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Originals

On contact interaction between a moving massive body and a linear elastic half space

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We study a class of problems involving the motion of a linear elastic body in frictional contact with a linear elastic half space. The dynamic effects considered are the inertial properties of the body regarded as rigid. We study only those regimes of contact interaction for which the slip velocity with the body taken as absolutely rigid and the time rate of change of the elastic displacements of points of the body and the half space that are on the contact surface are of the same order of magnitude. This work generalizes previous work on similar problems in that we simultaneously consider inertia forces of the body and the convective term in the slip-velocity due to the rigid-body velocity of the slider/indentor. Thus regimes of contact interaction investigated include rolling/sliding and shift-torsion type. We propose a variational formulation of the following two problems: (a) finite contact area and shift-torsion type of contact kinematics, (b) local contact area and general kinematics at the contact surface. Results for an elastic cylinder contacting an elastic half-plane are also given.

1 Introduction

Three-dimensional problems involving frictional-contact interaction between an elastic body moving on another elastic body are important both from theoretical and practical points of view. Their engineering importance derives from the need to simulate wear, fatigue, and energy losses in moving joints of different mechanical devices. Such problems are interesting mathematically due to the nonclassical boundary conditions involved and the evolutionary character of surface tractions and the slip velocity at the contacting surfaces.

The quasistatic indentation of a body into an elastic base in the presence of frictional forces has been studied by Spence [1] and Turner [2]. Mindlin and Deresiewicz [3] assumed local interaction between the contacting bodies and accounted for the possibility of slipping between them. Dundurs and Comninou

[4] studied nonlocal contact between surfaces of a semi-infinite cut in an infinite elastic medium when the interaction between them is shift/indentation/separation type, and Lubkin [5] has studied the torsion regime of interaction. Here torsion signifies the quasistatic rotation of the body about an axis perpendicular to the contact surface between the two bodies. Steady rolling/sliding problems have been investigated by DePater [6], Johnson [7], and Kalker [8], and Kalker [9] has also examined transient rolling/sliding problems. In all of these studies the effect of inertia forces of moving bodies on the contact interaction between them was neglected. Oden and Martins [10], Kikuchi and Oden [11], and Panagiotopoulos [12] have examined dynamic frictional contact problems. The numerical solution of contact problems for stick/shift oscillation regimes of contact is given in [10] and [11]. Wang and Knothe [13] have shown that the effect of dynamic terms in Lamé's equations of motions on the evolution of field variables at the contact surface is negligible for many problems of practical interest.

Here we study the problem of frictional contact interaction between a linear elastic body and a linear elastic half-space when inertial forces due to rigid motion of the body influence the tractions and the slip velocity of points on the contact surface. However, we neglect the dynamic terms in the Lamé equations for the elastic deformations of the body and the half-space. The expression for the slip velocity at a point on the contact surface has three dominant terms in it, namely, the slip velocity of a body point when the body is regarded as rigid, the local rate of change of the relative tangential displacement, and the convective rate of change of the relative tangential displacement of contacting points of the body and the half-space. Different regimes of contact interaction such as shift, torsion, gross sliding and stationary rolling etc. are special cases of the aforestated expression for the slip velocity. We consider two cases, namely, finite contact area but the convective rate of change of the relative tangential displacements is negligible, and local contact area (i.e. the characteristic dimension of the contact area is much smaller than a typical dimension of the body) and general kinematics at the contact surface. For the former case, the variational formulation of the problem is a minimax problem and for the latter it is a pure minimization problem. A major advantage of the variational formulation is that the problem of the minimization of a convex functional on a convex set is a standard problem in convex analysis and mathematical programming. Thus, known methods for the analytical and numerical solutions of the problems can be employed.

In deriving the variational formulation, the governing equations are discretized in time. Ideally one should derive a variational formulation of the non-discretized problem. However, this is difficult due to the simultaneous occurrence of nonlinear constraints of obstacle-type and nonconvexity of the action functional. Nonconvexity occurs in descriptions of propagation of patterns in space-time. This and the obstacle-constraints lead to collision-type events thus making the solution highly irregular. Mathematically, the nonconvex functional is no longer the tensor product of convex functionals as in the constraint-free case, and the problem becomes very difficult, if not impossible, to analyze.

We note that Point [14] has studied the peeling off of a thin viscoelastic solid glued to a fixed surface. She has proposed a phenomenological adhesion model in which the rate of adhesion is described by a function varying between 0 and 1; separation occurs when the function equals 0. The problem is treated as quasistatic; thus the crack-propagation during the peeling process is the only source of nonconvexity. Mathematically, the convexity comes from a characteristic function of a nonconvex set described by using the adhesion rate function in the action functional. Point approximates the characteristic function by smooth functions, shows the existence of a stationary point of the approximate functional and its convergence in a sufficiently weak topology. Our work is comparable to that of Point in the sense that in both papers the bodies are modeled as stationary and only finitely many degrees of freedom show propagation patterns: the rigid motion of the body in our work and the crack propagation in Point's work. Because of a simpler model used, Point proves the existence of weak solutions but we do not.

An example problem of an elastic cylinder rolling on an elastic half-plane is studied in detail, and numerical results for a range of values of different parameters are given.

2 Formulation of the problem

We use three rectangular Cartesian coordinate systems, as shown in Fig. 1, to study the contact interaction between an anisotropic and nonhomogeneous linear elastic body, usually referred to as an indentor/slider, and another isotropic and homogeneous linear elastic body with a flat top surface hereafter referred to as the base or the substrate. The bottom surface of the substrate is kept fixed. The coordinate system $OX_1X_2X_3$ is fixed in space, and the other two coordinate systems $C\overline{x}_1\overline{x}_2\overline{x}_3$ and $Cx_1x_2x_3$ have their origins at the center of mass C of the body and move with it. The coordinate axes $C\overline{x}_1\overline{x}_2\overline{x}_3$ always coincide with the principal axes of inertia of the undeformed body, and the coordinate axes $Cx_1x_2x_3$ always stay parallel to those of the system $OX_1X_2X_3$. In addition to the tractions $\boldsymbol{f}_s$ acting on the contact surface, we assume that the base deforms due to time-independent body forces $\boldsymbol{F}^-$ and the body due to external body forces $\boldsymbol{F}^+$ and inertia forces associated with its rigid motion.

The position of the body regarded as rigid is determined by the radius vector $\boldsymbol{R}$ of its center of mass with respect to the coordinate system $X_1X_2X_3$ and by the direction cosines α_{ij} $(i, j = 1, 2, 3)$ of the axes $O\overline{x}_1\overline{x}_2\overline{x}_3$ with respect to the axes $X_1X_2X_3$ or $Ox_1x_2x_3$. We assume that elastic deformations of the base and the body are infinitesimal so that linear kinematic relations apply. However, displacements of body points due to itsrigid motion may be finite. Suppose that for every body position $(\boldsymbol{R}, \boldsymbol{\alpha})$ we know some domain S_c in the plane $X_3 = 0$ which includes sets S_c^+ and S_c^- of surface particles, belonging to the contact surface in the deformed state, of the body and the substrate, respectively. Henceforth, superscripts + and – affixed to a quantity refer to the value of that quantity for the body and the substrate, respectively.

