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A THEOREM IN THE THEORY OF INCOMPRESSIBLE NAVIER-STOKES-FOURIER FLUIDS

Nota di R. C. Batra

Istituto Lombardo di Scienze e Lettere
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a uniform value θ_0? In an attempt to answer this question, I show that the velocity $\mathbf{v}$ and the temperature θ approach in L^2 norm the values $\mathbf{O}$ and θ_0, respectively, as $t \to \infty$, provided the shear viscosity, the conductivity and the specific heat of the fluid satisfy certain conditions. This result holds irrespectively of what the motion and the temperature is at the initial instant of time. Actually the nature of the result proved depends to a large degree on what smoothness assumptions one makes about the solution of the governing equations. For example, the assertion $(\mathbf{v}, \theta) \xrightarrow{L^2} (\mathbf{O}, \theta_0)$ as $t \to \infty$ holds if the solution $(\mathbf{v}, \theta)$ of the field equations satisfies the following conditions. For every $t > 0$, $\mathbf{v}$ and θ are square integrable over the region occupied by the fluid; $\mathbf{v}$ and θ are differentiable with respect to the spatial variables $\mathbf{x}$, and the derivatives with respect to $\mathbf{x}$ are square integrable. However, if we assume that $\mathbf{v}$ and θ be also uniformly continuous with respect to t, then we can prove a stronger result that $(\mathbf{v}, \theta) \to (\mathbf{O}, \theta_0)$ almost everywhere as $t \to \infty$. Since not much is known about the existence of solutions of the governing equations, it is not clear how smooth the solution would be if it existed for all times $t \geqq 0$.

It seems worth mentioning that, because of the possibility of the fluid's being set into oscillatory motion, the part of the boundary of the fluid on which adherence condition holds may vary with time. Here I make the somewhat plausible assumption that there exists a time T such that for all $t > T$, there is a material subsurface to which the fluid adheres. Because the result proved here is of asymptotic nature, without any loss in generality we may set $T = 0$. Similarly the part of the boundary on which the temperature is maintained at the value θ_0 may vary with time. In this case I do not require that there be a material subsurface on which the temperature is assigned for all times t.

The mechanical problem corresponding to the situation when the container is completely filled with the liquid has been solved by Kampé de Fériet [1]. He proved that the kinetic energy of any disturbance approaches zero exponentially. A significant difference between this case and the one I study here is that the total potential energy of the fluid remains constant for all $t > 0$ when the vessel is completely filled with the fluid. On the other hand, when the container is not completely filled, there is the possibility of the exchange of the mechanical energy between the kinetic and the potential parts and this complicates the problem. It is, perhaps, because of this exchange of mechanical energy between the kinetic and potential parts that it is hard to obtain a monotone decay of the kinetic energy. In any event, at present I am unable to obtain such a sharp estimate for the type of weak solutions whose existence is assumed.

Notations: I refer the deformation of the fluid to a fixed set of rectangular Cartesian axes. The vector $\mathbf{X}$ denotes the position of a material particle, and R denotes the region occupied by the fluid in the reference configuration. The vector $\mathbf{x} \equiv \chi(\mathbf{X}, t)$ gives the place occupied by $\mathbf{X}$, $\mathbf{v} \equiv \dot{\mathbf{x}}$ gives its velocity and the scalar $\theta = \theta(\mathbf{X}, t) > 0$ denotes the absolute temperature of $\mathbf{X}$ at time t. A comma followed by an index i designates differentiation with respect to x_i. The superposed dot stands for the material time derivative. The summation convention is used. The vector $\mathbf{n}$ designates the outer unit normal to the current configuration of the boundary ∂R of R. ∂R_1 and ∂R_2 denote subsets of ∂R, *i.e.* $\partial R_1 \subseteq \partial R$. By ∂R_1^c we shall mean $\partial R_1^c \equiv \partial R - \overline{\partial R_1}$.

2. - Formulation of the problem.

