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On the interaction between two circular voids in a nonlinear elastic solid

J. P. Zhang and **R. C. Batra,** Rolla, Missouri

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Summary. We study finite plane strain deformations of an infinite body containing two circular voids and made of a Blatz-Ko material. The body is subjected to either uniform tensile tractions at infinity or a uniform pressure on the void surfaces. In each case, the effect of varying the distance between the void centers on the deformations of the body is analyzed. When the voids are located close to each other, a uniform pressure on the void surfaces results in noncircular deformed voids, and for a fixed value of the pressure, the deformation induced increases as the voids get closer to each other. When the body is subjected to uniform tensile tractions at infinity, say along the x-axis, the voids are deformed into ellipsoids with major axes aligned along the x-axis.

1 Introduction

Because of the recent interest in ceramics and other parts manufactured by using powder metallurgy techniques, there have been many studies [1]–[5] concerning porous materials. Many [1]–[3] of these have assumed voids to be randomly distributed within the body and thus treated the porous medium as a continuum. Others [4]–[5] have studied deformations of a void in an otherwise homogeneous solid body. The body has been assumed to be made of either an elastic material or an elastic/plastic material. Of course, the determination of stresses at the tips of an ellipsoidal void in a linear elastic body goes back to the work of Inglis [6]. Here we aim at studying the interaction between two identical circular voids in a compressible, homogeneous and isotropic nonlinear elastic body made of a Blatz-Ko material [7] undergoing quasistatic plane strain deformations. In order to simplify the problem the body is assumed to be infinite. Either a uniform pressure is applied to the circular holes or a uniform tensile load is applied at infinity. In each case the effect of varying the distance between the circular voids on the deformations of their surfaces and of the adjoining material is studied. It is found that the two voids start interacting with each other when the distance between their centers is about five times the radius of each void.

2 Formulation of the problem

We use rectangular Cartesian coordinates with x_1-axis along the line joining the centers of two circular voids, x_2-axis perpendicular to it and the origin midway between the void centers to study finite plane strain deformations of a homogeneous infinite body having two identical circular cylindrical voids and made of an isotropic Blatz-Ko material. Assuming that the body deforms quasistatically, equations governing its mechanical deformations are

$$T_{i\alpha,\alpha} = 0, \quad i = 1, 2; \quad \alpha = 1, 2, \tag{1}$$

where $T_{i\alpha}$ is the first Piola-Kirchhoff stress tensor, a comma followed by an index α indicates partial differentiation with respect to X_α, and a repeated index implies summation over the range of the index. We assume that the deformations of the body are symmetrical with respect to the horizontal and vertical centroidal axes, and accordingly study deformations of the quarter of the body in the first quadrant. Because of the assumed symmetry of deformations, normal displacements and tangential tractions vanish on the left vertical and bottom horizontal surfaces.

Two different loadings are considered. In the first case,

$$T_{i\alpha}N_\alpha = -pn_i \tag{2.1}$$

on the surface of the circular void, and

$$|T_{i\alpha}N_\alpha| \to 0 \quad \text{as} \quad X_1{}^2 + X_2{}^2 \to \infty. \tag{2.2}$$

That is, a uniform pressure p is applied on the inner surface of the voids, and the bounding surfaces of the body are taken to be traction free. Here N_α and n_i denote, respectively, a unit outwardly directed normal to a bounding surface in the reference and present configurations. In the second case,

$$T_{i\alpha}N_\alpha = 0 \tag{3.1}$$

on the surface of the void, and

$$T_{i\alpha}N_\alpha = t\delta_{i1} \tag{3.2}$$

on the right surface at infinity. Thus the body is subjected at infinity to uniform tensile tractions t in the x_1-direction only, and the upper surface is assumed to be traction free. For a linear elastic body, Thompson [8] showed that for a precise statement of the problem, one should also specify the rates at which tractions approach their limiting values as $(X_1{}^2 + X_2{}^2)^{1/2} \to \infty$. One will assume that such should be the case for a nonlinear elastic body, too. However, for the problem at hand, there is no hope of proving an existence or uniqueness theorem at present, and we will eventually seek an approximate solution of the problem. Thus a specification of these rates of decay of tractions at infinity is not considered necessary.

The objective is to analyze deformations of the body and in particular that of the material adjoining the voids as a function of the distance $2h$ between their centers, and either pressure p applied to the voids or tractions t applied to the right bounding surface.

