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Free vibrations of a strain gradient beam by the method of initial values

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Abstract We extend the application of the method of initial values (also known as the transfer matrix method) to find frequencies of free vibrations of a strain-gradient-dependent Euler–Bernoulli beam (EBB) under different boundary conditions at the two end faces of the beam. For the classical EBB, we find the exact matricant or the carry-over matrix but it is numerically evaluated for the strain-gradient-dependent EBB. For the numerically evaluated matricant, it is found that ten iterations give converged values of the first six frequencies for the classical and the strain-gradient-dependent EBB. For the strain-gradient EBB, the sixth-order ordinary differential equation for the lateral deflection and three boundary conditions at each end have been derived by using the Hamilton principle. The material characteristic length is found to noticeably affect frequencies of free vibrations. Thus, the difference between frequencies of the classical and the strain-gradient-dependent EBB can be used to determine the value of the material characteristic length for a nanobeam for which length scale effects are believed to be dominant.

1 Introduction

The recent interest in nanotechnology has revived activity in gradient-dependent theories, which were developed, among others, by Toupin [1], Mindlin [2], Gurtin [3], and Eringen [4]. Gurtin [3] showed that the Clausius–Duhem inequality ruled out the dependence of the strain energy density for an elastic material upon the gradients of the deformation gradient (or the strain gradient). Batra [5] studied hyperelastic materials whose strain energy density depends upon the first- and the second-order gradients of the deformation, the temperature, its gradient, and the time rate of change of the temperature. He used an entropy inequality proposed by Green and Laws [6] to study its compatibility with the second law of thermodynamics. He showed that in such materials either thermal disturbances propagate with finite speed in the linear theory and the constitutive quantities do not depend upon the second-order gradients of the deformation or the constitutive quantities may depend upon the second-order gradients of the deformation, and in the linear theory, thermal disturbances do not propagate with finite speed. Dillon and Kratochvil [7] and Batra [8] used strain gradient theories to study elasto-plastic deformations. These theories introduce one or more material characteristic lengths, which are believed to be related to the atomic structure of the lattice; however, no such clear relation has been experimentally or theoretically established yet. It is outside the scope of this paper to review all works in strain gradient theories. The introduction of the length scale helps regularize an initial boundary value problem, and in numerical work, it facilitates finding solutions that are mesh-independent.

Peddieson et al. [9] used a simplified version of Eringen's non-local theory for elastic materials to study deformations of Euler–Bernoulli beams (EBBs) and derived a sixth-order ordinary differential equation for the

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lateral deflection of the beam. However, they used only two boundary conditions at each end of the beam and derived an expression for the deflection of a simply supported beam loaded by a uniformly distributed load. For typical values of the material characteristic length and the beams used as microelectromechanical devices, they found that the difference in the maximum deflections of the classical and the strain-gradient-dependent theories is about 1 %. However, for nanobeams, these differences could be significant.

Lam et al. [10] experimentally studied bending of cantilever beams made of epoxy and showed that for a fixed beam length the bending rigidity increased significantly with a decrease in the beam thickness, which could not be explained by considering traditional effects such as surface tension, and the flexibility of the support. They thus attributed the doubling of the bending rigidity of the beam with a decrease in its thickness to the presence of strain gradient effects. They derived the sixth-order ordinary differential equation for the transverse deflection of the beam and the pertinent six boundary conditions. They found analytical solutions of the boundary problem for loads applied at the tip of a cantilever beam, but did not study its vibrations.

Kong et al. [11] built upon the work of Lam et al. [10] and studied static deflections of a cantilever beam under a point load at its tip, and free vibrations of the cantilever beam. They showed that the choice of a higher-order boundary condition noticeably affects the tip deflection and the lowest four frequencies of free vibration. These values also depend upon the material characteristic length. The free vibration problem was numerically solved.

Using the displacement field appropriate for the Timoshenko beam (TB) theory, Asghari et al. [12] developed expressions for strains that contain terms corresponding to the von Karman nonlinearity. Using the Hamilton principle, they derived governing equations and boundary conditions for a gradient-dependent TB. They studied free vibrations of the TB with two axially immovable supports for which the governing equations could be reduced to two fourth-order differential equations in the lateral deflection and the angle of rotation of a transverse plane. By using the mode shapes corresponding to the fundamental frequency of a classical TB, they found the first two frequencies of a linear gradient-dependent TB. Results presented below for the EBB show that mode shapes of the classical and the strain-gradient-dependent beams are different.

