\mathscr{O}$, the full orthogonal group. We have

$$\check{\psi}(\boldsymbol{E}, \boldsymbol{w}_a) = \tilde{\psi}(\mathrm{tr}\, \boldsymbol{E}, \mathrm{tr}\, \boldsymbol{E}^2, \mathrm{tr}\, \boldsymbol{E}^3, \boldsymbol{w}_a \cdot \boldsymbol{w}_a, \boldsymbol{w}_a \cdot \boldsymbol{E}\boldsymbol{w}_a, \boldsymbol{w}_a \cdot \boldsymbol{E}^2 \boldsymbol{w}_a). \tag{4.1}$$

Alternatively, we may replace the set of independent variables by the set $\{I_1, I_2, I_3, J_{1a}, J_{2a}, J_{3a}\}$, where

$$I_1 = \mathrm{tr}\, \boldsymbol{E}, \quad I_2 = \tfrac{1}{2}\{(\mathrm{tr}\, \boldsymbol{E})^2 - \mathrm{tr}\,(\boldsymbol{E}^2)\}, \quad I_3 = \det \boldsymbol{E}, \tag{4.2}$$

$$J_{1a} = \boldsymbol{w}_a \cdot \boldsymbol{w}_a, \quad J_{2a} = \boldsymbol{w}_a \cdot \boldsymbol{E}\boldsymbol{w}_a, \quad J_{3a} = \boldsymbol{w}_a \cdot \boldsymbol{E}^2 \boldsymbol{w}_a,$$

so

$$\check{\psi}(\boldsymbol{E}, \boldsymbol{w}_a) = \bar{\psi}(I_1, I_2, I_3, J_{1a}, J_{2a}, J_{3a}). \tag{4.3}$$

Now (3.13) is equivalent to

$$\boldsymbol{S} = \bar{\varrho} \boldsymbol{F} \frac{\partial \psi}{\partial \boldsymbol{E}}, \qquad \tilde{\boldsymbol{T}} = \bar{\varrho} \frac{\partial \psi}{\partial \boldsymbol{E}}, \tag{4.4}$$

where $\tilde{\boldsymbol{T}}$ is the second Piola-Kirchhoff stress tensor. We may thus derive explicit representations for $\tilde{\boldsymbol{T}}$ and $\boldsymbol{h}_a$ from (3.18), (4.3), and (4.4). They are

$$\begin{aligned} \tilde{\boldsymbol{T}} = \bar{\varrho} \Bigg\{ & \left(\frac{\partial \psi}{\partial I_1} + I_1 \frac{\partial \psi}{\partial I_2} + I_2 \frac{\partial \psi}{\partial I_3}\right) \mathbf{1} + \left(\frac{\partial \psi}{\partial I_2} - I_1 \frac{\partial \psi}{\partial I_3}\right) \boldsymbol{E} + \frac{\partial \psi}{\partial I_3} \boldsymbol{E}^2 \\ & + \sum_a \left[\frac{\partial \psi}{\partial J_{2a}} \boldsymbol{w}_a \otimes \boldsymbol{w}_a + \frac{\partial \psi}{\partial J_{3a}} (\boldsymbol{w}_a \otimes \boldsymbol{E}\boldsymbol{w}_a + \boldsymbol{E}\boldsymbol{w}_a \otimes \boldsymbol{w}_a)\right] \Bigg\}, \end{aligned} \tag{4.5}$$

$$\boldsymbol{h}_a = 2\bar{\varrho} \left[\frac{\partial \psi}{\partial J_{1a}} \boldsymbol{w}_a + \frac{\partial \psi}{\partial J_{2a}} \boldsymbol{E}\boldsymbol{w}_a + \frac{\partial \psi}{\partial J_{3a}} \boldsymbol{E}^2 \boldsymbol{w}_a\right]. \tag{4.6}$$

The other vectors of interest in this theory are those for the heat flux $\boldsymbol{q}_a$ and the momentum exchange $\boldsymbol{m}_a^+$. These are not derivable from the potential $\check{\psi}$, and therefore require individual treatment. For the heat flux recall that we have

$$\boldsymbol{q}_a = \hat{\boldsymbol{q}}_a(\boldsymbol{F}, \phi_a, \mathrm{GRAD}\, \phi_a, \theta_a, \mathrm{GRAD}\, \theta_a). \tag{4.7}$$

For an isotropic material, the representation theorem for this function is

$$\begin{aligned} \boldsymbol{q}_a = -(& \varkappa_a^0\, \mathrm{GRAD}\, \theta_a + \varkappa_a^1 \boldsymbol{E}\, \mathrm{GRAD}\, \theta_a + \varkappa_a^2 \boldsymbol{E}^2\, \mathrm{GRAD}\, \theta_a \\ & + v_a^0\, \mathrm{GRAD}\, \phi_a + v_a^1 \boldsymbol{E}\, \mathrm{GRAD}\, \phi_a + v_a^2 \boldsymbol{E}^2\, \mathrm{GRAD}\, \phi_a), \end{aligned} \tag{4.8}$$