For plane strain deformations of a Blatz-Ko [7] material the strain energy density per unit reference volume is given by

$$W = \frac{\mu}{2}\left[\frac{(I-1)}{J^2} + 2J - 4\right], \tag{4}$$

where $J = \det \boldsymbol{F}$ is the Jacobian or the determinant of the deformation gradient $\boldsymbol{F}$ defined by

$$F_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha} = x_{i,\alpha}, \tag{5}$$

x_i being the present position of a material particle that occupied place X_α in the reference configuration,

$$I = \operatorname{tr}(\boldsymbol{F}\boldsymbol{F}^T) = \operatorname{tr}(\boldsymbol{F}^T\boldsymbol{F}) \tag{6}$$

is the first invariant of the left or right Cauchy-Green tensor, and μ is the shear modulus of the material at zero strain. For plane strain deformations of the Blatz-Ko material the Cauchy stress tensor $\boldsymbol{\sigma}$, related to the first Piola-Kirchhoff stress tensor $\boldsymbol{T}$ by

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{T}\boldsymbol{F}^T, \tag{7}$$

has the expression

$$\boldsymbol{\sigma} = \mu\left(1 - \frac{I}{J^3}\right)\boldsymbol{1} + \frac{\mu}{J^3}\boldsymbol{B}, \tag{8}$$

where

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T \tag{9}$$

is the left Cauchy-Green tensor. The stress-strain curve [7] and the strain energy density vs. the axial stretch curve for a Blatz-Ko material deformed in simple tension reveal that the axial stress reaches a limiting value with an increase in the axial stretch and equals nearly 0.95μ for axial stretches exceeding 2.5, but the strain energy density continues to increase monotonically with an increase in the axial stretch.

Substitution from Eqs. (5) through (9) into Eq. (1) gives two coupled highly nonlinear partial differential equations which ought to be solved, for x_1 and x_2, under the pertinent boundary conditions stated above. We are unable to solve these equations analytically, so we seek an approximate solution of the problem by the finite element method.

3 Numerical solution and results

The numerical solution of the problem necessitates the consideration of a finite region. The length of the finite region studied herein equals at least $26a$ in the x_1-direction and $20a$ in the x_2-direction; a being the radius of the circular void. However, for large values of h/a, where $2h$ equals the distance between void centers, the length of the finite region studied is suitably increased. That the region studied is adequate was verified by ensuring that the deformations of the body near the top surface were negligibly small and that of the material adjoining the right surface were also either negligibly small or essentially homogeneous, depending upon whether tractions there are prescribed to be zero or positive. The governing Eqs. (2) are first cast into their weak form by using the Galerkin approximation [9]. Note that the weak formulation of the problem incorporates natural boundary conditions. The resulting nonlinear algebraic equations are solved by the Newton-Raphson method. The applied load is divided into several steps, and within each load step, equilibrium iterations are used until, at each node point,

$$\frac{|\Delta \boldsymbol{u}|}{|\boldsymbol{u}|} \leqq 10^{-3}, \tag{10}$$

where $\Delta\boldsymbol{u}$ is the just computed increment in the displacement because of the unbalanced forces and $\boldsymbol{u}$ is the total computed displacement of that node. We note that even in the referential description of motion, the computation of forces at node points on the boundary from the surface tractions applied thereon must account for the deformations of the boundary.

The finite domain is divided into four-noded isoparametric quadrilateral elements, and 2×2 Gauss quadrature rule is used to evaluate various integrals numerically. The finite element mesh used is extremely fine in the region surrounding the void and gradually becomes coarser as we move away from the void surface. The developed code has been validated by analyzing the plane strain deformations of an infinite elastic body containing a circular void and studied analytically by Abeyaratne and Horgan [10]. A uniform pressure is applied to the void surface. This test problem is quite similar to the problem being studied herein. The maximum difference between the values of the hoop stress at any point on the inner surface of the void as obtained from the analytical and numerical solutions was found to be less than 1%; this validates the code.