Initial boundary value problems as well as eigenvalue problems can be numerically analyzed by using either the finite element method or a meshless method in which no nodal connectivity is needed. Here, we generalize the method of initial values (or the transfer matrix method [13]) to find frequencies of a strain-gradient-dependent Euler–Bernoulli beam (EBB). When the three material characteristic lengths are set equal to each other in Kong et al.'s work [11], their equations governing frequencies of the beam are identical with those for the strain gradient beam studied here. Our analysis technique does not involve complex arithmetic and gives eigenvalues and eigenvectors for different sets of boundary conditions at the two edges of the beam.

The rest of the paper is organized as follows. Section 2 reviews the method of initial values or the transfer matrix method, provides an exact expression for the carry-over matrix for the classical EBB, and applies the method to find frequencies and mode shapes of the EBB, which agree with the analytical solution of the problem. It is also shown in this section that for a carry-over matrix found by the method of successive approximations, ten terms in the expression for the carry-over matrix provide converged values of the first six frequencies. The method is applied to numerically evaluate frequencies of the strain gradient EBB in Sect. 3. Thus, we have successfully generalize the method of initial values previously used for studying vibrations of the classical EBB theory to the strain gradient EBB theory, which now can be solved by the method of initial values and the method of matricants. Conclusions of this work are summarized in Sect. 4.

2 The method of initial values

2.1 Brief review of the method

In this section, we briefly review the method of initial values and apply it to find frequencies and mode shapes of an Euler–Bernoulli beam (EBB). The solution of the initial value problem (IVP)

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \mathbf{A}\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{y}_0 \tag{1}$$

is

$$\mathbf{y}(x) = \mathbf{Y}(x, 0)\,\mathbf{y}_0, \tag{2}$$

where $\mathbf{Y}(x, 0)$ is the principal (or the carry-over) matrix and is unique for a given IVP. In Eqs. (1) and (2), $\mathbf{y}_0$ is the initial value of the unknown n-dimensional function $\mathbf{y}$ at the point $x = 0$, and $\mathbf{Y}(x, 0)$ is an $n \times n$ square matrix mapping the initial value $\mathbf{y}_0$ on to the current value $\mathbf{y}(x)$ of $\mathbf{y}$ at the point x. It is clear from Eq. (2) that if the principal matrix $\mathbf{Y}(x, 0)$ is known, then the solution of the IVP at the point x can be readily found.

For an IVP, the principal matrix $\mathbf{Y}(x, 0)$ can be found either analytically or by successive approximations as follows:

$$\mathbf{Y}(x, x_1) = \mathbf{I} + \int_{x_1}^{x} \mathbf{A}(\tau)\,\mathrm{d}\tau + \int_{x_1}^{x} \mathbf{A}(\tau) \int_{x_1}^{\tau} \mathbf{A}(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_{x_1}^{x} \mathbf{A}(\tau) \int_{x_1}^{\tau} \mathbf{A}(\sigma) \int_{x_1}^{\sigma} \mathbf{A}(\beta)\,\mathrm{d}\beta\,\mathrm{d}\sigma\,\mathrm{d}\tau + \cdots, \tag{3}$$

where $\mathbf{Y}(x, x_1)$ is called a matricant, and $\mathbf{A}$ is the matrix appearing in Eq. (1). Thus

$$\mathbf{Y}(x, x_n) = \mathbf{Y}(x, x_1)\mathbf{Y}(x_1, x_2)\mathbf{Y}(x_2, x_3)\cdots\mathbf{Y}(x_{n-1}, x_n), \tag{4.1}$$

$$\mathbf{Y}(x, x) = \mathbf{I}. \tag{4.2}$$

It is evident that the evaluation of the matricant by successive approximations is rather straight-forward. Thus, this method to evaluate frequencies is rather simple and may be advantageous over other numerical methods.