At a point of S_c^+, let

$$\boldsymbol{f}_s^+ = (p^+, \tau_{31}^+, \tau_{32}^+) \equiv (p^+, \boldsymbol{\tau}^+), \tag{2.1}$$

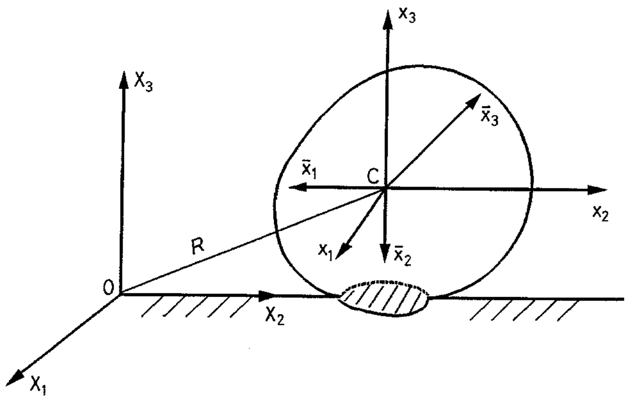

Fig. 1. Schematic sketch of the problem studied, and three sets of coordinate axes used

where p^+ is the normal pressure and $\boldsymbol{\tau}^+$ the tangential traction acting there. Then

$$p^+ = 0 \quad \text{if } X_3^+ - X_3^- > 0, \tag{2.2}$$

$$p^+ \geq 0 \quad \text{if } X_3^+ - X_3^- = 0, \tag{2.3}$$

$$|\boldsymbol{\tau}^+| \leq f(p^+) \quad \text{if } |\boldsymbol{s}| = 0, \tag{2.4}$$

$$\boldsymbol{\tau}^+ = -f(p^+)\frac{\boldsymbol{s}}{|\boldsymbol{s}|} \quad , \text{if } |\boldsymbol{s}| > 0. \tag{2.5}$$

Here X_3^+ and X_3^- are, respectively, the X_3-coordinate of points of the body and the substrate that have the same values of X_1 and X_2 coordinates, $\boldsymbol{s}$ is the slip velocity of a body point with respect to the base, and the function $f(p^+)$ characterizes the frictional force between the body and the substrate. When Coulomb's friction law is assumed to hold, $f(p^+) = \mu p^+$ where μ is the coefficient of friction between the two contacting bodies. Here we assume that f is a smooth function of p and $\partial f/\partial p$ is bounded. Condition (2.2) holds at points of S_c where the two bodies are not in contact with each other, and at the remainder of S_c the normal pressure is positive but the tangential tractions are given by (2.4) if there is no slipping between the two bodies, and by (2.5) when the body is slipping over the base. Let $x_3 = x_3\ (x_1, x_2, \boldsymbol{\alpha})$ be the parametric equation in the undeformed configuration, (see Fig. 2), of the surface of the body containing S_c^+. Then

$$X_3^+ - X_3^- = x_3(x_1, x_2, \boldsymbol{\alpha}) + R_3 + u_3^+ - u_3^-. \tag{2.6}$$

The slip velocity $\boldsymbol{s}$ can be written as

$$\boldsymbol{s} = \boldsymbol{v} + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}^+ - \boldsymbol{u}^-), \tag{2.7}$$

where $\boldsymbol{v}$ equals the slip velocity when both the body and the base are regarded as absolutely rigid. In linear elasticity, one usually sets $\frac{\mathrm{d}}{\mathrm{d}t}(\cdot) = \partial(\cdot)/\partial t$. However,

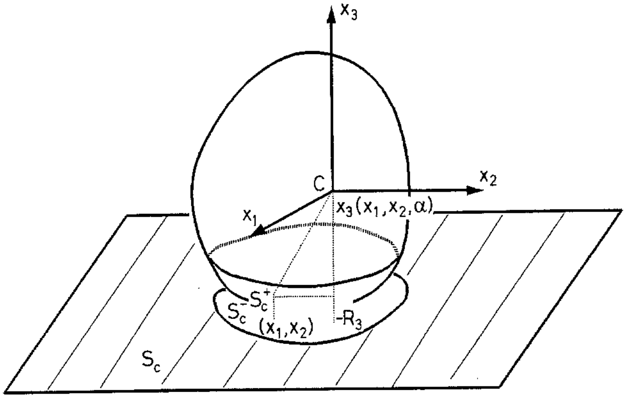

Fig. 2. The positions of surface points in the vicinity of the contact area

for the present problem, the deformations of the body consist of its infinitesimal elastic deformations superimposed upon its finite rigid displacements. Thus, for any function $h(x_1, x_2, x_3, t)$ defined on the contact surface,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial h}{\partial t} + V_1 \frac{\partial h}{\partial x_1} + V_2 \frac{\partial h}{\partial x_2}, \tag{2.8}$$

where V_1 and V_2 are the x_1- and x_2-components, respectively, of the velocity of a body point on the contact surface when the body is regarded as absolutely rigid. Hereafter $\boldsymbol{V}$ is referred to as the rigid-velocity of a body point. Note that $|V_3| \ll |V_1|$, $|V_3| \ll |V_2|$, and therefore the corresponding term involving V_3 is omitted in (2.8).

The substrate is assumed to be a linear elastic half space, therefore,

$$\boldsymbol{u}^- = \boldsymbol{u}_b^-(\boldsymbol{F}^-) + \boldsymbol{u}_t^-(\boldsymbol{f}_s^-), \tag{2.9}$$

where $\boldsymbol{u}_b^-$ and $\boldsymbol{u}_t^-$ denote displacements of a point of the substrate due to body force $\boldsymbol{F}^-$ and surface tractions $\boldsymbol{f}_s^-$, respectively. The functional relations $\boldsymbol{u}_b^-(\boldsymbol{F}^-)$ and $\boldsymbol{u}_t^-(\boldsymbol{f}_s^-)$ are determined, respectively, by Mindlin's [15] and Boussinesq-Cerruti's [15] solutions. Substitution from (2.8) and (2.9) into (2.7) yields

$$\boldsymbol{s} = \boldsymbol{v} + \frac{\partial}{\partial t}[\boldsymbol{u}^+ - \boldsymbol{u}_b^- - \boldsymbol{u}_t^-] + (\boldsymbol{V}^+ \cdot \mathrm{grad})\boldsymbol{u}^+ - (\boldsymbol{V}^- \cdot \mathrm{grad})\boldsymbol{u}^-. \tag{2.10}$$

Since $\| \mathrm{grad}\, \boldsymbol{u}^{\pm} \| \ll 1$, for every term on the right hand side of (2.10) to be of the same order of magnitude,

$$|\boldsymbol{v}| \ll |\boldsymbol{V}^{\pm}|, \tag{2.11}$$

and

$$\boldsymbol{V}^+ \approx \boldsymbol{V}^- \equiv \boldsymbol{V}. \tag{2.12}$$

The body force $\boldsymbol{F}^-$ is assumed not to depend upon time t, hence,

$$\frac{\partial \boldsymbol{u}_b^-}{\partial t} = 0. \tag{2.13}$$

Combining (2.10), (2.11), (2.12), and (2.13), we obtain

$$s = v + \frac{\partial}{\partial t}(u^+ - u_t^-) + (V \cdot \text{grad})(u^+ - u^-). \tag{2.14}$$

We now characterize contact regions according to the magnitude of V relative to that of v. In regimes of shift-torsion type we have

$$|V| \sim |v| \sim \left|\frac{\partial u^\pm}{\partial t}\right|, \tag{2.15}$$

and hence equation (2.14) reduces to

$$s = v + \frac{\partial}{\partial t}(u^+ - u_t^-). \tag{2.16}$$

When

$$|v| \sim |V| \gg \left|\frac{\partial u^\pm}{\partial t}\right|, \tag{2.17}$$

the effect of elastic deformations of the body and the substrate on the slip velocity is negligible, and

$$s = v. \tag{2.18}$$

In this case we call the sliding region as the region of gross sliding, and in steady regimes,

$$s = v + (V \cdot \text{grad})(u^+ - u^-). \tag{2.19}$$

Equations determining the rigid motion of the body are

$$M\ddot{R} = F + \int_{S_c} f_s^+ \, \mathrm{d}A, \tag{2.20}$$

$$I\dot{\Omega} + \Omega \times I\Omega = T + \int_{Sc} \bar{r} \times f_s^+ \mathrm{d}A, \tag{2.21}$$

where M is the mass of the body, F and T are resultant forces and moments due to external forces, except those on S_c, applied to the body, Ω is its angular velocity, I the inertia tensor with respect to principal axes embedded in the body, and $\bar{r}$ the position vector of a point in the body with respect to $O\bar{x}_1\bar{x}_2\bar{x}_3$ coordinate system.