where $\varkappa_a^\alpha, v_a^\alpha$ are functions of $\{I_1, I_2, I_3, J_{1a}, J_{2a}, J_{3a}, \phi_a, \theta_a, \mathrm{GRAD}\, \theta_a \cdot \mathrm{GRAD}\, \theta_a\}$. Assume that they do not in fact depend upon $\mathrm{GRAD}\, \theta_a \cdot \mathrm{GRAD}\, \theta_a$. However, $\boldsymbol{q}_a = 0$ in equilibrium. That then requires $v_a^\alpha \equiv 0$, so that

$$\boldsymbol{q}_a = -(\varkappa_a^0 \mathbf{1} + \varkappa_a^1 \boldsymbol{E} + \varkappa_a^2 \boldsymbol{E}^2)\, \mathrm{GRAD}\, \theta_a, \tag{4.9}$$

where

$$\varkappa_a^\alpha = \hat{\varkappa}_a^\alpha(I_1, I_2, I_3, J_{1a}, J_{2a}, J_{3a}, \phi_a, \theta_a), \tag{4.10}$$

with the property that, by (3.33),

$$\hat{\varkappa}_a^\alpha(I_1, I_2, I_3, J_{1a}, J_{2a}, J_{3a}, \phi_a, \theta_a) \geqq 0. \tag{4.11}$$

We henceforth write (4.11) as

$$\varkappa_a^\alpha \geqq 0. \tag{4.12}$$

The remaining functions in the theory retain the same type of functional dependence as is listed in (4.10).

4.1. A Simplified Theory. Even under conditions of maximal material symmetry, the equations we have derived are quite complicated. It is appropriate to search for a method of simplifying them without reducing their physical applicability. TRUESDELL & NOLL [1965] commented almost twenty years ago that though the general theory of nonlinear elasticity gave results which were physically reasonable, no simplified theory had ever been found which agreed quantitatively and qualitatively with experiment. That is still true, and since our equations generalize nonlinear elasticity, it should be true for them also. Nonetheless, there are simplified theories which are used in physical applications for purposes of mathematical expediency and are known to give reasonable results if used carefully. It is straightforward and useful to generalize them to the physical situation which we study here.

Let us consider (4.1). Assume $\breve{\psi}(\boldsymbol{E}, \boldsymbol{w}_a)$ has a minimum at $\boldsymbol{E} = 0$, $\boldsymbol{w}_a = 0$, and write the most general positive-definite form* for $\breve{\psi}$ with this property which is quadratic in the joint invariants of $\boldsymbol{E}$ and $\boldsymbol{w}_a$,

$$\bar{\varrho}\breve{\psi}(\boldsymbol{E}, \boldsymbol{w}_a) = \frac{\phi}{2}[\lambda(\operatorname{tr}\boldsymbol{E})^2 + 2\mu \operatorname{tr}(\boldsymbol{E}^2)] + \Sigma\tfrac{1}{2}(\alpha_a \boldsymbol{w}_a \cdot \boldsymbol{w}_a + \nu_a(\phi_a, \theta_a)), \tag{4.13}$$

where the multiplicative factor ϕ takes into account the fact that the voids have no free energy. Here

$$3\lambda + 2\mu > 0, \quad \mu > 0, \quad \alpha_a > 0 \tag{4.14}$$

are functions of $\{\phi_a, \theta_a\}$. We then have from (4.4) and (3.18)

$$\tilde{\boldsymbol{T}} = \phi[\lambda(\operatorname{tr}\boldsymbol{E})\,\boldsymbol{1} + 2\mu\boldsymbol{E}], \tag{4.15}$$

$$\boldsymbol{h}_a = \alpha_a \boldsymbol{w}_a. \tag{4.16}$$

Equation (4.9), taken to the same order of approximation as (4.15) and (4.16) is

$$q_a = -\varkappa_a \operatorname{GRAD} \theta_a, \tag{4.17}$$

with, by (4.11)

$$\varkappa_a(\phi_a, \theta_a) \geqq 0. \tag{4.18}$$

It is characteristic of the particular type of theory with one deformation but many temperatures, that the single relation for stress obtained from thermodynamics is of the form of (4.4), which gives only the total stress, while the

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* This form, whether or not thermodynamics is used, forces considerable uniqueness upon the theory. Furthermore, it *a priori* prevents interactions of the sort which would occur as a result of terms in which the strain and volume fraction gradients are multiplied. Our theory is closely related to that of the Mooney-Rivlin theory ([RIVLIN, 1948], RIVLIN & SAUNDERS [1951]). TRELOAR [1958] gives molecular motivation for such a theory.

stresses in all of the constituents are needed in the energy equations, say $(2.3)_5$. In general this is a profound difficulty in formulating mixture theories. That is not the case here, where in fact the difficulty is easily overcome. It is a direct consequence of $(2.5)_2$ that

$$\tilde{\boldsymbol{T}} = \tilde{\boldsymbol{T}}_1 + \tilde{\boldsymbol{T}}_2. \tag{4.19}$$

Now $\tilde{\boldsymbol{T}}$ is known from (4.15), so specification of $\tilde{\boldsymbol{T}}_1$ suffices to give $\tilde{\boldsymbol{T}}_2$. We rewrite (3.20) in the form

$$\bar{\varrho}\psi = \bar{\varrho}_1\psi_1 + \bar{\varrho}_2\psi_2 + \bar{\varrho}\bar{K}. \tag{4.20}$$