2.2 Application of the method to EBBs

For the sake of completeness, we first apply the method of initial values to study free vibrations of an EBB made of a linear elastic, isotropic, and homogeneous material governed by

$$\rho \frac{\partial^2 \bar{v}}{\partial t^2} = \frac{\partial^2 \bar{M}}{\partial x^2}, \tag{5.1}$$

$$\bar{M} = -EI \frac{\partial^2 \bar{v}}{\partial x^2}, \tag{5.2}$$

where $\bar{v}(x, t)$ is the transverse deflection of a point x $(0 \le x \le L)$ at time t, L is the beam length, $\bar{M}$ is the bending moment, ρ equals the mass density (mass/length) of the material of the beam, E is Young's modulus, and I is the second moment of area of cross-section about the neutral axis. We study beams for which EI and ρ are constants, which will be the case for a homogeneous beam of uniform cross-section. Equations (5.1) and (5.2) are supplemented by initial and boundary conditions. In terms of the rotation, $\bar{\varphi}(x, t)$, of a cross-section about the y-axis and the shear force, $\bar{T}(x, t)$, defined by

$$\bar{\varphi}(x, t) = \frac{\partial \bar{v}(x, t)}{\partial x}, \quad \bar{T}(x, t) = \frac{\partial \bar{M}(x, t)}{\partial x}, \tag{6}$$

Eq. (5) becomes

$$\rho \frac{\partial^2 \bar{v}}{\partial t^2} = \frac{\partial \bar{T}}{\partial x}, \quad \bar{M} = -EI \frac{\partial \bar{\varphi}}{\partial x}. \tag{7}$$

For time-harmonic vibrations of the beam, we seek solutions of the form

$$\begin{aligned} \bar{v}(x, t) = v(x)e^{i\omega t}, &\quad \bar{\varphi}(x, t) = \varphi(x)e^{i\omega t}, \\ \bar{M}(x, t) = M(x)e^{i\omega t}, &\quad \bar{T}(x, t) = T(x)e^{i\omega t}, \end{aligned} \tag{8}$$

where $i = \sqrt{-1}$ and ω is a natural frequency of the beam. Substitution from Eq. (8) into Eq. (7) gives the following first-order ordinary differential equation for the four-dimensional vector $\mathbf{y} = (v, \varphi, M, T)$:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \mathbf{A}\mathbf{y}, \tag{9}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{EI} & 0 \\ 0 & 0 & 0 & 1 \\ -\rho\omega^2 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v \\ \varphi \\ M \\ T \end{Bmatrix}, \tag{10}$$

and $\mathbf{y}_0 = \mathbf{y}(0)$ depends upon boundary conditions prescribed at the end $x = 0$ of the beam.

We first solve the problem analytically by finding the principal matrix, and then approximately by using Eqs. (3) and (4).

2.3 Solution of the beam problem with analytical principal matrix

For Eqs. (9) and (10), the analytically found expressions for elements of the principal matrix are listed below:

$$Y_{11} = Y_{22} = Y_{33} = Y_{44} = \frac{1}{2}(\cos\alpha + \cosh\alpha), \tag{11.1}$$

$$Y_{12} = -EIY_{23} = Y_{34} = -\frac{Y_{41}}{EI\mu^2} = \frac{\sin\alpha + \sinh\alpha}{2\sqrt{\mu}}, \tag{11.2}$$

$$Y_{13} = Y_{24} = \frac{Y_{31}}{(EI\mu)^2} = \frac{Y_{42}}{(EI\mu)^2} = \frac{\cos\alpha - \cosh\alpha}{2EI\mu}, \tag{11.3}$$

$$Y_{14} = \frac{Y_{32}}{(\mu EI)^2} = -\frac{Y_{43}}{EI\mu^2} = \frac{\sin\alpha - \sinh\alpha}{2EI\mu^{3/2}}, \tag{11.4}$$

where

$$\alpha = x\sqrt{\mu}, \quad \mu^2 = \frac{\rho\omega^2}{EI}, \tag{12}$$

and we have written Y_{ab} $(a, b = 1, 2, 3, 4)$ for $Y_{ab}(x, 0)$. Thus

$$\mathbf{y}(x) = \mathbf{Y}(x, 0)\,\mathbf{y}(0). \tag{13}$$

2.3.1 Beam clamped at both ends

For a beam clamped at the two ends, the boundary conditions are

$$v(0) = \varphi(0) = v(L) = \varphi(L) = 0. \tag{14}$$

Substitution from Eq. (14) into Eq. (13) and setting $x = L$ give

$$\begin{bmatrix} 0 \\ 0 \\ M(L) \\ T(L) \end{bmatrix} = \begin{bmatrix} Y_{11}(L,0) & Y_{12}(L,0) & Y_{13}(L,0) & Y_{14}(L,0) \\ Y_{21}(L,0) & Y_{22}(L,0) & Y_{23}(L,0) & Y_{24}(L,0) \\ Y_{31}(L,0) & Y_{32}(L,0) & Y_{33}(L,0) & Y_{34}(L,0) \\ Y_{41}(L,0) & Y_{42}(L,0) & Y_{43}(L,0) & Y_{44}(L,0) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ M(0) \\ T(0) \end{bmatrix}. \tag{15}$$