The thermomechanical deformations of an incompressible Navier-Stokes-Fourier fluid subject to lamellar body forces are governed by

$$(1) \qquad \begin{cases} d_{ii} = 0\,, \\ \varrho \dot{v}_i = t_{ij,j} - \varrho \Omega_{,i}\,, \\ \dot{\varepsilon} = - q_{i,i} + t_{ij} d_{ij} - r(\theta, \theta_0)(\theta - \theta_0)\,, \end{cases}$$

where

$$(2) \qquad \begin{cases} d_{ij} \equiv \frac{1}{2}(v_{i,j} + v_{j,i}) = v_{(i,j)} \\ t_{ij} = - p \delta_{ij} + 2\mu(\theta) d_{ij}\,, \\ q_i = - k(\theta)\theta_{,i}\,. \end{cases}$$

Here, ϱ. ε, t_{ij}, μ and k denote, respectively, the mass density per unit volume, the internal energy density per unit volume, the Cauchy stress tensor, the shear viscosity and the thermal conductivity of the fluid. Both μ and k are functions of the temperature θ. The scalar field p gives the arbitrary hydrostatic pressure and $\Omega = \Omega(\mathbf{x})$ gives the potential of the body forces. Ω is assumed to be a non-negative, bounded, differentiable function with the property that $\varrho\Omega_{,i}$ is square integrable over R. I remark that gravity is included as a special case when Ω is linear in $\mathbf{x}$. In the energy equation $(1)_3$, $\mathbf{q}$ denotes the heat flux and the source term $r(\theta, \theta_0)\,(\theta - \theta_0)$ accounts for the change of the internal energy due to exchange of heat by radiation into the surroundings. Quite often, radiation is either neglected or is assumed to be an arbitrary function of place and time. Here r is assumed to be bounded; in particular, it may vanish identically. In order that heat may radiate out of body if it is at a temperature higher than that of the surrondings, the function r should be non-negative. I assume that by an appropriate choice of the scales for various quantities, the above equations (1) and (2) and the new quantities introduced below are put in non-dimensional form. It may be remarked that the fluid need not be homogeneous. I assume that the density, the shear viscosity and the thermal conductivity are continuous and bounded functions so that the various integrals considered below do exist. For use in the discussion to follow, I introduce the Helmholtz free energy function φ, defined by

$$\varphi(\theta, \mathbf{X}) \equiv \varepsilon(\theta, \mathbf{X}) - \theta\eta(\theta, \mathbf{X})\,, \tag{3}$$

where η is the entropy density per unit volume. It is useful to recall that

$$\frac{\partial \varepsilon}{\partial \theta} = \theta \frac{\partial \eta}{\partial \theta} \equiv c(\theta, \mathbf{X})\,,$$

where c is the specific heat. Therefore, for the case at hand,

$$\dot{\varepsilon} = \theta\dot{\eta}\,. \tag{4}$$

Taking the inner product of $(1)_2$ with $\mathbf{v}$ integrating the resulting equation, $(1)_1$ and $(1)_3$ over the region $\chi(R, t)$, using the divergence theorem and the relations (2), we get

$$
(5)\quad \begin{cases} \oint \mathbf{v}\cdot\mathbf{n}=0\,, \\[2mm] \dfrac{d}{dt}\dfrac{1}{2}\displaystyle\int \varrho\,\mathbf{v}\cdot\mathbf{v}\;dV=\oint t_{ij}\,n_j\,v_i\,dA-2\int\mu\,d_{ij}\,d_{ij}\,dV-\int\varrho\dot{\Omega}dV\,, \\[2mm] \dfrac{d}{dt}\displaystyle\int \varepsilon\,dV=\oint k\,\theta_{,i}\,n_i\,dA+2\int\mu\,d_{ij}\,d_{ij}\,dV-\int r\,(\theta-\theta_0)\,dV\,. \end{cases}
$$

Substituting for $\dot{\varepsilon}$ from $(1)_3$ in (4), integrating the resulting equation over the region $\chi(R,t)$ and making an obvious simplification, we obtain

$$
(6)\qquad \frac{d}{dt}\int \eta dV=\oint\frac{k}{\theta}\,\theta_{,i}\,n_i\,dA+ \\
+\int\left[\frac{k}{\theta^2}\,\theta_{,i}\,\theta_{,i}+\frac{2\mu}{\theta}\,d_{ij}\,d_{ij}-\frac{r}{\theta}\,(\theta-\theta_0)\right]dV\,.
$$

If the fields $(\mathbf{v},\theta)$ satisfy (1) and the prescribed side conditions such as initial conditions and boundary conditions then also $(\mathbf{v},\theta)$ satisfies (5) but the converse is by no means true. For $(\mathbf{v},\theta)$ to be a solution of (1), d_{ij} and $\theta_{,i}$ should be differentiable with respect to $\mathbf{x}$, but for $(\mathbf{v},\theta)$ to be a solution of (5), d_{ij} and $\theta_{,i}$ should exist almost everywhere in $\chi\,(R,t)$ and should be square-integrable.