We have not done so here, but is a straightforward task to develop a thermodynamic theory for the matrix only. A significant result of this theory is

$$\tilde{\boldsymbol{T}}_1 = \bar{\varrho}_1 \frac{\partial \psi_1}{\partial \boldsymbol{E}}. \tag{4.21}$$

Furthermore, it is clear from (3.2) that $\tilde{\boldsymbol{T}}_1$ will be independent of the properties of the inclusions. Since the behavior of the mixture under stress is expected to be principally a property of the behavior of the matrix, we assume

$$\bar{\varrho}_1\check{\psi}_1(\boldsymbol{E}, \boldsymbol{w}_1) = \frac{\phi_1}{2}[\lambda(\operatorname{tr}\boldsymbol{E})^2 + 2\mu \operatorname{tr}(\boldsymbol{E}^2)] + \tfrac{1}{2}(\alpha_1 \boldsymbol{w}_1 \cdot \boldsymbol{w}_1 + \nu_1(\phi_1, \theta_1)), \tag{4.22}$$

with λ and μ functions of θ_1. It then follows that

$$\tilde{\boldsymbol{T}}_1 = \phi_1[\lambda(\operatorname{tr}\boldsymbol{E})\,\mathbf{1} + 2\mu\boldsymbol{E}], \tag{4.23}$$

and

$$\tilde{\boldsymbol{T}}_2 = \phi_2[\lambda(\operatorname{tr}\boldsymbol{E})\,\mathbf{1} + 2\mu\boldsymbol{E}]. \tag{4.24}$$

For the internal energy e_a we have arguments closely analogous to those for the free energy ψ_a; that is, by (3.38),

$$e_a = \check{e}_a(\boldsymbol{E}, \phi_a, \boldsymbol{w}_a, \theta_a), \tag{4.25}$$

and so

$$\dot{e}_a = \frac{\partial e_a}{\partial \boldsymbol{E}} \cdot \dot{\boldsymbol{E}} + \frac{\partial e_a}{\partial \phi_a}\dot{\phi}_a + \frac{\partial e_a}{\partial \boldsymbol{w}_a} \cdot \dot{\boldsymbol{w}}_a + \frac{\partial e_a}{\partial \theta_a}\dot{\theta}_a, \tag{4.26}$$

or, with the obvious changes in notation,

$$\dot{e}_a = \boldsymbol{C}_a^E \cdot \dot{\boldsymbol{E}} + c_a^\phi \dot{\phi}_a + \boldsymbol{c}_a^w \cdot \dot{\boldsymbol{w}}_a + c_a^\theta \dot{\theta}_a. \tag{4.27}$$

Here, c_a^θ is the *specific heat** of constituent a, while the other quantities are what

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* This corresponds to the "specific heat at constant volume" defined in many books on physics.

were called in the older literature *latent heats.** {See, for example, TRUESDELL & BHARATHA [1977].} Thus, the tensor C_a^E might be called the *latent heat with respect to strain.* However, because the body is isotropic, and because of the degree of approximation taken in the other equations of this theory, the tensor C_a^E and the vector c_a^w have special forms. They are

$$C_a^E = C_a^{E0}\mathbf{1} + C_a^{E1}\boldsymbol{E}, \tag{4.28}$$

$$\boldsymbol{c}_a^w = c_a^w \boldsymbol{w}_a. \tag{4.29}$$

It is seen here that C_a^{E1} and c_a^w have the dimensions which might be expected of latent heats of their types.

We now formulate constitutive equations for the exchanges of momentum, equilibrated inertia, and energy. Let us consider the momentum exchange function $\boldsymbol{m}_a^+$. In elementary treatments for diffusing mixtures this term is taken to be proportional to a diffusion velocity. Since here we have a nondiffusing material, a reasonable first try for a constitutive equation might be

$$\boldsymbol{m}_a^+ = 0. \tag{4.30}$$

Likewise, it is a reasonable first approximation to take

$$g_a^+ = 0. \tag{4.31}$$

The energy interaction e_a^+, however, is another matter. In situations where the body is being heated quickly, it is to be expected that the temperatures of the matrix and the inclusions will differ quite radically. In that case an energy exchange caused by conduction between the constituents, and *possibly radiation also,* will occur. An appropriate constitutive equation for these phenomena is

$$e_1^+ = -e_2^+ = c^c(\theta_2 - \theta_1) + c^r(\theta_2 - \theta_1)^N, \quad c^c > 0, \quad c^r > 0, \quad N > 1, \tag{4.32}$$

with N an odd integer. The special case of this equation $c^r = 0$ is commonly called "NEWTON'S Law of cooling". The inertial coefficient k_a in $(2.3)_4$ is still the subject of research. Most commonly it is taken to be a constant, often zero. NUNZIATO & COWIN [1979] use as motivation arguments of KNOWLES & JAKUB [1979] which lead to a considerably more complicated form.