Thus

$$Y_{13}(L,0)M(0) + Y_{14}(L,0)T(0) = 0, \tag{16.1}$$

$$Y_{23}(L,0)M(0) + Y_{24}(L,0)T(0) = 0. \tag{16.2}$$

For Eq. (16) to have a non-trivial solution for $M(0)$ and $T(0)$, we must have

$$det\begin{bmatrix} Y_{13}(L,0) & Y_{14}(L,0) \\ Y_{23}(L,0) & Y_{24}(L,0) \end{bmatrix} = 0, \tag{17}$$

or equivalently

$$\cos L\sqrt{\mu}\cosh L\sqrt{\mu} - 1 = 0, \tag{18}$$

which is the characteristic equation for the determination of the frequency ω and is the same as that given in [14]. Solving Eq. (16.1) for $M(0)$ and substituting for $\mathbf{y}(0)$ in Eq. (13), we get the following expression for the corresponding mode shape:

$$v(x) = A\left(-Y_{14}(L,0)Y_{13}(x,0) + Y_{14}(x,0)Y_{13}(L,0)\right). \tag{19}$$

Here and below, A is a constant that can be evaluated by suitably normalizing the mode shape.

2.3.2 Cantilever beam

For the cantilever beam clamped at the end $x = 0$, the boundary conditions are

$$v(0) = 0, \quad \varphi(0) = 0, \quad M(L) = 0, \quad T(L) = 0. \tag{20}$$

The characteristic equation for finding frequencies is obtained from the following two equations:

$$Y_{33}(L,0)M(0) + Y_{34}(L,0)T(0) = 0, \tag{21.1}$$
$$Y_{43}(L,0)M(0) + Y_{44}(L,0)T(0) = 0. \tag{21.2}$$

The requirement that Eqs. (21) have a non-trivial solution for $M(0)$ and $T(0)$ gives the following equation for finding ω:

$$\cos L\sqrt{\mu}\cosh L\sqrt{\mu} + 1 = 0. \tag{22}$$

The corresponding mode shapes are given by

$$v(x) = A\left(Y_{13}(x,0)Y_{34}(L,0) - Y_{14}(x,0)Y_{33}(L,0)\right). \tag{23}$$

Equations (22) and (23) agree with the analytical solution of the problem [14].

2.3.3 Simply supported beam

Boundary conditions for a simply supported beam are

$$v(0) = v(L) = M(0) = M(L) = 0. \tag{24}$$

The equation for the natural frequencies is obtained by requiring that the following two equations:

$$Y_{12}(L,0)\varphi(0) + Y_{14}(L,0)T(0) = 0, \tag{25.1}$$
$$Y_{32}(L,0)\varphi(0) + Y_{34}(L,0)T(0) = 0 \tag{25.2}$$

have a non-trivial solution. The characteristic equation is

$$\sin L\sqrt{\mu}\sinh L\sqrt{\mu} = 0, \tag{26}$$

and the corresponding mode shapes are given by

$$v(x) = A\left(Y_{14}(x,0)Y_{12}(L,0) - Y_{12}(x,0)Y_{14}(L,0)\right). \tag{27}$$

Equations (26) and (27) can also be found in [14].

2.3.4 Free-free beam

For a beam with both ends free (e.g., for a beam hanging in air), the boundary conditions are

$$T(0) = M(0) = T(L) = M(L) = 0. \tag{28}$$

The equation for the determination of the natural frequencies is obtained from the following two homogeneous equations:

$$Y_{31}(L,0)v(0) + Y_{32}(L,0)\varphi(0) = 0, \tag{29.1}$$
$$Y_{41}(L,0)v(0) + Y_{42}(L,0)\varphi(0) = 0. \tag{29.2}$$

For Eqs. (29.1, 2) to have a non-trivial solution for $v(0)$ and $\varphi(0)$, we must have

$$\cos L\sqrt{\mu}\cosh L\sqrt{\mu} - 1 = 0, \tag{30}$$

which is exactly the same as Eq. (18). That is, the free-free beam and the beam clamped at both ends have the same frequencies; however, the mode shapes are different. Mode shapes of the free-free beam are given by

$$v(x) = A\left(Y_{11}(L,x)Y_{32}(L,0) - Y_{12}(L,x)Y_{31}(L,0)\right). \tag{31}$$

Equations (30) and (31) agree with those given in [14].