Our main tools in the proof of the theorem stated below are the inequalities due to Poincaré and Korn. I recall these inequalities. For differentiable functions f defined on R such that $f\in L^2(R)$ and $f_{,i}\in L^2(R)$, we have Poincaré's inequality [3, p. 355]

$$
(7)\qquad \int_R f^2\,dV\leqq p_1\left[\int_R f_{,i}f_{,i}\,dV+\oint_{\partial R} f^2\,dA\right],
$$

and if $f=0$ on a part ∂R_0 of the boundary of R, then the inequality (7) can be sharpened to [4]

$$
(8)\,(^1)\qquad \int_R f^2\,dV\leqq p_2\int_R f_{,i}f_{,i}\,dV\,.
$$

(1) I am grateful to Professor C. M. Dafermos for providing me a proof of (8) and to Professor J. L. Ericksen for bringing to my attention the reference [4].

Here the constant p_1 depends only upon R whereas p_2 depends on R and ∂R_0. For differentiable vector valued functions $\mathbf{u}$ defined on R such that $\mathbf{u} \in L^2(R)$, $u_{i,j} \in L^2(R)$ and $\mathbf{u} = \mathbf{0}$ on a part ∂R_0 of the boundary of R, we have Korn's inequality [4]

$$\int_R u_{i,j}\, u_{i,j}\, dV \leqq k_1 \int_R u_{(i,j)}\, u_{(i,j)}\, dV\,, \tag{9}$$

where k_1 is a function of R and ∂R_0.

In what follows I assume that the problem outlined in section 1, or equivalently the equation (1) under the boundary conditions (10) and suitable initial conditions, has a weak solution in the sense that for every $t > 0$,

(i) the mapping χ of the reference configuration R into the present configuration is continuously differentiable with respect to $\mathbf{X}$ and t, $\mathbf{v}$ is differentiable with respect to $\mathbf{x}$ and $\mathbf{v} \in L^2(R)$, $v_{i,j} \in L^2(R)$,

(ii) θ is diferentiable with respect to $\mathbf{x}$, $\theta \in L^2(R)$, $\theta_{,i} \in L^2(R)$,

(iii) $\int \varrho\, \mathbf{v} \cdot \mathbf{v}\, dV$ and $\int \varepsilon\, dV$ are differentiable with respect to time t,

(iv) $(\mathbf{v}, \theta)$ satisfy (5), boundary conditions (10) and suitable initial conditions,

(v) the real valued functions q_1 and p_2 (or q_2) appearing in the inequalities below are bounded.

This definition of the weak solution differs from the one often used in that (iii) is usually not required to hold [2]. I now state the theorem I wish to prove below.

THEOREM: Let R be an open, connected and bounded region with a smooth boundary [3]. Then the weak solution $(\mathbf{v}, \theta)$ of (1) under the boundary conditions

$$\left\{\begin{array}{ll} \mathbf{v}(\mathbf{x}, t) = \mathbf{0}\,, & (\mathbf{x}, t) \in \chi(\partial R_1(t), t) \times (0, t), \\ t_{ij}\, n_j = p_0\, n_i\,, & (\mathbf{x}, t) \in \chi(\partial R_1{}^c, t) \times (0, t)\,, \\ \theta(\mathbf{x}, t) = \theta_0\,, & (\mathbf{x}, t) \in \chi(\partial R_2(t), t) \times (0, t)\,, \\ q_i\, n_i = b(\theta, \theta_0)\, (\theta - \theta_0)\,, & (\mathbf{x}, t) \in \chi(\partial R_1{}^c, t) \times (0, t)\,, \quad [4] \end{array}\right. \tag{10}$$

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[2] *E.g.* see Ladyzhenskaya [2].

[3] The region is assumed to be smooth enough to apply the inequalities (7), (8) and (9). For details concerning this, see [3, 4].

[4] The function $b(\theta, \theta_0)$ is assumed to be bounded.

satisfies

$$
(11)\qquad \begin{cases} \mathbf{v} \xrightarrow{L^2} \mathbf{0}\,,\ t \to \infty\,, \\ \theta \xrightarrow{L^2} \theta_0\,,\ t \to \infty\,, \\ \lim\limits_{t \to \infty} \int \varrho \Omega \, dV \text{ exists,} \end{cases}
$$