Thus, *modulo* specific functional forms for scalar-valued constitutive equations, which should be determined by experiment, we have specific equations of mass, motion and energy for an isotropic material of the type described.

It is appropriate to collect the equations of this theory in one place. We have *conservation of mass*

$$\bar{\varrho}_a = \varrho_a \det \boldsymbol{F}, \tag{4.33}$$

balance of momentum

$$\bar{\varrho}\dot{\boldsymbol{v}} = \bar{\varrho}\boldsymbol{b} + \mathrm{DIV}\,\{\phi\boldsymbol{F}[\lambda(\mathrm{tr}\,\boldsymbol{E})\,\mathbf{1} + 2\mu\boldsymbol{E}]\}, \tag{4.34}$$

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* Most recent books use the term "latent heat" in quite another sense.

*balance of equilibrated inertia**

$$\bar{\varrho}_a k_a \ddot{\phi}_a = \operatorname{DIV}(\alpha_a \operatorname{GRAD} \phi_a) - \frac{\partial \psi_a}{\partial \phi_a} - \delta_a \dot{\phi}_a, \tag{4.35}$$

and *balance of energy*

$$\begin{aligned} c^c(\theta_2 - \theta_1) + c^r(\theta_2 - \theta_1)^N = {} & \bar{\varrho}_1(\boldsymbol{C}_1^E \cdot \dot{\boldsymbol{E}} + c_1^\phi \dot{\phi}_1 + \boldsymbol{c}_1^w \cdot \dot{\boldsymbol{w}}_1 + c_1^\theta \dot{\theta}_1) \\ & - \phi_1 \boldsymbol{F}[\lambda(\operatorname{tr} \boldsymbol{E})\, \boldsymbol{1} + 2\mu \boldsymbol{E}] \cdot \dot{\boldsymbol{F}}^T + \operatorname{DIV}(-\varkappa_1 \operatorname{GRAD} \theta_1) - \bar{\varrho}_1 r_1 \\ & - \dot{\phi}_1 \left(\frac{\partial \psi_1}{\partial \phi_1} - \delta_1 \dot{\phi}_1 \right) - \alpha_1 \operatorname{GRAD} \phi_1 \cdot \operatorname{GRAD} \dot{\phi}_1, \end{aligned} \tag{4.36}$$

$$\begin{aligned} c^c(\theta_1 - \theta_2) + c^r(\theta_1 - \theta_2)^N = {} & \bar{\varrho}_2(\boldsymbol{C}_2^E \cdot \dot{\boldsymbol{E}} + c_2^\phi \dot{\phi}_2 + \boldsymbol{c}_2^w \cdot \dot{\boldsymbol{w}}_2 + c_2^\theta \dot{\theta}_2) \\ & - \phi_2 \boldsymbol{F}[\lambda(\operatorname{tr} \boldsymbol{E})\, \boldsymbol{1} + 2\mu \boldsymbol{E}] \cdot \dot{\boldsymbol{F}}^T + \operatorname{DIV}(-\varkappa_2 \operatorname{GRAD} \theta_2) - \bar{\varrho}_2 r_2 \\ & - \dot{\phi}_2 \left(\frac{\partial \psi_2}{\partial \phi_2} - \delta_2 \dot{\phi}_2 \right) - \alpha_2 \operatorname{GRAD} \phi_2 \cdot \operatorname{GRAD} \dot{\phi}_2. \end{aligned} \tag{4.37}$$

4.2. Thermal Stresses. In the previous sections, we have taken the free energy ψ to be a quadratic function of the strain $\boldsymbol{E}$ and the volume fraction gradients $\boldsymbol{w}_a$, while letting temperature and volume fraction serve as parameters in constitutive equations. In order to describe thermal stresses, we must give an explicit role to temperatures in ψ. {A succinct exposition of this matter in the classical linear case is given by CARLSON [1972].} Thus, we assume that** $\check{\psi}(\boldsymbol{E}, \boldsymbol{w}_a, \theta_a)$ has a minimum at $\boldsymbol{E} = 0$, $\boldsymbol{w}_a = 0$, $\theta_a = \bar{\theta}$. We write the most general positive-definite form for $\check{\psi}$ which has this property and which is quadratic in $\theta_a - \bar{\theta}$ and the joint invariants of $\boldsymbol{E}$ and $\boldsymbol{w}_a$,

$$\begin{aligned} \bar{\varrho}\check{\psi}(\boldsymbol{E}, \boldsymbol{w}_a) = {} & \frac{\phi}{2}[\lambda(\operatorname{tr} \boldsymbol{E})^2 + 2\mu \operatorname{tr}(\boldsymbol{E}^2) + \Sigma(2m_a(\theta_a - \bar{\theta}) \operatorname{tr} \boldsymbol{E} + n_a(\theta_a - \bar{\theta})^2)] \\ & + \Sigma\, \tfrac{1}{2}(\alpha_a \boldsymbol{w}_a \cdot \boldsymbol{w}_a + \nu_a(\phi_a)), \end{aligned} \tag{4.38}$$

with

$$3\lambda + 2\mu > 0, \quad \mu > 0, \quad \alpha_a > 0, \tag{4.39}$$

$$n_a > 0, \quad (3\lambda + 2\mu)\, n_a \geqq 3m_a. \tag{4.40}$$

Here, λ, μ, m_a, and n_a are constants, while α_a and ν_a may be functions of ϕ_a. The

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* Here we have left the dependence of the Helmholtz free energies on the constituent volume fractions to be determined by experiment. There are at least two theoretical works which address this matter. The work of NUNZIATO & COWIN [1979], which again appeals to the work of KNOWLES & JAKUB [1979] for restrictions on the form of those derivatives, and that of COWIN & NUNZIATO [1983], which takes such derivatives to be affine forms in the volume fractions.