The deformations of the body are governed by

$$(A_{ijk\ell}\varepsilon_{k\ell})_{,j} + F_i^+ = \varrho a_i^+, \quad (i = 1, 2, 3), \tag{2.22}$$

$$\varepsilon_{k\ell} = (u_{k,\ell}^+ + u_{\ell,k}^+)/2, \quad A_{ijk\ell} = A_{jik\ell} = A_{ij\ell k}. \tag{2.23}$$

Here $A_{ijk\ell}$ are elastic moduli of the material of the body, ϱ is its mass density, a repeated index implies summation over the range of the index, and

$$a^+ = \dot{\Omega} \times \bar{r} + \Omega \times (\Omega \times \bar{r}) + \ddot{R}. \tag{2.24}$$

3 A solution technique for the problem

Recalling that we are interested in finding tractions at the contact surface and elastic displacements of the body at any time t, we first write the governing equations in the finite difference form, and then give an equivalent variational formulation of the problem. Knowing the solution at time t, we show how to find the solution at time $t+\Delta t$. Since the initial conditions are known, therefore, the solution can be obtained by marching forward in time. Let the interval $[0, T]$ of interest be divided into N equal subintervals of length $\Delta t = T/N$, and denote $g(K\Delta t)$ by g_K. We use the backward-difference method to approximate the time derivatives of g, i.e.,

$$\dot{g}\big((K+1)\Delta t\big) = (g_{K+1} - g_K)/\Delta t, \tag{3.1}$$

$$\ddot{g}\big((K+1)\Delta t\big) = (g_{K+1} - 2g_K + g_{K-1}/\Delta t^2. \tag{3.2}$$

With such substitutions, the equations of motion (2.22), and boundary conditions (2.2) – (2.5) can be written as follows.

$$\begin{aligned}\frac{\partial}{\partial x_j}\big(A_{ijmn}\varepsilon_{mn}(u^+_{K+1})\big) + (F^+_{K+1})_i - \varrho\Big\{\big[\dot{\boldsymbol{\Omega}}(\boldsymbol{\alpha}_{K+1}, \boldsymbol{\alpha}_K, \boldsymbol{\alpha}_{K-1}) \times \bar{\boldsymbol{r}}\big]\\ + \boldsymbol{\Omega}(\boldsymbol{\alpha}_{K+1}, \boldsymbol{\alpha}_K) \times \big[\boldsymbol{\Omega}(\boldsymbol{\alpha}_{K+1}, \boldsymbol{\alpha}_K) \times \bar{\boldsymbol{r}}\big]\Big\}_i - \\ \varrho(\boldsymbol{R}_{K+1} - 2\boldsymbol{R}_K + \boldsymbol{R}_{K-1})_i/\Delta t^2 = 0, \quad (i = 1, 2, 3),\end{aligned} \tag{3.3}$$

$$p^+_{K+1} = 0 \quad \text{if } \big(X^+_{K+1}\big)_3 - \big(X^-_{K+1}\big)_3 > 0, \tag{3.4}$$

$$p^+_{K+1} \geq 0 \quad \text{if } \big(X^+_{K+1}\big)_3 - \big(X^-_{K+1}\big)_3 = 0, \tag{3.5}$$

$$|\boldsymbol{\tau}^+_{K+1}| \leq f(p^+_{K+1}) \quad \text{if } |\Delta s_{K+1}| = 0, \tag{3.6}$$

$$\boldsymbol{\tau}^+_{K+1} = -f(p^+_{K+1})\Delta s_{K+1}/|\Delta s_{K+1}| \quad \text{if } |\Delta s_{K+1}| > 0, \tag{3.7}$$

where

$$\begin{aligned}\Delta s_{K+1} = &\Delta t\big[\boldsymbol{v}_{K+1} + (\boldsymbol{V}_{K+1} \cdot \text{grad})(\boldsymbol{u}^+_{K+1} - (\boldsymbol{u}^-)_{K+1})\big]\\ &+ \boldsymbol{u}^+_{K+1} - \boldsymbol{u}^+_K - (\boldsymbol{u}^-_t)_{K+1} + (\boldsymbol{u}^-_t)_K,\end{aligned} \tag{3.8}$$

is obtained from (2.14) for a general contact regime. We note that the displacements $\boldsymbol{u}^-_{K+1}$ of the substrate can be expressed as functions of p_{K+1} and $\boldsymbol{\tau}_{K+1}$ by Boussinesq-Cerruti's formulae [15]. Equations (2.20) and (2.21) for the determination of $\boldsymbol{R}$ and $\boldsymbol{\alpha}$ corresponding to the rigid motion of the body take the form

$$M\frac{\boldsymbol{R}_{K+1} - 2\boldsymbol{R}_K + \boldsymbol{R}_{K-1}}{\Delta t^2} = \boldsymbol{F}_{K+1} + \int_{S_c} (f^+_s)_{K+1}\,\mathrm{d}A, \tag{3.9}$$

$$\begin{aligned}&\boldsymbol{I}\dot{\boldsymbol{\Omega}}(\boldsymbol{\alpha}_{K+1}, \boldsymbol{\alpha}_K, \boldsymbol{\alpha}_{K-1}) + \boldsymbol{\Omega}(\boldsymbol{\alpha}_{K+1}, \boldsymbol{\alpha}_K) \times \boldsymbol{I}\boldsymbol{\Omega}(\boldsymbol{\alpha}_{K+1}, \boldsymbol{\alpha}_K) = \\ &\boldsymbol{T}_{K+1} + \int_{S_c} \overline{\boldsymbol{r}}_{K+1} \times (\boldsymbol{f}_s^+)_{K+1}\, \mathrm{d}A.\end{aligned} \tag{3.10}$$

We first give a variational formulation of the problem defined by equations (3.3) through (3.10) for regimes of shift-torsion type for which the slip velocity is given by equation (2.16). Subsequently, we will study the case of general motion but involving local interaction at the contact surface. We introduce the following functionals.

$$\mathcal{F}(\boldsymbol{u}_{K+1}^+) = \int_{\mathfrak{R}} \left[\frac{1}{2} A_{ijmn} \varepsilon_{ij}(\boldsymbol{u}_{K+1}^+) \varepsilon_{mn}(\boldsymbol{u}_{K+1}^+) - (F_{K+1}^*)_i (u_{K+1}^+)_i\right] \mathrm{d}V, \tag{3.11}$$

$$\begin{aligned}\mathcal{E}\big((\boldsymbol{f}_s^-)_{K+1}, \boldsymbol{u}_{K+1}^+\big) = \int_{Sc} \Big[& 0.5(\boldsymbol{u}_t \cdot \boldsymbol{f}_s^-)_{K+1} + (\boldsymbol{u}_b \cdot \boldsymbol{f}_s^-)_{K+1} \\ & + \big((u_{K+1}^+)_3 + x_3(x_1, x_2, \boldsymbol{\alpha}_{K+1}) + (R_3)_{K+1}\big) p_{K+1}^- \\ & + (\boldsymbol{u}_k^- - \boldsymbol{u}_K^+ - \boldsymbol{u}_{K+1}^+ - \boldsymbol{v}_{K+1}\Delta t) \cdot \boldsymbol{\tau}_{K+1}^- \Big] \mathrm{d}A,\end{aligned} \tag{3.12}$$

$$L(\boldsymbol{f}_s^-, \boldsymbol{u}^+) = \mathcal{F}(\boldsymbol{u}^+) - \mathcal{E}(\boldsymbol{f}_s^-, \boldsymbol{u}^+), \tag{3.13}$$

where $\boldsymbol{F}^*$ equals the sum of the inertia forces and body forces. The volume integration in (3.11) and henceforth is over the present region $\mathfrak{R}$ occupied by the body. In order to shorten the notation we drop the index $K+1$ but keep the index K. We investigate the following minimax problem. Find

$$\sup_{f_s^-} \inf_{u^+} L(\boldsymbol{f}_s^-, \boldsymbol{u}^+) \quad \forall \quad \boldsymbol{f}_s^- \in \mathfrak{Z}, \tag{3.14}$$

where

$$\mathfrak{Z} = \{(p, \boldsymbol{\tau}) \mid p \geq 0, \quad |\boldsymbol{\tau}| \leq f(p)\}, \tag{3.15}$$

and prove its equivalence to the problem defined by equations (3.3) – (3.7).