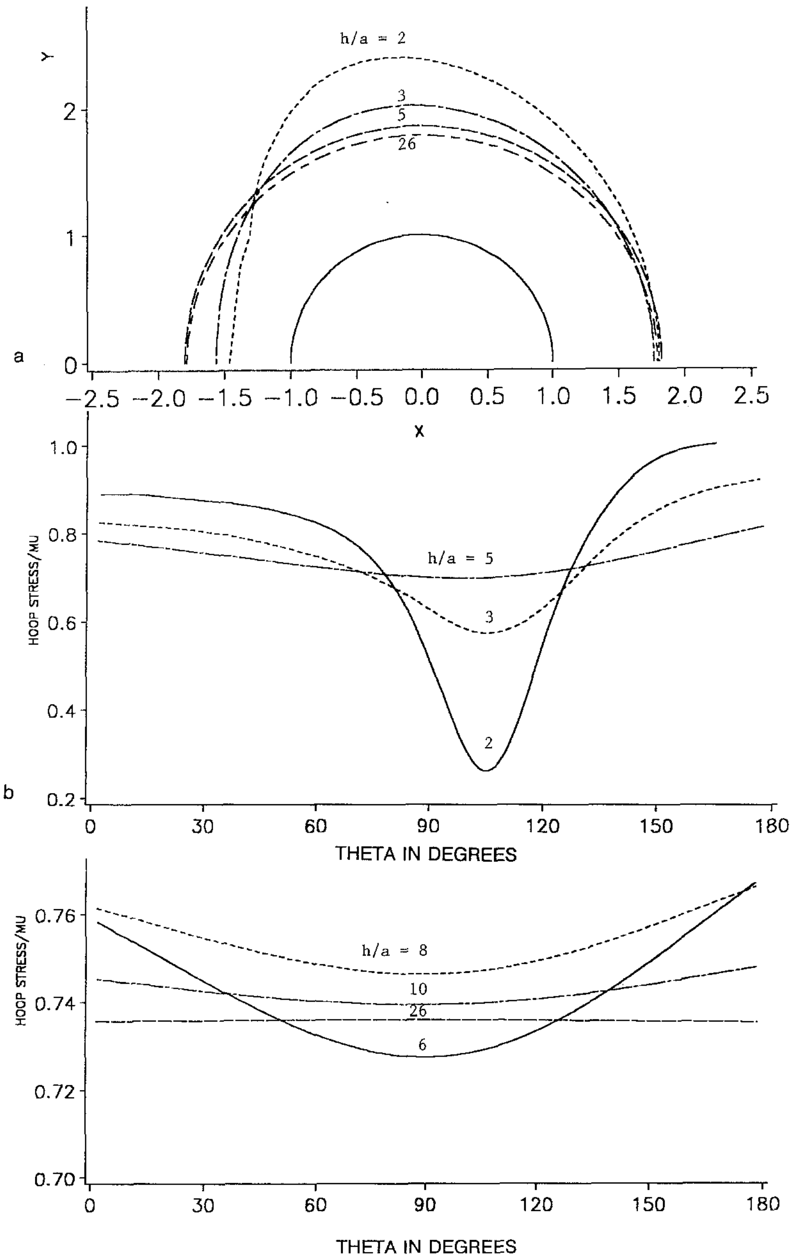


Fig. 1. a Deformed shapes of the initially circular void for $h/a = 2$, 3, 5, and 26 when void surfaces are subjected to pressure $p = 1.5\mu$. The void in the undeformed configuration, represented by the solid curve, is shown for reference. **b** Distribution of the hoop stress on the inner surface of the deformed void for different values of h/a when pressure $p = 1.5\mu$ is applied to the void surfaces

3.1 Results for pressure applied to void surfaces

We first present and discuss results for the case when pressure is applied to the void surface. When there is no interaction between the two voids, one expects that circular voids will be deformed into circular voids. We have plotted in Fig. 1 the deformed shapes of voids and the distribution of the hoop stress on their inner surfaces for four different values of h/a with the value of the pressure p kept fixed at $p/\mu = 1.5$. It is clear that the deformed shapes of the void for $h/a = 2$ and 3 are noncircular, and that for $h/a = 26$ is circular. The abscissa is measured from the undeformed position of the void center. The distribution of the hoop stress $\sigma_{\theta\theta}$ on the inner surface of the