provided

$$
(12)\qquad \begin{cases} \text{(i)} \quad c_1 \equiv \inf\limits_{\mathbf{X}} \varrho(\mathbf{X}) > 0\,, \\ \text{(ii)} \quad c_2 \equiv \sup\limits_{\mathbf{X}} \varrho(\mathbf{X}) \text{ is finite,} \\ \text{(iii)} \quad c_3 \equiv \inf\limits_{\theta, \mathbf{X}} \mu(\theta, \mathbf{X}) > 0\,, \\ \text{(iv)} \quad c_4 \equiv \inf\limits_{\theta, \mathbf{X}} \dfrac{c(\theta, \mathbf{X})}{\theta} > 0\,, \\ \text{(v)} \quad \bigcap\limits_{t=0}^{\infty} \partial R_1(t) \neq \varnothing \text{ in the sense that the 2-dimensional} \\ \qquad \text{measure of } \bigcap\limits_{t=0}^{\infty} \partial R_1(t) \text{ is nonzero, and} \\ \text{(vi)} \quad \text{one of the following two holds} \\ \text{(a)} \quad \bigcap\limits_{t=0}^{\infty} \partial R_2(t) \neq \varnothing\,,\ \inf\limits_{\theta, \mathbf{X}} \theta_0 \dfrac{k}{\theta^2} > 0, \text{ or } \inf\limits_{\theta, \mathbf{X}} \dfrac{r(\theta, \theta_0)}{\theta} > 0\,, \\ \text{(b)} \quad \inf\limits_{\theta, \mathbf{X}} \left(\dfrac{b(\theta, \theta_0)}{\theta}\,,\ \dfrac{\theta_0\, k\,(\theta)}{\theta^2} \right) > 0\,, \text{ or } \inf\limits_{\theta, \mathbf{X}} \dfrac{r(\theta, \theta_0)}{\theta} > 0\,. \end{cases}
$$

In (12), the infimum or supremum is taken over all possible values of the temperature θ. However, if in a particular problem, we have *a priori* knowledge of the range of values over which θ can vary, then we can take the infimum or supremum with θ varying over that range only.

The assumption $(12)_5$ and the boundary condition $(10)_1$ imply that $\mathbf{v}$ satisfies conditions sufficient for Poincaré's inequality (8) and Korn's inequality (9) to hold. Combining these two inequalities yields

$$
(13)\qquad \int\limits_{\chi(R,t)} \mathbf{v} \cdot \mathbf{v} \, dV \leq q_1 \int\limits_{\chi(R,t)} d_{ij}\, d_{ij}\, dV\,,
$$

where q_1 depends on R and ∂R_1. If ∂R_1 varies with time t, then q_1 can be regarded as a real-valued function of time t. I assume this function to be bouded above for every t and, with some ambiguity in notation, denote its least upper bound also by q_1. I shall use a similar notation below and therefore the various factors occurring in the inequalities below are to be understood in the sense of suprema of their values if either ∂R_1 or ∂R_2 is a function of time t.

3. - Proof of the theorem.

I first note that because of $(5)_1$ and the boundary conditions $(10)_{1,2}$, the first integral on the right-hand side of $(5)_2$ vanishes. Now rewriting $(5)_2$ as

$$\frac{d}{dt}\int\left(\frac{\varrho}{2}\,\mathbf{v}\cdot\mathbf{v}+\varrho\,\Omega\right)dV=-\,2\int\mu d_{ij}\,d_{ij}\,dV\leqq 0\,,$$

we see that the non-negative function $\int\varrho\left(\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v}+\Omega\right)dV$ is non-increasing and therefore

$$\lim_{t\to\infty}\int\left(\frac{\varrho}{2}\,\mathbf{v}\cdot\mathbf{v}+\varrho\,\Omega\right)dV \text{ exists.} \tag{14}$$

Integrating both sides of $(5)_2$ with respect to t in the time interval $(0, T)$ where T is arbitrary, we get

$$\int\frac{\varrho}{2}\,\mathbf{v}\cdot\mathbf{v}\,dV\Big|_{t=T}=\int\frac{\varrho}{2}\,\mathbf{v}\cdot\mathbf{v}\,dV\Big|_{t=0}-\\ -\,2\int_0^T dt\int\mu d_{ij}\,d_{ij}\,dV-\int\varrho\Omega\,dV\Big|_0^T. \tag{15}$$