Since there are no constraints imposed on $\boldsymbol{u}^+$, therefore,

$$\begin{aligned}\delta_{u^+} L = -\int_{\mathfrak{R}} \left[\frac{\partial}{\partial x_j}\big(A_{ijmn}\varepsilon_{mn}(\boldsymbol{u}_0^+)\big) + F_i^*\right] (u_i^+ - u_{oi}^+)\, \mathrm{d}V + \\ \int_{S_c} \left[\boldsymbol{f}_s^+(\boldsymbol{u}_0^+) + \boldsymbol{f}_s^-\right] \cdot (\boldsymbol{u}^+ - \boldsymbol{u}_0^+)\, \mathrm{d}A = 0.\end{aligned} \tag{3.16}$$

This implies that satisfaction of Lamé's equations (3.3) and

$$\boldsymbol{f}_s^+(\boldsymbol{u}_o^+) = -\boldsymbol{f}_s^- \quad \text{on } S_c \tag{3.17}$$

are necessary and sufficient conditions for $\boldsymbol{u}_o^+$ to be a solution of the problem (3.14).

Consider now supremum in (3.14) and the variation of L with respect to f_s^-. For the sake of simplicity we first demonstrate basic calculations for the case

$$f(p) = g(x_1, x_2), \tag{3.18}$$

and then consider the general case. Recalling that $f(p)$ is the limiting frictional force between two contacting bodies, equation (3.18) models the case when a thin layer of material, such as a lubricant, adjoining the contact surface has undergone plastic deformations.

Note that the first term under the integral sign on the right-hand side of equation (3.12) is quadratic in f_s^- due to the linear dependence of $\boldsymbol{u}_t$ on f_s^- and represents the potential energy of the substrate. Hence this term is positive and the functional $\mathscr{E}$ is convex. With $f(p)$ given by (3.18), the set $\mathfrak{Z}$ of admissible f_s^- is convex. Therefore, the necessary and sufficient condition for supremum of L with respect to f_s^- is

$$\delta_{f_s^-}\mathscr{E} \geq 0 \quad \forall \quad f_s^- \in \mathfrak{Z}, \tag{3.19}$$

which can be written as

$$\begin{aligned} \delta_{f_s^-}\mathscr{E} = \int_{S_c} \big\{ \big(-(u_t^-)_3 - (u_b^-)_3 + (u_o)_3^+ + x_3(x_1, x_2, \alpha) + R_3\big)(p - p_o) + \\ (\boldsymbol{u}_t^- + \boldsymbol{u}_b^- - \boldsymbol{u}_o^+ - \boldsymbol{u}_K^- + \boldsymbol{u}_K^+ - \boldsymbol{v}\Delta t) \cdot (\boldsymbol{\tau} - \boldsymbol{\tau}_o) \big\} \, \mathrm{d}A \geq 0 \quad \forall \quad f_s^- \in \mathfrak{Z}. \end{aligned} \tag{3.20}$$

Consider now the variation in (3.20) with respect to p only. In this case the problem defined by (3.20) becomes: find p_o such that

$$\ell(p_o) = \inf_{p \geq 0} \ell(p) \tag{3.21}$$

where

$$\ell(p) = \int_{S_c} \big(-(u_t^-)_3 - (u_b^-)_3 + (u_o)_3^+ + x_3(x_1, x_2, \boldsymbol{\alpha}) + R_3\big) p \, \mathrm{d}A. \tag{3.22}$$

If $(\boldsymbol{u}_o^+, p_o)$ is the solution of the boundary-value problem defined by equations (2.22), (2.2) and (2.3), then for any $p \geq 0$ the following relations are valid

$$\ell(p) \geq 0, \quad \ell(p_o) = 0. \tag{3.23}$$

Therefore, inequalities (3.20) and (3.21) are satisfied. Suppose now that inequality (3.20) holds. Thus, inequality (3.21) is satisfied, and since $\ell(0) = 0$, we have

$$\ell(p_o) \leq 0. \tag{3.24}$$

Suppose that $\ell(p_o) < 0$. Then for $p = \lambda p_o$, $0 < \lambda < 1$, we get a contradiction with (3.21). Hence $\ell(p_o) = 0$, and $\ell(p) \geq 0 \; \forall \; p \geq 0$. This is the integral form of boundary conditions (2.2) and (2.3), or equivalently (3.4) and (3.5).

We now consider the variation of L with respect to $\boldsymbol{\tau}$ only and rewrite inequality (3.20) as

$$\delta_{\boldsymbol{\tau}}\mathscr{E} = -\int_{S_c} \Delta s_o \cdot (\boldsymbol{\tau} - \boldsymbol{\tau}_o) \, \mathrm{d}A \geq 0 \quad \forall \quad |\boldsymbol{\tau}| \leq g, \tag{3.25}$$

where Δs_o is evaluated at $f_s^- = (f_s^-)_o$. This is equivalent to the variational problem of finding $\boldsymbol{\tau}_o$ such that

$$m(\boldsymbol{\tau}_o) = \sup_{|\boldsymbol{\tau}|\leq g} m(\boldsymbol{\tau}), \tag{3.26}$$

where

$$m(\boldsymbol{\tau}) = \int_{S_c} \Delta \boldsymbol{s}(f_{o_s}^-) \cdot \boldsymbol{\tau}\, \mathrm{d}A. \tag{3.27}$$

If function $\boldsymbol{\tau}_o$ is a solution of equations (3.6) and (3.7), we have

$$m(\boldsymbol{\tau}) \leq m(\boldsymbol{\tau}_o) = \int_{S_c} g|\Delta \boldsymbol{s}(f_{o_s}^-)|\, \mathrm{d}A \quad \forall \quad |\boldsymbol{\tau}| \leq g. \tag{3.28}$$

Thus the variational inequality (3.20) is satisfied. Suppose now that the function $\boldsymbol{\tau}_o$ satisfies (3.20) and hence (3.26). Using results from convex analysis [16], we have

$$\sup_{|\boldsymbol{\tau}|\leq g} \int_{S_c} \boldsymbol{\tau} \cdot \Delta \boldsymbol{s}(f_{o_s}^-)\, \mathrm{d}A = \int_{S_c} g|\Delta \boldsymbol{s} \cdot (f_{o_s}^-)|\, \mathrm{d}A, \tag{3.29}$$

which together with equation (3.26) gives

$$\int_{S_c} g|\Delta \boldsymbol{s} \cdot (f_{o_s}^-)|\, \mathrm{d}A = \int_{S_c} \Delta \boldsymbol{s}_o \cdot \boldsymbol{\tau}_o\, \mathrm{d}A. \tag{3.30}$$

Relation (3.30) together with the inequality $|\boldsymbol{\tau}_o| \leq g$ is the integral form of boundary conditions (3.6) and (3.7).