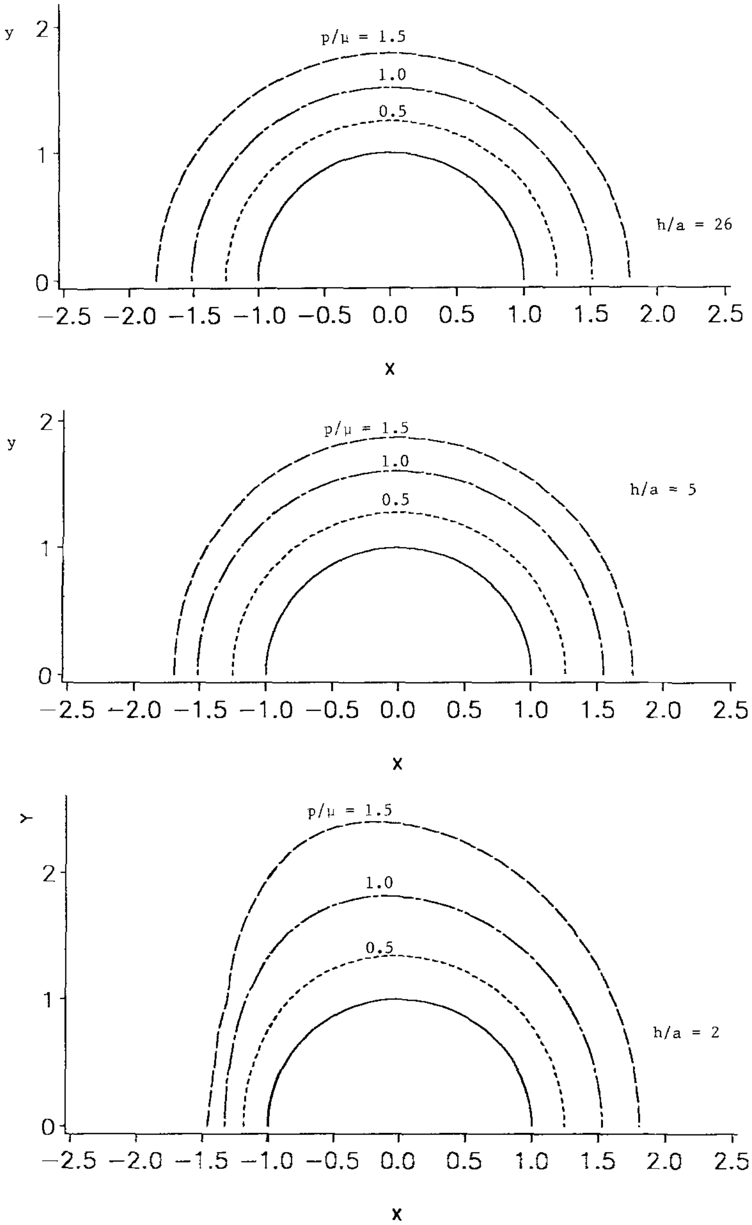
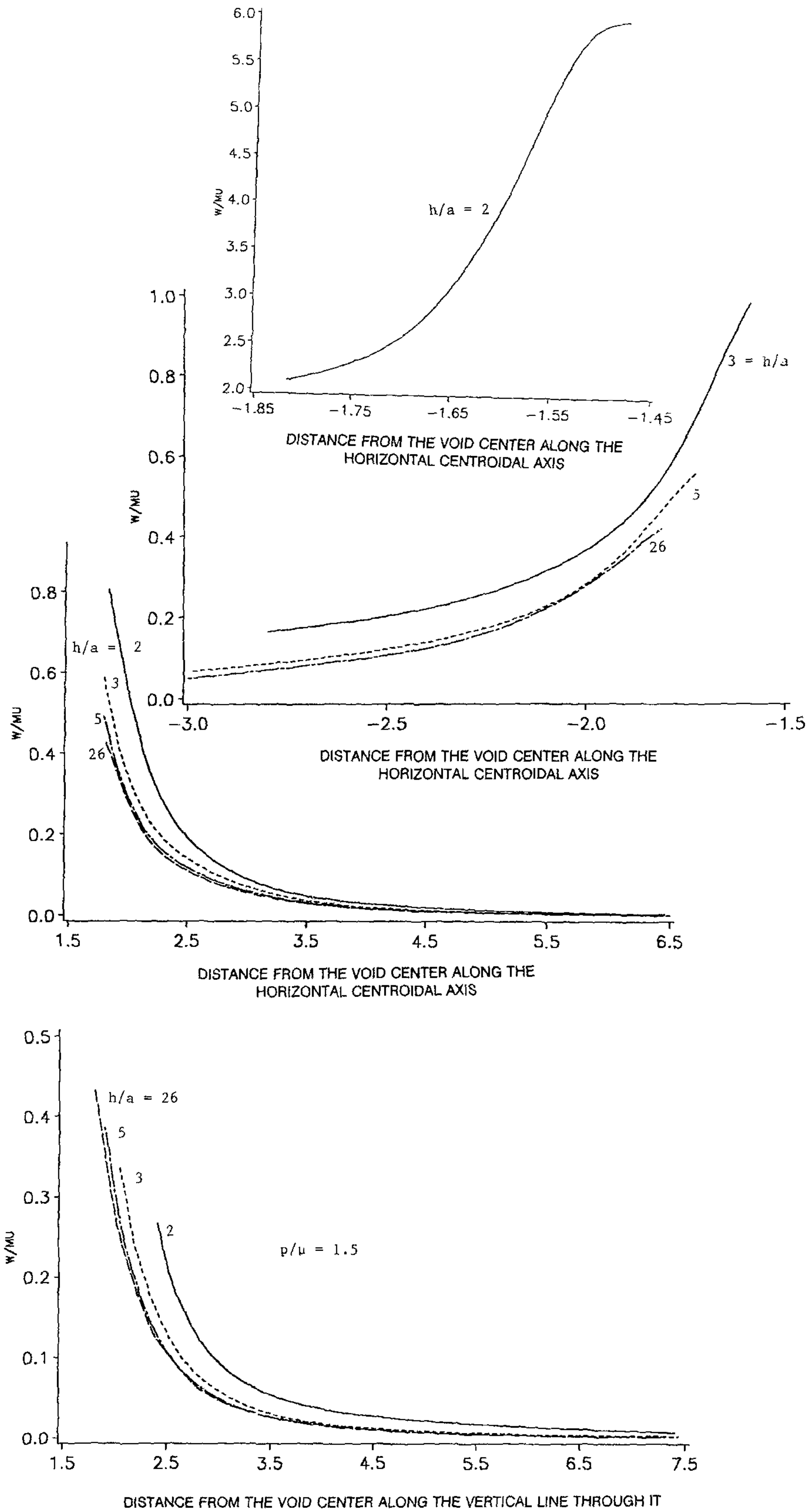