Since $\int\varrho\Omega\,dV$ is a bounded function of t, (15) implies that

$$\int\mu d_{ij}\,d_{ij}\,dV\in L^1\,(0,\infty)\,, \tag{16}$$

for otherwise $\int\frac{\varrho}{2}\,\mathbf{v}\cdot\mathbf{v}\,dV$ would become negative for sŏme large value of t, which is impossible. Using $(12)_{1,2}$ and (13), we get the following string of inequalities:

$$\int \frac{\varrho}{2} \mathbf{v} \cdot \mathbf{v} \, dV \leqq \frac{c_2}{2} \int \mathbf{v} \cdot \mathbf{v} \, dV \leqq \frac{c_2 q_1}{2} \int d_{ij} d_{ij} \, dV \leqq$$

$$\leqq \frac{q_1 c_2}{2 c_3} \int \mu d_{ij} d_{ij} \, dV ,$$

and these with (16) imply that

$$I(t) \equiv \int \frac{\varrho}{2} \mathbf{v} \cdot \mathbf{v} \, dV \in L^1 (0, \infty) . \tag{17}$$

With the smoothness assumptions we have on Ω, $\varrho\dot{\Omega} \in L^1 (\chi (R, t))$. Using this, the assumption that Ω is bounded and (16), we conclude from $(5)_2$ that

$$\dot{I} \in L^1 (0, \infty) .$$

Thus $I(t)$ is uniformly continuous in t and

$$I(t) \to 0 \quad \text{as} \quad t \to \infty \tag{18}$$

now follows from (17). $(11)_1$ is an immediate consequence of (18) and $(12)_1$ and $(11)_3$ follows from (18) and (14). Combining $(5)_2$, $(5)_3$ and (6) suitably, we can verify the following

$$\frac{d}{dt} \int_{\chi(R,t)} (\varepsilon - \theta_0 \eta + \varrho \mathbf{v} \cdot \mathbf{v} + 2 \varrho \Omega) \, dV = - \oint_{\chi(\partial R_2^c, t)} \frac{b}{\theta} (\theta - \theta_0)^2 \, dA - \tag{19}$$

$$\int_{\chi(R,t)} \left| \frac{\theta_0 k}{\theta^2} \theta_{,i} \theta_{,i} + 2\mu \left(1 + \frac{\theta_0}{\theta}\right) d_{ij} d_{ij} + \frac{r}{\theta} (\theta - \theta_0)^2 \right| dV .$$

If r, b, k and μ are non-negative functions, it is clear that the right-hand side is aways non-positive. Following Ericksen [5], I introduce a finite Taylor expansion in the temperature for φ, obtaining thereby

$$\varepsilon - \theta_0 \eta = \varphi_0 + K (\theta - \theta_0)^2$$

where

$$\varphi_0 \equiv \varphi (\theta_0) , \text{ and}$$

$$K \equiv - \frac{1}{2} \frac{\partial^2 \varphi}{\partial^2 \theta} (\theta^*) ,$$

θ^* being a value of the temperature between θ and θ_0. An alternative expression for K is

$$K = \frac{c(\theta^*)}{\theta^*} .$$

On the assumption that the specific heat is always positive, K would be positive. Since $\dot{\varphi}_0 = 0$, we conclude that

$$\frac{d}{dt} \int_{\chi(R,t)} [K(\theta - \theta_0)^2 + \varrho \, \mathbf{v} \cdot \mathbf{v} + 2\, \varrho \Omega] \, dV \leqq 0 .$$

Thus, with the assumptions that $\mu > 0$, $k > 0$, b and r are non-negative functions, the non-negative function

$$\int [K(\theta - \theta_0)^2 + \varrho \, \mathbf{v} \cdot \mathbf{v} + 2\, \varrho \Omega] \, dV$$

is monotonically non-increasing and hence must converge as $t \to \infty$.

This along with $(12)_4$ and (14) implies that

$$\lim_{t \to \infty} \int (\theta - \theta_0)^2 \, dV \text{ exists} . \tag{20}$$

When $\bigcap\limits_{t=0}^{\infty} \partial R_2(t) \neq \emptyset$, we can use (8) to arrive at

$$\theta_0 \int \frac{k}{\theta^2} \theta_{,i} \theta_{,i} \, dV = \theta_0 \int \frac{k}{\theta^2} (\theta - \theta_0)_{,i} (\theta - \theta_0)_{,i} \, dV ,$$

$$\geqq \left(\inf_{\theta, \mathbf{X}} \frac{\theta_0 \, k}{\theta^2} \right) \frac{1}{p_2} \int (\theta - \theta_0)^2 \, dV ,$$

and thus obtain the following from (19):

$$\frac{d}{dt} \int (K(\theta - \theta_0)^2 + \varrho \mathbf{v} \cdot \mathbf{v} + 2\, \varrho \Omega) \, dV \leqq - q_3 \int (\theta - \theta_0)^2 \, dV - 2 \int \mu d_{ij} \, d_{ij} \, dV, \tag{21}$$