We now study the case of general friction law and introduce the sequence of minimax problems: find

$$\sup_{f_{s(n)}^-} \inf_{\boldsymbol{u}_{(n)}^+} L(f_{s(n)}^-, \boldsymbol{u}_{(n)}^+) \quad \forall \quad f_{s(n)}^- \in \mathfrak{Z}_{(n)} \tag{3.31}$$

where

$$\mathfrak{Z}_{(n)} = \left\{(p_{(n)}, \boldsymbol{\tau}_{(n)}) \mid p_{(n)} \geq 0, \quad |\boldsymbol{\tau}_{(n)}| \leq f(p_{(n-1)})\right\}, \quad n = 1, 2, \ldots . \tag{3.32}$$

That is, the pressure field computed at the $(n-1)$st iteration determines the surface tractions for the nth iteration. Because of the results proved above, for every n, the minimax problem (3.31) is equivalent to a boundary-value problem. Convergence conditions restrict the dependence of the limiting frictional force upon the contact pressure. Thus $|f(p_n)|$ needs to be bounded above by a constant which depends upon the geometry of the body and the elastic moduli of the body and the substrate. For Coulomb's friction law, this restricts the value of the coefficient of friction. We note that iterative processes similar to that defined by (3.31) and (3.32) have been employed by Panagiotopoulos [12], Kravchuk [17], and Kalker [8]. We refer the reader to these references for details of the convergence of the iterative process.

4 Local contact between isotropic and homogeneous bodies

We now consider the case when the characteristic dimension of the domain S_c is much smaller than a typical dimension of the body. Suppose that the external forces act at points far from the contact area and do not influence much the elastic displacements at points of the body that are close to the contact surface. Due to the localized nature of the contact area, we can use a half-space approximation (e.g. see Kalker [8]) for displacements of body points adjacent to the contact surface. Thus

$$u_3^+ - u_3^- = B_{31}(p) + \kappa B_{32}(\boldsymbol{\tau}) - (u_b^-)_3, \tag{4.1}$$

$$u_\beta^+ - (u_t^-)_\beta = (\kappa \boldsymbol{B}_{21}(p) + \boldsymbol{B}_{22}(\boldsymbol{\tau}))_\beta, \quad \beta = 1, 2, \tag{4.2}$$

where B_{31}, B_{32}, $\boldsymbol{B}_{21}$, etc. are Boussinesq-Cerruti's integral operators [15], and

$$\kappa = \frac{1-2\nu^+}{G^+} - \frac{1-2\nu^-}{G^-}, \tag{4.3}$$

$\nu^\pm$ and $G^\pm$ being Poisson's ratio and the shear modulus, respectively, for the material of the body and the substrate.

Substitution from (4.2) into (3.8) yields

$$\Delta \boldsymbol{s} = \Delta \boldsymbol{v}^* + \boldsymbol{B}(\boldsymbol{\tau}) - \boldsymbol{B}^*(\boldsymbol{\tau}) \tag{4.4}$$

where

$$\Delta \boldsymbol{v}^* = \boldsymbol{v}\Delta t + \kappa(\Delta t \boldsymbol{V} \cdot \operatorname{grad})\boldsymbol{B}_{21}(p) + \kappa \boldsymbol{B}_{21}(p) - \boldsymbol{u}^+ + \boldsymbol{u}_t^-, \tag{4.5}$$

$$\boldsymbol{B}^*(\cdot) = -\Delta t(\boldsymbol{V} \cdot \operatorname{grad})\boldsymbol{B}_{22}(\cdot), \quad \boldsymbol{B}(\cdot) = \boldsymbol{B}_{22}(\cdot). \tag{4.6}$$

We do not need equations (3.3) since displacements within the body caused by surface tractions at the contact surface are determined by Boussinesq-Cerruti's formulae [15].

We now obtain the variational problem for the determination of tractions at the contact surface. For this case we have a pure minimization problem. When $\kappa = 0$ the problem for the determination of the pressure field at the contact surface can be considered separately from that for the determination of friction forces there and has the form

$$\inf_{p \ge 0} \left\{ f(p) = \int_{S_c} \left[\frac{1}{2} B_{11}(p) p + \left(u_3^+ + x_3(x_1, x_2, \boldsymbol{\alpha}) + R_3 - u_3^- \right) p \right] \mathrm{d}A \right\}. \tag{4.7}$$

The functional $f(p)$ is convex and, therefore, the inequality $\delta f \ge 0 \, \forall \, p$ is the necessary and sufficient condition for f to take the minimum value. This inequality is similar to (3.21). Hence by following arguments similar to those given for inequality (3.21) we can show the equivalence of problem (4.7) to the boundary conditions at the contact surface.

Having determined the pressure field at the contact surface from (4.7), the tangential tractions there can be found from

$$\inf_{|\boldsymbol{\tau}|\le f(p)} \left\{ g(\boldsymbol{\tau}) = \int_{S_c} \left[f(p)|\Delta \boldsymbol{s}(\boldsymbol{\tau})| - \boldsymbol{\tau}\cdot\Delta \boldsymbol{s}(\boldsymbol{\tau}) \right] \mathrm{d}A \right\} \tag{4.8}$$

where $\Delta \boldsymbol{s}$ is given by (4.4). Regarding (4.4) as a constraint, we introduce the Lagrangian

$$\begin{aligned}\mathfrak{L}(\boldsymbol{\tau}, \Delta \boldsymbol{s}, \boldsymbol{\lambda}) = \int_{S_c} & \left[f(p)\,|\,\Delta \boldsymbol{s}(\boldsymbol{\tau})| - \boldsymbol{\tau}\cdot\Delta \boldsymbol{v}^* - \boldsymbol{\tau}\cdot B(\boldsymbol{\tau}) \right. \\ & \left. -\boldsymbol{\lambda}\cdot\left(\Delta \boldsymbol{s} - \Delta \boldsymbol{v}^* - \boldsymbol{B}(\boldsymbol{\tau}) + \boldsymbol{B}^*(\boldsymbol{\tau})\right)\right] \mathrm{d}A\end{aligned} \tag{4.9}$$

where $\boldsymbol{\lambda}$ is a Lagrange multiplier. By varying the argument $\boldsymbol{\tau}$ in $\mathfrak{L}(\boldsymbol{\tau}, \Delta \boldsymbol{s}, \boldsymbol{\lambda})$, we obtain the following condition for the minimum of $\mathfrak{L}(\boldsymbol{\tau}, \Delta \boldsymbol{s}, \boldsymbol{\lambda})$ at the point $\boldsymbol{\tau} = \boldsymbol{\tau}_o$, $\boldsymbol{s} = \boldsymbol{s}_o$.

$$-\int_{S_c} \boldsymbol{b}\cdot(\boldsymbol{\tau} - \boldsymbol{\tau}_o)\,\mathrm{d}A \ge 0 \quad \forall \quad |\boldsymbol{\tau}| \le f(p), \tag{4.10}$$

where

$$\boldsymbol{b} = \Delta \boldsymbol{s}_0 + \boldsymbol{B}(\boldsymbol{\tau}_o - \boldsymbol{\lambda}) + \boldsymbol{B}^*(\boldsymbol{\tau}_o - \boldsymbol{\lambda}). \tag{4.11}$$

The inequality (4.10) is similar to (3.25) studied above, and therefore we have

$$\int_{S_c} \boldsymbol{\tau}_o\cdot\boldsymbol{b}\,\mathrm{d}A = \int_{S_c} f(p)|\boldsymbol{b}|\,\mathrm{d}A, \tag{4.12}$$

or in local form

$$f(p)|\boldsymbol{b}| - \boldsymbol{\tau}_o\cdot\boldsymbol{b} = 0. \tag{4.13}$$