Fig. 2. Deformed shapes of the initially circular void for different values of the internal pressure for $h/a = 2, 5$, and 26


W/MU
h/a = 2
DISTANCE FROM THE VOID CENTER ALONG THE HORIZONTAL CENTROIDAL AXIS
3 = h/a
5
26
DISTANCE FROM THE VOID CENTER ALONG THE HORIZONTAL CENTROIDAL AXIS
h/a = 2
3
5
26
DISTANCE FROM THE VOID CENTER ALONG THE HORIZONTAL CENTROIDAL AXIS
h/a = 26
5
3
2
p/μ = 1.5
DISTANCE FROM THE VOID CENTER ALONG THE VERTICAL LINE THROUGH IT

deformed void, depicted in Fig. 1, shows that for large values of h/a, when there is no interaction between the voids, the hoop stress stays essentially constant on the inner surface of the void. However, for smaller values of h/a signifying interaction between the voids the hoop stress on the inner surface of the void is nonuniformly distributed, and the point where the hoop stress is lowest depends upon the distance between the voids. These plots suggest that the voids start interacting when $h/a \leqq 10$. The angular position of a point is measured counterclockwise from the horizontal axis. Figure 2 exhibits the deformed shapes of the void for $h/a = 2$, 5, and 26 and for different values of the pressure p. For $h/a = 2$, as the applied pressure increases the left void surface becomes increasingly vertical with its curvature at the point of intersection with the horizontal axis approaching zero. However, for $h/a = 5$ and 26, the deformed surfaces of the void appear circular for all three values of p considered. A closer examination indicates that for $h/a = 5$ and for $p = 1.0\mu$ and 1.5μ the deformed shape of the inner surface of the void deviates somewhat from a circle.

In linear elasticity one often invokes Saint-Venant's Principle, which states that in a body subjected to surface tractions whose resultant force and moment vanish, the strain energy density will decay to zero as one moves away from the point of application of the tractions. Toupin [11], Knowles [12], Sternberg [13], and Batra [14] have given precise statements of the principle and have also derived expressions for the decay rate of the strain energy density. In order to ascertain whether or not such a principle holds for the nonlinear elastic problem being studied herein, we have plotted in Fig. 3 the distribution of the strain-energy density W per unit volume in the reference configuration on the horizontal centroidal axis and also on the vertical line passing through the void center. In these plots, the distance of a point is measured from the center of the undeformed void and has been nondimensionalized by using the radius of the undeformed void as the length scale; points to the left of the void center have been assigned negative values of the abscissa. It is clear that the strain energy density does not decay that rapidly to the left of the void because of the interaction between the two voids. For $h/a = 2$, the strain energy density W takes on very high values at points to the left of the void surface as compared to the case when $h/a \geqq 3$. However, to the right of the void, the strain energy density decays to almost zero within a distance of three times the void radius. The same holds for the decay of the strain energy density along the vertical line through the void center. Thus, except for points between the two voids, the strain energy density does decay quite rapidly.

Abeyaratne and Horgan [10] found that for axisymmetric deformations of an infinite body made of a Blatz-Ko material containing a circular cavity and loaded internally by a uniform pressure p, equations governing the deformations of the body lose ellipticity when the internal pressure p equals 1.021μ. Knowles and Sternberg [15] proved that these equations are elliptic at a point, provided that

$$2 - \sqrt{3} < \lambda_1/\lambda_2 < 2 + \sqrt{3}, \tag{11}$$

where λ_1 and λ_2 are the principal stretches. Our numerical results for $h/a = 26$ give that inequalities (11) are violated somewhere in the body when $p = 0.95\mu$. During the solution of the problem, we employed load steps equal to 0.05μ. The loss of ellipticity occurred first at points on the inner surface of the void and, upon further loading, spread outward. When two voids are

Fig. 3. Distribution of the strain energy density per unit reference volume on the horizontal centroidal axis and also on the vertical line through the void center for different values of h/a and pressure $p = 1.5\mu$ applied to the void surfaces

located close to each other, say $h/a = 2, 3$, etc., the ellipticity condition did not fail simultaneously at all points on the inner surface. Rather, it first failed at points on the inner surface of the void that are close to the other void and, upon further loading, the ellipticity condition also failed at other points on the inner surface of the void.