** We have taken the obvious small liberty with the notation of the previous section.

arguments in this case are quite similar to those in the previous section. That is,

$$\tilde{\boldsymbol{T}} = \tilde{\boldsymbol{T}}_1 + \tilde{\boldsymbol{T}}_2, \tag{4.41}$$

$$\tilde{\boldsymbol{T}}_1 = \bar{\varrho}_1 \frac{\partial \psi_1}{\partial \boldsymbol{E}}, \tag{4.42}$$

so it is plausible to assume

$$\begin{aligned}\bar{\varrho}_1 \psi_1(\boldsymbol{E}, \boldsymbol{w}_1, \theta_1) = \tfrac{1}{2}\phi_1\,(\lambda(\operatorname{tr}\boldsymbol{E})^2 + 2\mu \operatorname{tr}(\boldsymbol{E}^2) + \Sigma[2m_1(\theta_1 - \bar{\theta}) \operatorname{tr}\boldsymbol{E} \\ + n_1(\theta_1 - \bar{\theta})^2]) + \Sigma\,\tfrac{1}{2}(\alpha_1 \boldsymbol{w}_1 \cdot \boldsymbol{w}_1 + \nu_1(\phi_1)).\end{aligned} \tag{4.43}$$

It then follows that

$$\tilde{\boldsymbol{T}}_a = \phi_a[\lambda(\operatorname{tr}\boldsymbol{E})\,\boldsymbol{1} + 2\mu\boldsymbol{E} + m_a(\theta_a - \bar{\theta})\,\boldsymbol{1}]. \tag{4.44}$$

There is no effect on the equations for equilibrated inertia. The field equations are exactly the same as (4.33)–(4.37), except that

$$\text{DIV}\,\{\phi\boldsymbol{F}\,\Sigma m_a(\theta_a - \bar{\theta})\} \tag{4.45}$$

is added to the right side of (4.34),

$$-\phi_1 m_1(\theta_1 - \bar{\theta})\,\boldsymbol{F} \cdot \dot{\boldsymbol{F}}^T \tag{4.46}$$

is added to the right side of (4.36), and

$$-\phi_2 m_2(\bar{\theta}_2 - \theta)\,\boldsymbol{F} \cdot \dot{\boldsymbol{F}}^T \tag{4.47}$$

is added to the right side of (4.37).

5. Constitutive Equations for a Transversely Isotropic Material

Consider a set of rectangular Cartesian material co-ordinates. Let the material be transversely isotropic about the X_3 axis. This means that the material isotropy group $\mathscr{G}$ consists of the identity $\boldsymbol{1}$ and the tensors which have the following matrices with respect to that co-ordinate system:

$$\begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{5.1}$$

together with their products. The result corresponding to (4.3) is then

$$\begin{aligned}\check{\psi}(\boldsymbol{E}, \boldsymbol{w}_a) = \overline{\psi}(&E_{\alpha\alpha}, E_{\alpha\beta}E_{\alpha\beta}, E_{33}, E_{3\alpha}E_{\alpha 3}, E_{3\alpha}E_{\alpha\beta}E_{\beta 3}, \\ &w_{3a}, w_{\alpha a}E_{\alpha 3}, w_{\alpha a}E_{\alpha\beta}E_{\beta 3}, w_{\alpha a}w_{\alpha a}, w_{\alpha a}E_{\alpha\beta}w_{\beta a}),\end{aligned} \tag{5.2}$$

where each of $\{\alpha, \beta\}$ takes the values $\{1, 2\}$, and the usual summation convention is observed for Greek indices.

We define

$$\psi^l = \bar{\varrho}\frac{\partial \bar{\psi}}{\partial I_l}, \tag{5.3}$$

where I_l is the l^{th} entry in the list of independent variables on the right hand side of (5.2). We then have

$$\begin{aligned} \tilde{T}_{\alpha\beta} &= \psi^1\,\delta_{\alpha\beta} + 2\psi^2 E_{\alpha\beta} + \psi^5 E_{\alpha 3}E_{\beta 3} + \sum_a (\psi_a^8 w_{\alpha a}E_{\beta 3} + \psi_a^{10} w_{\alpha a} w_{\beta a}),\\ \tilde{T}_{\alpha 3} &= \psi^4 E_{\alpha 3} + \psi^5(E_{\alpha\beta}E_{\beta 3}) + \sum_a (\psi^7 w_{\alpha a} + \psi_a^8 E_{\alpha\beta} w_{\beta a}),\\ \tilde{T}_{33} &= \psi^3,\\ h_{\alpha a} &= (\psi_a^7\,\delta_{\alpha\beta} + \psi_a^8 E_{\alpha\beta})\,E_{\beta 3} + 2(\psi_a^9\,\delta_{\alpha\beta} + \psi_a^{10}E_{\alpha\beta})\,w_{\beta a},\\ h_{3a} &= \psi^6. \end{aligned} \tag{5.4}$$