If we vary the argument $\Delta \boldsymbol{s}$ in $\mathfrak{L}(\boldsymbol{\tau}, \Delta \boldsymbol{s}, \boldsymbol{\lambda})$, the condition for the minimum of $\mathfrak{L}$ becomes

$$\int_{S_c} f(p)\left[|\Delta \boldsymbol{s}| - \boldsymbol{\lambda}\cdot\Delta \boldsymbol{s}\right] \mathrm{d}A - \int_{S_c} \left[f(p)|\Delta \boldsymbol{s}_o| - \boldsymbol{\lambda}\cdot\Delta \boldsymbol{s}_o\right] \mathrm{d}A \ge 0. \tag{4.14}$$

Assume now that for some $\Delta \boldsymbol{s} = \Delta \boldsymbol{s}^*$ the first term on the left-hand side of the inequality (4.14) is negative. Then for $\Delta \boldsymbol{s} = \omega\Delta \boldsymbol{s}^*$, $\omega > 0$, the sign of the left-hand side of the inequality (4.14) is determined by the first term when ω is sufficiently large. This contradicts the inequality. Hence

$$\int_{S_c} \left[f(p)|\Delta \boldsymbol{s}| - \boldsymbol{\lambda}\cdot\Delta \boldsymbol{s}\right] \mathrm{d}A \ge 0 \quad \forall \quad \Delta \boldsymbol{s} \tag{4.15}$$

or

$$|\boldsymbol{\lambda}| \le f(p). \tag{4.16}$$

For sufficiently small ω the sign of the left-hand side of (4.14) is governed by the second term, and therefore

$$\int_{S_c} \left[f(p)|\Delta \boldsymbol{s}_o| - \boldsymbol{\lambda} \cdot \Delta \boldsymbol{s}_o\right] \, \mathrm{d}A \leq 0 \tag{4.17}$$

which holds only if

$$\int_{S_c} \left[f(p)|\Delta \boldsymbol{s}_o| - \boldsymbol{\lambda} \cdot \Delta \boldsymbol{s}_o\right] \, \mathrm{d}A = 0, \tag{4.18}$$

or equivalently

$$\Delta \boldsymbol{s}_o + \boldsymbol{B}^*(\boldsymbol{\tau}_o - \boldsymbol{\lambda}) = \alpha \boldsymbol{\tau}_o, \quad \alpha \geq 0, \tag{4.19}$$

$$\alpha(|\boldsymbol{\tau}_o| - f(p)) \equiv 0 \quad \text{in } S_c. \tag{4.20}$$

We now consider the quadratic form

$$W(\boldsymbol{\lambda} - \boldsymbol{\tau}_o, \boldsymbol{\lambda} - \boldsymbol{\tau}_o) = \int_{S_c} (\boldsymbol{\lambda} - \boldsymbol{\tau}_o) \cdot \left[\boldsymbol{B}(\boldsymbol{\lambda} - \boldsymbol{\tau}_o) - \boldsymbol{B}^*(\boldsymbol{\lambda} - \boldsymbol{\tau}_o)\right] \, \mathrm{d}A. \tag{4.21}$$

Since operator $\boldsymbol{B}$ is positive and operator $\boldsymbol{B}^*$ is skew symmetric, therefore,

$$W(\lambda - \gamma_o, \lambda - \gamma_o) \geq 0. \tag{4.22}$$

Substitution from (4.19) into (4.21) yields

$$W(\boldsymbol{\lambda} - \boldsymbol{\tau}_o, \boldsymbol{\lambda} - \boldsymbol{\tau}_o) = \int_{S_c} \alpha(\boldsymbol{\lambda} \cdot \boldsymbol{\tau}_0 - \boldsymbol{\tau}_o \cdot \boldsymbol{\tau}_o) \, \mathrm{d}A + \int_{S_c} (\boldsymbol{\tau}_0 \cdot \boldsymbol{s}_o - \boldsymbol{\lambda} \cdot \boldsymbol{s}_o) \, \mathrm{d}A \geq 0. \tag{4.23}$$

Recalling (4.16), (4.19), and (4.20), we conclude that the inequality (4.23) is satisfied only if

$$\int_{S_c} \left[f(p)|\Delta \boldsymbol{s}_o| - \Delta \boldsymbol{s}_o \cdot \boldsymbol{\tau}_o\right] \, \mathrm{d}A = 0. \tag{4.24}$$

Equation (4.24) together with $|\boldsymbol{\tau}_o| \leq f(p)$ is equivalent to the integral form of boundary conditions (2.4) and (2.5).

The inverse assertion that the solution $\boldsymbol{\tau}_o$ of the boundary-value problem minimizes $g(\boldsymbol{\tau})$ follows from the inequalities

$$g(\boldsymbol{\tau}_o) = 0, \quad g(\boldsymbol{\tau}) \geq 0 \quad \forall \quad |\boldsymbol{\tau}| \leq f(p). \tag{4.25}$$

For the general case when $\kappa = \frac{1-2\nu^+}{G^+} - \frac{1-2\nu^-}{G^-} \neq 0$, we construct the following sequence of variational problems:

$$\inf_{p_m \geq 0} \left\{ f(p_m, \boldsymbol{\tau}_{m-1}) = \right. \tag{4.26}$$

$$\left. \int_{S_c} \left[\frac{1}{2} B_{11}(p_m) p_m + \left((u_t)_3^- - u_3^+ - x_3(x_1, x_2, \boldsymbol{\alpha}) - R_3 - \kappa B_{12}(\boldsymbol{\tau}_{m-1})\right) p_m\right] \, \mathrm{d}A \right\}$$

$$\inf_{|\boldsymbol{\tau}_m|\le f(p_m)} \left\{ g(\boldsymbol{\tau}_m, p_m) = \int_{S_c} \left[f(p_m)|\Delta s(\boldsymbol{\tau}_m, p_m)| - \boldsymbol{\tau}_m \cdot \Delta s(\boldsymbol{\tau}_m, p_m) \right] \mathrm{d}A \right\}, \quad (4.27)$$

$m = 1, 2, \ldots$. The iterative process (4.26) and (4.27) is based on the observation that in known solutions of frictional contact problems (e.g. see Johnson [7]), the influence of frictional force, within every iteration, on the difference between the normal displacements of points of the body and the substrate that are on the contact surface is weak. The convergence of the iterative process follows from the works of Panagiotopoulos [12] and Kravchuk [17] which deals with quasistatic frictional contact problems involving two elastic bodies. In order for the iterative process to converge, $|\kappa \partial f/\partial p|$ should be bounded which holds since κ defined by (4.3) is finite and $\partial f/\partial p$ has been assumed to be bounded. We note that for Coulomb's law, $\partial f/\partial p$ equals the coefficient of friction.

5 An example

In order to illustrate some of the aforestated results we study the rolling/sliding ofa cylinder on a half-space, and assume that both are made of the same isotropic and homogeneous material. Also deformations involved are small so that linear elasticity theory can be used. We assume that, in addition to the surface tractions on the contact surface, a time dependent horizontal force T_{ex} and a driving moment M_{ex} are applied at the center of the cylinder (Fig. 3). We study the quasistationary problem and assume that the term involving the time derivatives in the expression for the slip velocity is negligible, but the distribution of contact forces evolves in time due to the inertia properties of the body. Also, Coulomb's friction law with constant coefficient of friction μ is assumed.