where

$$q_3 = \inf_{\theta, \mathbf{X}} \left(\frac{\theta_0 k}{p_2 \theta^2} + \frac{r}{\theta} \right),$$

which by assumption is positive. If instead, the second alternative in $(12)_6$ holds, then with the definition

$$c_5 \equiv \inf_{\theta, \mathbf{X}} \left(\frac{b}{\theta}, \ \frac{\theta_0 k}{\theta^2} \right),$$

we have

$$\text{(22)} \qquad \oint_{\chi(\partial R_2^c, t)} \frac{b}{\theta} (\theta - \theta_0)^2 \, dA + \int_{\chi(R,t)} \theta_0 \frac{k}{\theta^2} \theta_{,i} \theta_{,i} \, dV \geqq$$

$$\geqq c_5 \left| \oint_{\chi(\partial R_2^c, t)} (\theta - \theta_0)^2 \, dA + \int_{\chi(R,t)} (\theta - \theta_0)_{,i} (\theta - \theta)_{,i} \, dV \right| .$$

Since $\theta = \theta_0$ on $\chi(\partial R_2, t)$, we can replace the region of integration $\chi(\partial R_2^c, t)$ by $\chi(\partial R_2, t)$ in the first integral on the right-hand side of (22). Now the use of Poincaré's inequality in the form (7) gives

$$\oint_{\chi \partial (R_2^c, t)} \frac{b}{\theta} (\theta - \theta_0)^2 \, dA + \int_{\chi(R,t)} \theta_0 \frac{k}{\theta^2} \theta_{,i} \theta_{,i} \, dV \geqq \frac{c_5}{q_2} \int_{\chi(R,t)} (\theta - \theta_0)^2 \, dV,$$

and therefore

$$\text{(23)} \qquad \oint_{\chi(\partial R_2^c, t)} \frac{b}{\theta} (\theta - \theta_0)^2 \, dA + \int \left\{ \theta_0 \frac{k}{\theta^2} \theta_{,i} \theta_{,i} + \frac{r}{\theta} (\theta - \theta_0)^2 \right\} dV \geqq$$

$$\geqq q_4 \int_{\chi(R,t)} (\theta - \theta_0)^2 \, dV,$$

where

$$q_4 \equiv \inf_{\theta, \mathbf{X}} \left(\frac{c_5}{q_2} + \frac{r}{\theta} \right),$$

and by assumption we have $q_4 > 0$. (23) and (19) again give (21) with

q_3 replaced by q_4. Now an argument similar to the one used in concluding (17) from $(5)_2$ yields

$$\int (\theta - \theta_0)^2 \, dV \in L^1(0, \infty)$$

and this along with (20) implies that

$$\int (\theta - \theta_0)^2 \, dV \to 0 \quad \text{as} \quad t \to \infty .$$

Now $(11)_2$ follows immediately once we note that

$$\left| \int (\theta^2 - \theta_0^2) \, dV \right| \leqq \int (\theta - \theta_0)^2 \, dV + 2\theta_0 \left| \int (\theta - \theta_0) \, dV \right| ,$$

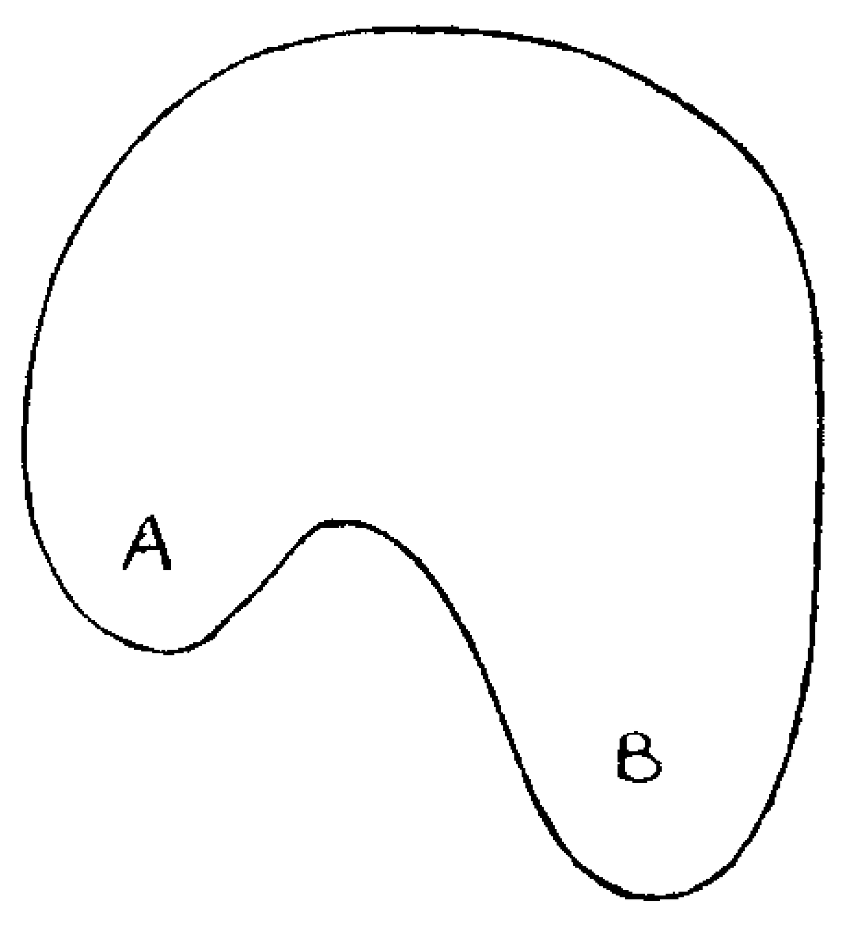
$$\leqq \int (\theta - \theta_0)^2 \, dV + 2\theta_0 \left(\int d\,V \right)^{1/2} \left(\int (\theta - \theta_0)^2 \, dV \right)^{1/2} .$$