The general equation for the heat flux (corresponding to (4.9)) in this case is

$$\begin{aligned} q_{\alpha a} &= -(\varkappa_a^0 \theta_{a,\alpha} + \varkappa_a^1 E_{\alpha\beta}\theta_{a,\beta}),\\ q_{3a} &= -\varkappa_a \theta_{a,3}, \end{aligned} \tag{5.5}$$

where

$$\theta_{a,\beta} \equiv \frac{\partial \theta_a}{\partial X_\beta},$$

etc., and the $\varkappa$'s are again non-negative.

5.1. A Simplified Theory. Here, we write out the equations for the transversely isotropic case, which correspond to the simplified theory for the isotropic case given in the previous section. We have

$$\begin{aligned} \bar{\varrho}\check{\psi}(\boldsymbol{E}, \boldsymbol{w}_a) = \frac{\phi}{2}\,&[c^1 E_{\alpha\alpha}E_{\beta\beta} + c^2 E_{\alpha\beta}E_{\alpha\beta} + c^3 E_{\alpha\alpha}E_{33} + c^4 E_{33}^2 + c^5 E_{3\alpha}E_{\alpha 3}]\\ &+ \sum_a (c_a^6 E_{\alpha\alpha} w_{3a} + c_a^7 E_{33} w_{3a} + c_a^8 w_{3a}^2 + c_a^9 E_{\alpha 3} w_{\alpha a}\\ &+ c_a^{10} w_{\alpha a} w_{\alpha a}) + \nu(\phi_a, \theta_a). \end{aligned} \tag{5.6}$$

Therefore*

$$\begin{aligned} \tilde{T}_{\alpha\beta} &= \phi[c^1 E_{\gamma\gamma}\,\delta_{\alpha\beta} + c^2 E_{\alpha\beta} + \tfrac{1}{2}c^3 E_{33}\,\delta_{\alpha\beta}] + \sum_a c_a^6\,\delta_{\alpha\beta} w_{3a},\\ \tilde{T}_{\alpha 3} = \tilde{T}_{3\alpha} &= \phi[c^5 E_{3\alpha}] + \sum_a c_a^9 w_{\alpha a},\\ \tilde{T}_{33} &= \phi[c^3 E_{\alpha\alpha} + 2c^4 E_{33}] + \sum_a c_a^7 w_{3a},\\ h_{a\alpha} &= c_a^9 E_{\alpha 3} + 2c_a^{10} w_{\alpha a},\\ h_{a3} &= c_a^6 E_{\alpha\alpha} + c_a^7 E_{33} + 2c_a^8 w_{3a}. \end{aligned} \tag{5.7}$$

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* We have here taken account of the fact that $E_{\alpha\beta}$ and $w_{\alpha a}w_{\beta a}$ are symmetric.

The heat flux is

$$q_{a\alpha} = -\varkappa_a \theta_{a,\alpha}, \qquad q_{3\alpha} = -\varkappa_{a3}\theta_{a,3}. \tag{5.8}$$

The stresses in the constituents are of the forms (5.6), that is,

$$\begin{aligned} \tilde{T}_{a\alpha\beta} &= \phi_a[c^1 E_{\gamma\gamma}\,\delta_{\alpha\beta} + c^2 E_{\alpha\beta} + \tfrac{1}{2}c^3 E_{33}\,\delta_{\alpha\beta}] + c_a^6\,\delta_{\alpha\beta} w_{3a}, \\ \tilde{T}_{a\alpha 3} &= \tilde{T}_{a3\alpha} = \phi_a[c^5 E_{3\alpha}] + c_a^9 w_{\alpha a}, \\ \tilde{T}_{a33} &= \tfrac{1}{2}\phi_a[c^3 E_{\alpha\alpha} + 2c^4 E_{33}] + c_a^7 w_{3a}, \end{aligned} \tag{5.9}$$

and the rate of change of internal energy is

$$\begin{aligned} \dot{e}_a = {} & C_a^E \dot{E}_{\alpha\alpha} + 2c_{a3}^E E_{\alpha 3}\dot{E}_{\alpha 3} + c_{a33}^E \dot{E}_{33} + c_a^\phi \dot{\phi}_a \\ & + c_a^w w_a \dot{w}_{a\alpha} + c_{a3}^w \dot{w}_{a3} + c_a^\theta \dot{\theta}_a. \end{aligned} \tag{5.10}$$

The equations (4.30)–(4.32) do not change, *i.e.*,

$$\boldsymbol{m}_a^+ = 0, \qquad g_a^+ = 0, \tag{5.11}$$

$$e_1^+ = -e_2^+ = c^c(\theta_2 - \theta_1) + c^r(\theta_2 - \theta_1)^N. \tag{5.12}$$