The plot of σ_{22} at points on the horizontal centroidal axis and to the right of the void surface for $h/a = 5$ and $p = 1.5\mu$ indicated that σ_{22} stayed continuous even after the ellipticity condition had failed. The ellipticity condition failed to hold at points on the horizontal centroidal axis for which $x_1 < 2.19a$, but of course greater than the deformed position of the right void tip. The numerical computations progressed smoothly, even when the ellipticity condition failed.

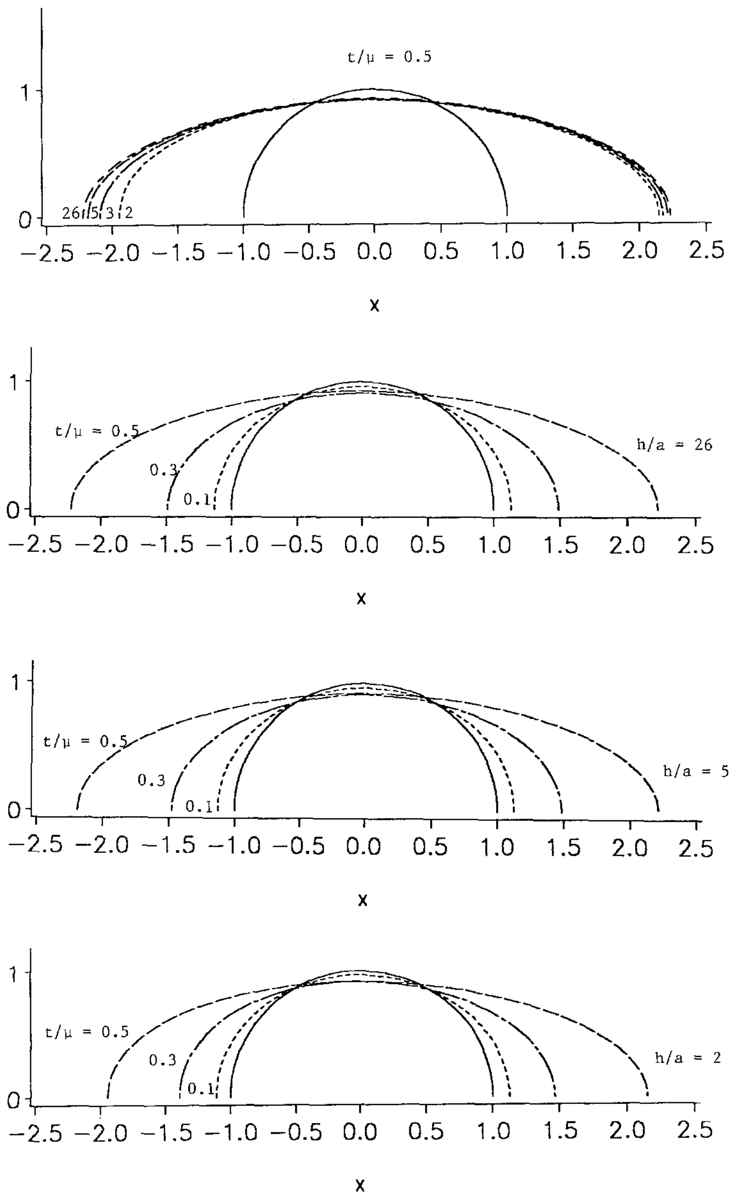


Fig. 4. Deformed shapes of the initially circular void for $h/a = 2, 3, 5$, and 26 when void surfaces are traction free and a uniform tensile traction $t = 0.5\mu$ is applied to the vertical surfaces, and for different values of the traction t when $h/a = 2, 5$, and 26