To obtain the last inequality, we have used the Cauchy-Schwarz inequality. This completes the proof of the theorem.

Now consider the case when the solution $(\mathbf{v}, \theta)$ of (5) and (10) is also required to be smooth in the following sense: the fields $(\mathbf{v}, \theta)$ are uniformly continuous with respect to t. For such solutions of (5) and (10), the assertion that $(\mathbf{v}, \theta) \to 0$ almost everywhere as $t \to \infty$ can be proved from $(11)_{1,2}$ and the definitions of limit and of uniform continuity. I remark that the weak solutions are not generally assumed to have this additional smoothness.

4. - Remarks.

Even though I have shown above that the potential energy of the fluid approaches a limit as $t \to \infty$, depending upon the shape of the container, this limiting value can be different for different sizes of the initial disturbances. For example, if the vessel has a shape like the one sketched below one could start with all the fluid in the valley A and end up with some fluid in B or vice versa. One possibility is that, in the process of the transfer of the fluid from A to B, the fluid would come in contact with some part of the wall it had not previously touched, and then the bounding layer of the fluid would stay there forever afterward. Should this happen, it is conceivable that, at some point on the wall connecting A and B, the thickness of the layer of the fluid connecting the fluid in A and B might decrease as t increases, ultimately becoming zero as $t \to \infty$. Such a motion of the fluid would be topological for every t but in the limit as $t \to \infty$, become non-topological. I remark that the preceding analysis covers such a possibility.



Dussan [6] has studied the smoothness of the velocity field and surface tractions in the neighborhood of the line common to three materials, *e.g.* where the interface between two immiscible fluids meets a solid wall. Her study shows that the velocity field and the surface tractions may not be well defined in the neighborhood of this common line. Also the assumption that $\chi \in C^1$ and the adherence condition at the solid walls are incompatible with the assumption that the common line moves. Her basic assumptions in arriving at these results are that a point on the fluid-fluid interface travels to the common line in a finite interval of time and the fluids adhere to the solid walls. In the problem studied here, I have neglected the deformations of the other fluid (*e.g.* air) which may be contained in the container besides the Navier-Stokes-Fourier fluid. Moreover this other fluid is supposed to exert a uniform pressure p_0 on that part of the boundary of the Navier-Stokes-Fourier fluid which is not in contact with the walls of the container. Thus the assumptions made here are not the same as those made by Dussan [6]. The preceding remarks should make clear that the assumptions $\chi \in C^1$ and ∂R_1 is a function of time t are not mutually inconsistent.