5.2. Thermal Stresses. The arguments for this case are easy extensions of the previous ones. We have a free energy of essentially the form (5.6), that is,

$$\begin{aligned} \bar{\varrho}\check{\psi}(\boldsymbol{E}, \boldsymbol{w}_a, \theta_a) = {} & \frac{\phi}{2}\,[c^1 E_{\alpha\alpha}E_{\beta\beta} + c^2 E_{\alpha\beta}E_{\alpha\beta} + c^3 E_{\alpha\alpha}E_{33} + c^4 E_{33}^2 + c^5 E_{3\alpha}E_{\alpha 3} \\ & + 2\sum_a (m_a^1(\theta_a - \bar{\theta})\,E_{\alpha\alpha} + m_a^2(\theta_a - \bar{\theta})\,E_{33} + n_a(\theta_a - \bar{\theta})^2)] \\ & + \sum_a (c_a^6 E_{\alpha\alpha} w_{3a} + c_a^7 E_{33} w_{3a} + c_a^8 w_{3a}^2 + c_a^9 E_{\alpha 3} w_{\alpha a} + c_a^{10} w_{\alpha a} w_{\alpha a}) \\ & + \nu(\phi_a). \end{aligned} \tag{5.13}$$

This leads to the same equations as in the previous section, except that the stresses (and therefore the balance equations for momentum and energy) must be adjusted to account for additional terms of the forms

$$\begin{aligned} \tilde{T}_{\alpha\beta} &\sim \phi \sum_a m_a^1(\theta_a - \bar{\theta})\,\delta_{\alpha\beta}, \\ \tilde{T}_{33} &\sim \phi \sum_a m_a^2(\theta_a - \bar{\theta}), \end{aligned} \tag{5.14}$$

$$\begin{aligned} \tilde{T}_{a\alpha\beta} &\sim \phi_a m_a^1(\theta_a - \bar{\theta})\,\delta_{\alpha\beta}, \\ \tilde{T}_{a33} &\sim \phi_a m_a^2(\theta_a - \bar{\theta}). \end{aligned} \tag{5.15}$$

6. Relation to the P-α Model for Porous Solids

A model which is widely used in calculations for porous solids is the *P*-α model. {This model was formulated by HERRMANN [1969]. A useful synopsis is presented by KIPP & LAWRENCE [1982].} The model we have presented here is much more general than the *P*-α model in that it is a three-dimensional theory which includes dissipation and which accounts for inclusions.* This theory reduces to the *P*-α model. To see that, we consider the one-dimensional case of the theory for isotropic materials set forth in § 4. We assume the body has no inclusions, so $\phi = \phi_1$ and $\phi_2 = 0$. The only case of interest is $0 < \phi < 1$. Then the function ψ in (3.22) depends only on $\{E, \phi, \boldsymbol{w}_a, \theta, \gamma\}$. Moreover, let us drop the dependence of this function on $\boldsymbol{w}_a$, so we have

$$\psi = \check{\psi}(E, \phi, \theta, \gamma). \tag{6.1}$$

We replace the stress by the pressure

$$\tilde{\boldsymbol{T}} \to -\tilde{p}; \tag{6.2}$$

then (4.4) becomes

$$\tilde{p} = -\bar{\varrho}\frac{\partial\psi}{\partial E}, \tag{6.3}$$

while (3.12) is

$$\eta = -\frac{\partial\psi}{\partial\theta}. \tag{6.4}$$

The above contain constitutive information only. The field equation for equilibrated inertia is also of interest. By (4.35) it is

$$\bar{\varrho}k\ddot{\phi} + \delta\dot{\phi} + \frac{\partial\check{\psi}}{\partial\phi} = 0. \tag{6.5}$$

The relations (3.23)–(3.24) between deformation gradient and strain become

$$F = \sqrt{2E + 1}, \tag{6.6}$$

so the equation of mass balance is

$$\bar{\gamma}\bar{\phi} = \gamma\phi\sqrt{2E + 1}, \tag{6.7}$$

Thus, *in this one-dimensional case*, we may eliminate the strain E from the function $\hat{\psi}$ in (6.1) in favor of the intrinsic mass density γ. We define

$$\psi(\phi, \theta, \gamma) = \check{\psi}(\hat{E}(\phi, \gamma), \phi, \theta, \gamma), \tag{6.8}$$

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* The model thus allows for hysteresis. For a detailed discussion of this matter, see PASSMAN [1984].