3.2 Results for tensile tractions applied at right bounding surfaces

In this case the circular void is deformed into an ellipsoid regardless of the magnitude of tensile tractions applied to the vertical bounding surfaces. However, the lengths of the major and minor axes of the ellipsoid do depend upon the distance between the void centers. Figure 4 depicts, for $t/\mu = 0.5$, the deformed shapes of the initially circular void surfaces for $h/a = 2, 3, 5$, and 26, and for different values of the traction when $h/a = 2, 5$, and 26. It is evident that the length of the minor axis of the deformed ellipse does not depend much upon h/a. For $h/a = 5$ and 26, the deformed shapes of the void surface are virtually unchanged for each value of the applied traction considered. However, for $h/a = 2$, the deformed shape of the void surface is unsymmetrical about the vertical line passing through the void center; the void tip to the right of the center moves farther to the right than the displacement to the left of the other void tip. The interaction between the two voids is thus manifested in the asymmetry, about the vertical line passing through the void center, of the deformations of the void surface. In Fig. 5 we have plotted the distribution of the hoop stress on the inner surface of the void for $h/a = 2, 3, 5$, and 26. Since the curvature of the deformed void surface at $\theta = 90°$ decreases and that at $\theta = 0°$ and 180° increases, the hoop stress is positive in the region near $\theta = 90°$, and negative in the region close to $\theta = 0°$ and 180°. The hoop stress is symmetric about $\theta = 90°$ for $h/a = 26$, but loses this symmetry as h/a becomes small because of the interaction between the two voids.

The variations of the strain energy density per unit reference volume on the horizontal centroidal axis and also on the vertical line through the void center for $t = 0.5\mu$ and for $h/a = 2, 3$, 5, and 26 are plotted in Fig. 6. The abscissa indicates the position of a point relative to the center of the undeformed void; points to the left of the void center have negative values of the abscissa. Recall that the farthest point to the left of the void center is on the vertical centroidal axis. The values of the strain energy density W at points between the two voids are quite small as compared to those on the vertical line through the void center or at points on the horizontal centroidal axis, but to the right of the void surface. On the horizontal centroidal axis, W approaches asymptotically its limiting value at the right bounding surface where tractions are applied. It

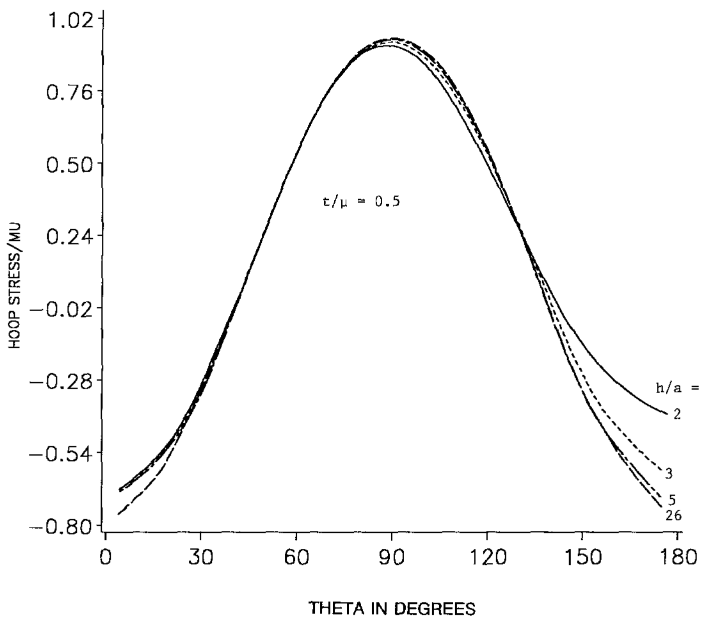


Fig. 5. Distribution of the hoop stress on the inner surface of the deformed void for different values of h/a and with $t = 0.5\mu$