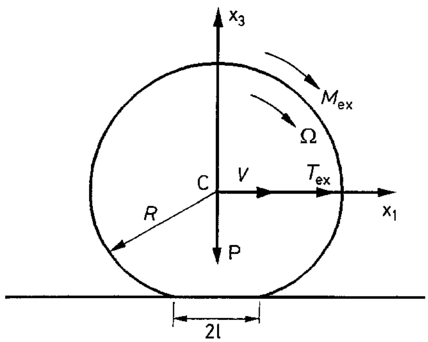

Fig. 3. An elastic cylinder rolling and sliding on an elastic half-space

The distribution of surface tractions at the contact surface can be obtained by using Carter-Poritsky's solution (e.g. see Johnson [7], Kalker [8]). This solution, valid for the stationary case needs either the resultant frictional force or the rigid-slip velocity to be prescribed. For the present problem these values are obtained from the equations of motion of the cylinder. The Carter-Poritsky solution gives

$$\tau_{13} = \begin{cases} -\mu p_o \left(1-\left(\frac{x_1}{\ell}\right)^2\right)^{1/2} & \text{if } -\ell \leq x_1 \leq \ell_a, \\ -\mu p_o \left\{\left(1-\left(\frac{x_1}{\ell}\right)^2\right)^{1/2} - \frac{\ell-d}{\ell}\left(1-\left(\frac{x_1+d}{\ell-d}\right)^2\right)^{1/2}\right\}, & \ell_a \leq x_1 \leq \ell, \end{cases} \tag{5.1}$$

$$\tau_{23} = 0, \tag{5.2}$$

$$v_1^o = V - \Omega R = -\mu\varepsilon V\left[1 - \left(1 + \frac{F}{\mu P}\right)^{1/2}\right], \quad \varepsilon = \frac{\ell}{R}, \tag{5.3}$$

where ℓ is half-length of the contact zone, p_o is the maximum pressure at the contact surface, ℓ_a is the x_1-coordinate of the edge of the adhesion zone, V is the x_1-component of the rigid-velocity of a cylinder point, R is the cylinder radius, P is the total force acting normal to the contact surface, and F is the resultant frictional force. Besides this, we have

$$\ell = \left[\frac{4PR(1-\nu^2)}{\pi E}\right]^{1/2}, \tag{5.4}$$

$$p_o = \left[\frac{PE}{\pi R(1-\nu^2)}\right]^{1/2}, \tag{5.5}$$

$$\ell_a = 2d - \ell, \quad d = \ell\left(1 - \left(1 + \frac{F}{\mu P}\right)^{1/2}\right). \tag{5.6}$$

For the present problem, equations of motion (2.20) and (2.21) take the form

$$M\dot{V} = -F + T_{ex}(t), \tag{5.7}$$

$$I\dot{\Omega} = FR + M_{ex}(t), \tag{5.8}$$

where $I = I_2$, the mass moment of inertia about the axis of the cylinder. Instead of equation (5.8) it is more convenient to consider the following equation (5.9) for the rigid-slip velocity obtained by combining equations (5.3), (5.7), and (5.8)

$$M\dot{v}_1^o = -(1+\eta)F + T_{ex} - \eta\frac{M_{ex}}{R}, \quad \eta \equiv \frac{MR^2}{I}. \tag{5.9}$$

Solving equation (5.3) for F and substituting the result into (5.7) and (5.9), we arrive at

$$M\dot{V} = \mu P\left[\left(1+\frac{v_x^o}{\mu V\varepsilon}\right)^{1/2} - 1\right] + T_{ex}, \tag{5.10}$$

$$M\dot{v}_1^o = (1+\eta)\mu P\left[1-\left(1+\frac{v_x^o}{\mu V\varepsilon}\right)^{1/2}\right] + T_{ex} - \eta\frac{M_{ex}}{R}. \tag{5.11}$$

Once nonlinear equations (5.10) and (5.11) have been solved for V and v_1^o, we can find F from equation (5.3), d from $(5.6)_2$, ℓ from (5.4), ℓ_a from $(5.6)_1$, p_o from (5.5), and $\boldsymbol{\tau}$ from (5.1) and (5.2). Hence the problem has been completely solved. We note that a numerical solution of equations (5.10) and (5.11) can be obtained by using an iterative method.

We present below detailed results for the case

$$T_{ex} = 0, \quad M_{ex}(t) = M_o + \gamma t,$$

where γ is a constant. Figures 4a through 4c exhibit the evolution in time of the dimensionless friction force $\overline{F} \equiv |F|/\mu P$, half-length of adhesion zone $\overline{\ell} \equiv (1-d/\ell)$, and the rigid-slip velocity $\overline{v} = |v_1^o|/\mu\varepsilon V(0)$ for different values of $\overline{\gamma} = \gamma MV(0)/(\mu P)^2 R$. In these figures, the abscissa represents the nondimensional time $\overline{t} = t\mu P/MV(0)$. The evolution of $\overline{F}$, $\overline{\ell}$ and $\overline{v}$ for different values of the dimensionless initial moment $\overline{M_o} = M_o/\mu PR$ is plotted in Figs. 5a through 5c. It is clear that the resultant frictional force and the magnitude of the rigid-slip velocity increase with time when a driving moment is applied to the cylinder. Consequently, the length of the adhesion zone decreases with time and hence that of the slipping zone increases. For the same value of $\overline{t}$, the magnitudes of $\overline{F}$, $\overline{v}$ increase, and that of $\overline{\ell}$ decreases with an increase in the value of $\overline{M_o}$. Figure 6 depicts the influence of inertial properties of the cylinder upon the solution of the problem. The cases $\eta = 1, 2$ and ∞ correspond, respectively, to a hollow cylindrical ring, a homogeneous solid cylinder and a cylinder with all of its mass concentrated at its centroidal axis. It is clear that the mass distribution in the cylinder influences noticeably the evolution of the frictional force at the contact surface.

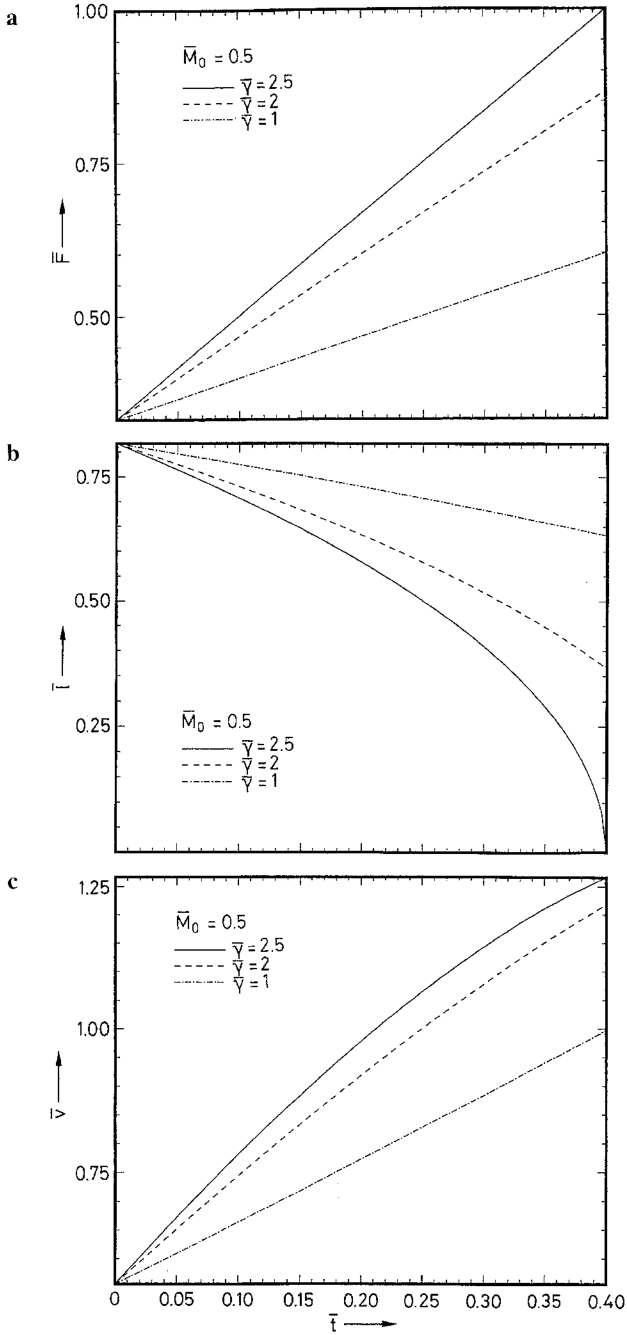

Fig. 4. Evolution of (**a**) friction force, (**b**) half-length of adhesion zone, and (**c**) the rigid slip velocity for different values of the rate of increase of the applied moment