where $\hat{E}(\phi, \gamma)$ is the function obtained by solving (6.7) for E. We then have

$$\frac{\partial \psi}{\partial E} = \partial_1 \psi, \tag{6.9}$$

$$\frac{\partial \psi}{\partial \phi} = \partial_1 \psi \frac{\partial E}{\partial \phi} + \partial_2 \psi, \tag{6.10}$$

$$\frac{\partial \psi}{\partial \theta} = \partial_3 \psi, \tag{6.11}$$

where $\partial_n \psi$ is the partial derivative of $\hat{\psi}$ with respect to the n^{th} variable. The pressure $\tilde{p}$ is then

$$\tilde{p} = \partial_1 \hat{\psi}. \tag{6.12}$$

It is also useful to define the pressure associated with the Cauchy stress

$$T \to -p \tag{6.13}$$

so that

$$p = \frac{\bar{\varrho}}{\varrho} \tilde{p} \tag{6.14}$$

and thus

$$p = \frac{\bar{\varrho}}{\varrho} \partial_1 \hat{\psi}. \tag{6.15}$$

Recall that stresses (and therefore pressures) are defined as force per unit *total* area. Pressures expressed as forces *per unit area of solid* are given by

$$\tilde{p} = \bar{\phi} \tilde{p}_s, \quad p = \phi p_s, \tag{6.16}$$

so that (6.12) and (6.15) are

$$\tilde{p}_s = -\frac{\bar{\varrho}}{\bar{\phi}} \partial_1 \hat{\psi}, \quad p_s = -\frac{\bar{\varrho}^2}{\varrho} \partial_1 \hat{\psi}. \tag{6.17}$$

Both (4.13) and (5.6) indicate that the free energy should have the form

$$\bar{\varrho} \bar{\psi} = \phi \zeta(\theta, \gamma)\, \xi(E) + \nu(\phi, \theta, \gamma). \tag{6.18}$$

Then by (6.17)

$$p_s = -\frac{\bar{\varrho}}{\gamma \phi} \zeta \frac{d\xi}{dE}, \tag{6.19}$$

$$= \hat{p}_s(\gamma, \phi, \theta), \tag{6.20}$$

Let us consider (6.5), and neglect inertia and dissipation, so that $k = 0$, $\delta = 0$. Then

$$\frac{\partial \check{\psi}}{\partial \phi} = 0, \tag{6.21}$$

or, by (6.10) and (6.18)

$$\zeta \xi(\hat{E}) - \zeta \xi' \left(\frac{\bar{\gamma} \phi}{\gamma \phi} \right)^2 + \frac{\partial \nu}{\partial \phi} = 0. \tag{6.22}$$

Under moderate conditions of smoothness this equation may be solved to yield

$$\phi = \hat{\phi}(\gamma, \theta). \tag{6.23}$$

Substituting (6.23) into (6.20) gives

$$p_s = \hat{p}_s(\gamma, \hat{\phi}(\gamma, \theta), \theta), \tag{6.24}$$

so that, with the obvious redefinition of the function $\hat{p}_s$,

$$p_s = \bar{p}_s(\gamma, \theta). \tag{6.25}$$

It is usual to assume that functions of the type $\bar{p}_s$ satisfy

$$\frac{\partial \bar{p}_s}{\partial \gamma} \neq 0 \tag{6.26}$$

almost everywhere on their domain. We make that assumption here. Then we may solve (6.25) for γ, obtaining

$$\gamma = \tilde{\gamma}(p_s, \theta). \tag{6.27}$$

If we substitute (6.27) into (6.23), we obtain

$$\phi = \tilde{\phi}(p_s, \theta). \tag{6.28}$$

The equations (6.25) and (6.28) are the same as those of HERRMANN {[1969], equations (3) and (5)} which defined the *P*-α theory, except that HERRMANN, apparently motivated by the Mie-Grüneisen equation of state,* takes the internal energy e to be an independent variable. Two comments are appropriate here. The first is that the original papers of MIE and GRÜNEISEN assume conditions so restrictive that the internal energy is *proportional* to the temperature. Furthermore, the equation is often interpreted in that way by those who apply it. Second, by (2.13) and (3.12),

$$e = \psi - \theta \frac{\partial \psi}{\partial \theta}, \tag{6.29}$$

so by (6.21) and our assumptions about the functional dependence of ψ, we have

$$e = \hat{e}(\gamma, \theta). \tag{6.30}$$

Unless the assumption of MIE and GRÜNEISEN that the energy is a function only of the temperature holds, e must satisfy

$$\frac{\partial e}{\partial \gamma} \neq 0 \tag{6.31}$$

almost everywhere on its domain. Then we may write

$$\theta = \hat{\theta}(\gamma, e). \tag{6.32}$$

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* HERRMANN cites RICE, MCQUEEN, & WALSH [1958].

Substituting (6.32) into (6.25) gives

$$p_s = \bar{p}_s(\gamma, \phi, \hat{\theta}(\gamma, e)) = f(\gamma, \phi, e), \tag{6.33}$$

while (6.23) and (6.32) give

$$\phi = \hat{\phi}(\gamma, \hat{\theta}(\gamma, e)), \tag{6.34}$$

which, when substituted into (6.33), gives

$$p_s = h(\gamma, e). \tag{6.35}$$

This is the relation of HERRMANN [1969]. Assume

$$\frac{\partial h}{\partial \gamma} \neq 0 \tag{6.36}$$

almost everywhere on the domain of h. Then by (6.33)

$$\gamma = \gamma_w(p_s, e), \tag{6.37}$$

which, when substituted into (6.34), gives

$$\phi = \hat{\phi}(\gamma_w(p_s, e), \hat{\theta}(\gamma_w(p_s, e), e)), \tag{6.38}$$

or

$$\phi = g(p_s, e), \tag{6.39}$$

which is the relation (5) of HERRMANN [1969].

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