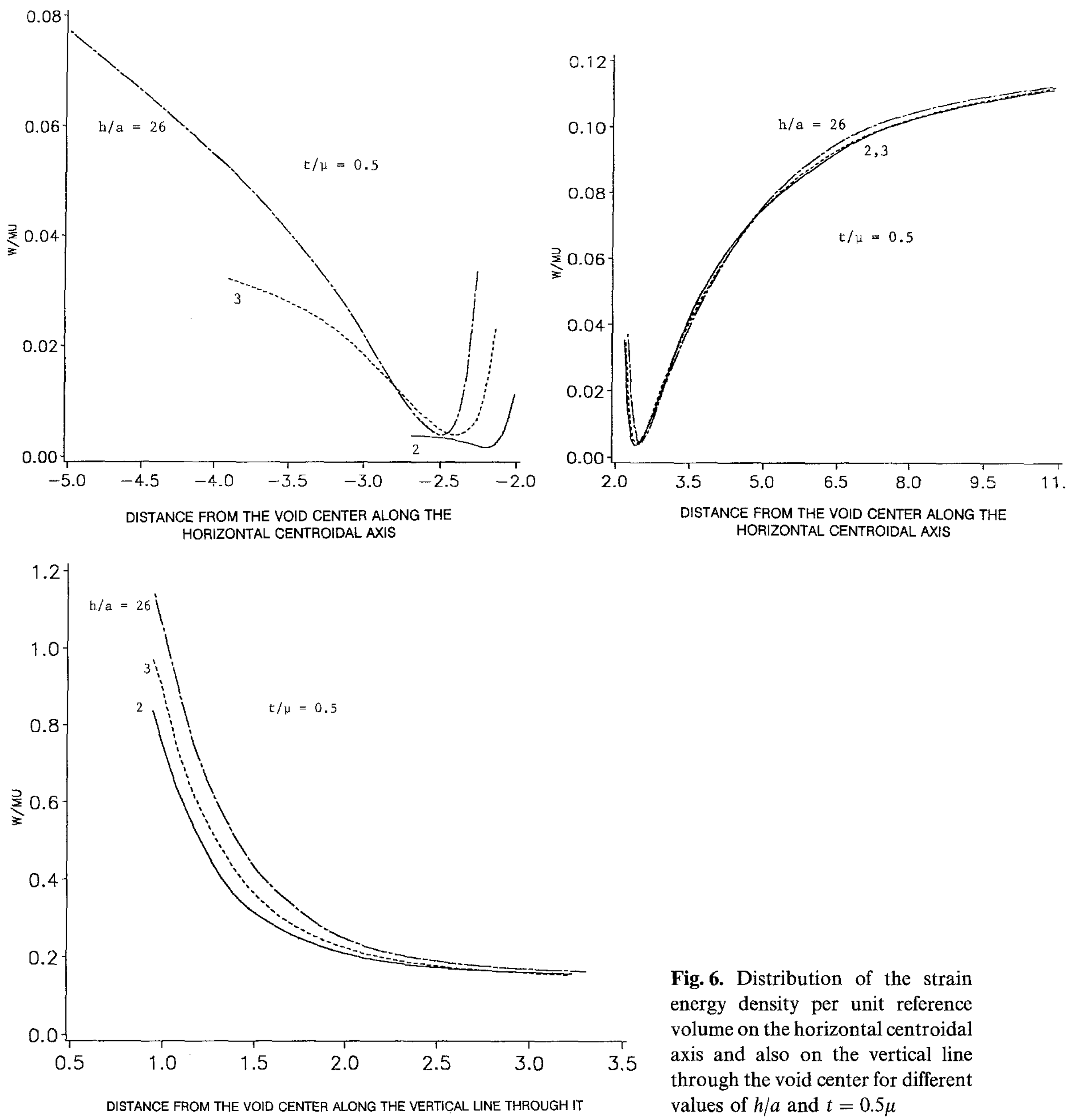


Fig. 6. Distribution of the strain energy density per unit reference volume on the horizontal centroidal axis and also on the vertical line through the void center for different values of h/a and $t = 0.5\mu$

should be evident from the plots of W in Fig. 6 and of the hoop stress given in Fig. 5 that the deformations of the body are more severe at points adjoining the vertex of the minor axis of the deformed ellipsoid than at points near the tips of the ellipsoid on its major axis.

4 Conclusions

We have studied finite plane strain deformations of a body made of a Blatz-Ko material and containing two circular voids in the undeformed configuration. The body is deformed either by applying a uniform pressure to the void surfaces or by applying uniform tensile tractions to the vertical surfaces. For the latter case, the voids are deformed into ellipsoids, irrespective of the distance between their centers. The hoop stress is compressive at points on the void surface which are near the vertices of the major axis of the deformed elliptical surface, but tensile elsewhere. The deformed void surface is symmetrical about the vertical line passing through the center of the void for $h/a \geqq 3$, but nonsymmetrical for h/a $= 2$. Here $2h$ is the distance between void centers, and a equals the radius of the undeformed void.

When a uniform pressure p is applied to void surfaces, they are deformed into circles for $h/a > 5$, but noncircular shapes for smaller values of h/a. The hoop stress at points on the void surface is uniform for $h/a = 26$, but nonuniform for $h/a \leqq 10$. For $h/a = 26$, the ellipticity condition failed for $p = 0.95\mu$, compared to the analytical value of 1.021μ. For other values of h/a, the loss of ellipticity of the governing equations occurred at different values of p. The normal stress σ_{22} was found to be continuous on the horizontal centroidal axis, even when the ellipticity condition had failed at its connected subinterval.

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Authors' address: J. P. Zhang and R. C. Batra, Department of Mechanical and Aerospace Engineering and Engineering Mechanics, University of Missouri – Rolla, Rolla, MO 65401-0249, U.S.A.