It seems worth mentioning that the above analysis holds for the case when the shear viscosity μ is also a function of $\mathbf{d}$ provided μ is bounded below, *i.e.* $(12)_3$ holds. Two other possible generalizations of the result presented here are to Reiner-Rivlin fluids and to incompressible fluids of second grade. I now explore conditions which would be sufficient for the above analysis to apply in these two cases. To do so for Reiner-Rivlin fluids, I exploit the fact that the crucial step in the proof of the above theorem is the following: The dissipation function $\Phi \equiv tr(\mathbf{td})$ is bounded below by $tr\,\mathbf{d}^2$. For Reiner-Rivlin fluids, the Cauchy stress is given by (5)

$$\mathbf{t} = -p\mathbf{1} + f_1(II, III)\mathbf{d} + f_2(II, III)\ \mathbf{d}^2,$$

where p, the hydrostatic pressure, is an arbitrary function of $\mathbf{x}$ and t, and where II and III denote the second and the third invariants of $\mathbf{d}$. The first invariant $I\ (= tr\ \mathbf{d})$ is identically zero. Thus for the dissipation function Φ we obtain

$$\Phi = f_1\, tr\ \mathbf{d}^2 + f_2\, tr\ \mathbf{d}^3,$$

$$= (f_1 + II\, f_2 - f_2)\, tr\ \mathbf{d}^2 + f_2\, (tr\ \mathbf{d}^2 + tr\ \mathbf{d}^3 - II\, tr\ \mathbf{d}^2)\,.$$

The Hamilton-Cayley theorem, *viz*

$$\mathbf{d}^3 - I\ \mathbf{d}^2 + II\ \mathbf{d} - III\ \mathbf{1} = \mathbf{0}\,,$$

and the fact that $I \equiv 0$, imply

$$tr\,\mathbf{d}^4 = -\ II\ tr\ \mathbf{d}^2.$$

Thus

$$tr\ \mathbf{d}^2 + tr\ \mathbf{d}^3 - II\ tr\ \mathbf{d}^2 = tr\ (\mathbf{d}^2 + \mathbf{d}^3 + \mathbf{d}^4)\,.$$

The observation that the right-hand side is always non-negative for every $\mathbf{d}$ immediately leads to the conclusions that the inequalities

(24) $$f_1 - f_2 + II\, f_2 \geqq \text{const}\ \alpha > 0, \quad f_2 \geqq 0\,,\ \forall\, \mathbf{d}$$

are sufficient for the validity of the following inequality:

$$\Phi \geqq \alpha\, tr\ \mathbf{d}^2.$$

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(5) I assume that f_1 and f_2 are bounded.

Hence the theorem holds for Reiner-Rivlin fluids if f_1 and f_2 satisfy (24).

The constitutive equation of incompressible fluids of second grade is [7, Chapter VI]

$$\mathbf{t} = -p\,\mathbf{1} + \mu\,\mathbf{A}_1 + \alpha_1\,\mathbf{A}_2 + \alpha_2\,\mathbf{A}_1^2\,,$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are first and second Rivlin-Ericksen tensors; μ, α_1 and α_2 are constants and p, the hydrostatic pressure, is an arbitrary function of $\mathbf{x}$ and t. Recalling that [7, § II.11]

$$\mathbf{A}_1 = 2\,\mathbf{d}\,,$$

$$\mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1 \text{ grad } \mathbf{v} + (\mathbf{A}_1 \text{ grad } \mathbf{v})^T\,,$$

we obtain

$$tr\,(\mathbf{td}) = 2\mu\ tr\,\mathbf{d}^2 + \alpha_1\,\dot{\overline{tr\,\mathbf{d}^2}} + 4\,(\alpha_1 + \alpha_2)\,tr\,\mathbf{d}^3.$$

Following the steps necessary to derive $(5)_2$ from $(1)_2$, we arrive at

$$\frac{d}{dt}\int\left[\frac{\varrho}{2}\,(\mathbf{v}\cdot\mathbf{v} + 2\Omega) + \alpha_1\ tr\,\mathbf{d}^2\right] dV = - \int\,[2\mu\ tr\,\mathbf{d}^2 + 4\,(\alpha_1+\alpha_2)\,tr\,\mathbf{d}^3]\,dV. \tag{25}$$

Now reasoning similar to the one presented in section 3 shows that the inequalities

$$\mu > 0\,, \qquad \alpha_1 > 0\,, \qquad \alpha_1 + \alpha_2 = 0\,, \tag{26}$$

are sufficient for the solution $\mathbf{v}$ of $(5)_1$ and (25) under the boundary conditions $(10)_{1,2}$ and suitable initial conditions to exhibit the following behavior:

$$\begin{cases} \int v^2\,dV \to 0\,,\ t\to\infty \\ \int tr\,\mathbf{d}^2 \to 0\,,\ t\to\infty \\ \lim \int \varrho\Omega dV \text{ exists as } t\to\infty \end{cases} \tag{27}$$

provided $(12)_5$ holds. That (26) leads to (27) complements the known result [7, § VI.5] that the condition $\alpha_1 < 0$ implies some sort of instability.

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