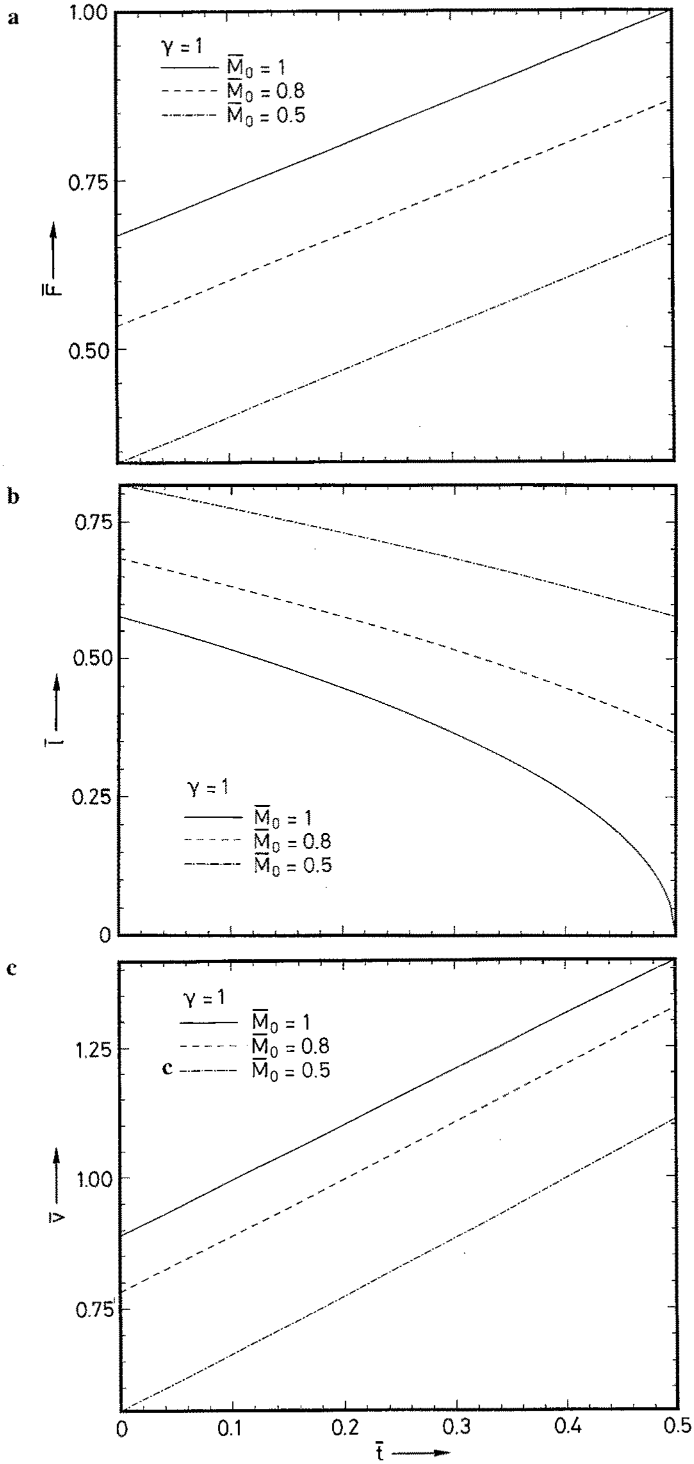

Fig. 5. Evolution of (**a**) friction force, (**b**) adhesion zone, and (**c**) the rigid slip velocity for different initial values of the applied moment.

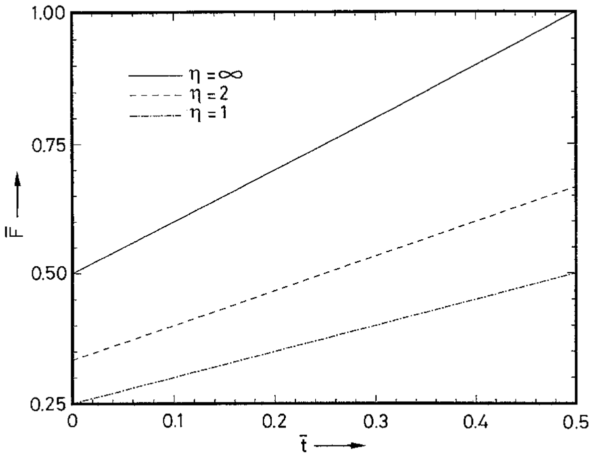

Fig. 6. Evolution of the frictional force for a homogeneous cylinder, a non-homogeneous cylinder with entire mass concentrated at the center, and a homogeneous circular cylindrical ring.

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References

1. Spence, D. A.: The Hertz contact problem with finite friction, J. of Elasticity 5 (1975) 297–319
2. Turner, J. R.: The frictional unloading problem on linear elastic half-space, J. Inst. Maths. Appl. 24 (1979) 439–469
3. Mindlin, R. D.; Deresiewicz, H.: Elastic spheres in contact under varying oblique forces, Journ. Appl. Mech. 20 (1953) 327–344
4. Dundurs, J.; Comninou, M.: An educational elasticity problem with friction, Part 3, General load paths, Journ. Appl. Mech. 50 (1983) 77–94
5. Lubkin, J. L.: Torsion of elastic spheres in contact, Journ. Appl. Mech. 18 (1951) 183–187
6. DePater, A. D.: In *Rolling Contact Phenomena,* ed. Bidwell, New York: Elsevier
7. Johnson, K. L.: Contact Mechanics, Cambridge University Press (1985)
8. Kalker, J. J.: Three-Dimensional Elastic Bodies in Rolling Contact, Kluwer (1990)
9. Kalker, J. J.: Transient phenomenon in two elastic cylinders rolling over each other with dry friction, J. Appl. Mech. 37 (1970) 677–689
10. Oden, J. T.; Martins, J. A. C.: Models and computational methods for dynamic friction phenomena, Comput. Meth. Appl. Mech. Engr. 52 (1985) 527–634
11. Kikuchi, N.; Oden, J. T.: Contact Problems in Elasticity, A Study of Variational Inequalities and Finite Element Methods, SIAM Studies in Applied Mathematics, Philadelphia (1988)

12. Panagiotopoulos, P. D.: Inequality Problems in Mechanics and Applications. Convex and Nonconvex Energy Functions, Birkhäuser Verlag, Basel, Boston (1985)
13. Wang, G.; Knothe, K.: The influence of inertia forces on steady state rolling contact, Acta Mechanica 79 (1989) 221 – 232
14. Point, N.: Unilateral Contact with Adherence, Mathematical Methods in the Appl. Sciences, 10 (1988) 367 – 381
15. Love, A. E. H.: A Treatise on the Mathematical Theory of Elasticity, 4th Edition, New York: Dover (1944)
16. Ekeland, I.; Temam, R.: Convex Analysis and Variational Problems, North Holland, Amsterdam (1976)
17. Kravchuk, A. S.: On the theory of contact problems taking account of friction on the contact surface, PMM (Appl. Math. Mech.) 44 (1980) 83 – 88

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