2.3.5 Clamped-simply supported beam

For a beam clamped at the left end and simply supported at the right end, the boundary conditions are

$$v(0) = \varphi(0) = M(L) = v(L) = 0. \tag{32}$$

The equation for the determination of natural frequencies derived from

$$det \begin{bmatrix} Y_{13}(L,0) & Y_{14}(L,0) \\ Y_{33}(L,0) & Y_{34}(L,0) \end{bmatrix} = 0 \tag{33}$$

is

$$\cos\alpha \sinh\alpha - \sin\alpha \cosh\alpha = 0, \tag{34.1}$$

and the corresponding mode shapes are given by

$$v(x) = A\left(Y_{13}(L,x)Y_{14}(L,0) - Y_{14}(L,x)Y_{13}(L,0)\right). \tag{34.2}$$

Equations (34.1) and (34.2) are also given in [14].

2.4 Solution of the beam problem with numerically evaluated principal matrix

In some problems (e.g., for the strain gradient beam studied in the next section), the principal matrix cannot be analytically evaluated. We, therefore, solve a beam problem by using the matricant to ascertain the effect of the number of terms in Eqs. (4.1, 2) on the accuracy of the solution. For $n = 8$ in Eq. (4), we get

$$Y_{11} = Y_{22} = Y_{33} = Y_{44} = \frac{\mu^2(x-x_1)^4\left(\mu^2(x-x_1)^4 + 1{,}680\right)}{40{,}320} + 1, \tag{35.1}$$

$$Y_{12} = Y_{34} = -Y_{23} = -\frac{1}{\mu^2}Y_{41} = \frac{1}{120}\mu^2(x-x_1)^5 + x - x_1, \tag{35.2}$$

$$Y_{13} = Y_{24} = \frac{1}{\mu^2}Y_{31} = \frac{1}{\mu^2}Y_{42} = \frac{1}{720}(x-x_1)^2\left(-\mu^2(x-x_1)^4 - 360\right), \tag{35.3}$$

$$Y_{21} = -Y_{32} = Y_{43} = \mu^2 Y_{14} = \frac{\mu^2(x-x_1)^3\left(\mu^2(x-x_1)^4 + 840\right)}{5{,}040}, \tag{35.4}$$

where we have omitted the arguments (x, x_1) of $Y_{ab}(a, b = 1, 2, 3, 4)$.

For a beam clamped at both ends, we calculate the value of the matricant for 8 intervals by using Eq. (4.1) and compute frequencies by using Eq. (17). The first six non-dimensional frequencies

$$\Omega_i = \omega_i L^2\sqrt{\frac{\rho}{EI}},\ i = 1, 2, \ldots, \tag{36}$$

are listed in Table 1. As the number of terms in Eq. (4.1) is increased from 8 to 10, there is no perceptible difference in the values of the first six frequencies up to six significant digits signifying that the frequencies have converged. We have also listed in the Table frequencies derived from the analytical expression, Eq. (18). It is clear that frequencies obtained by keeping 8 terms in Eq. (4.1) agree very well with those deduced from Eq. (18).

For other boundary conditions, eight terms in Eq. (4.1) were found to give converged values of the first six frequencies, which are compared with their analytical counterparts in Table 2. Frequencies of a free-free EB beam were also computed but are omitted since they equal those of a clamped–clamped beam. The analytical frequencies were computed by using the closed-form expressions given in [14].

For each set of boundary conditions considered, frequencies computed by retaining 8 terms in Eq. (4.1) match very well with the corresponding analytical results.

Table 1 Frequencies of an EB beam clamped at both ends

Frequency\number of terms	5	6	8	10	Analytical
Ω_1	22.3754	22.3730	22.3733	22.3733	22.3733
Ω_2	61.7893	61.6586	61.6730	61.6728	61.6728
Ω_3	122.4670	120.7890	120.9060	120.9030	120.9034
Ω_4	210.9680	199.8050	199.857	199.860	199.859
Ω_5	376.1830	302.4200	298.277	298.569	298.555
Ω_6	484.4390	453.7300	414.3790	417.159	416.9907

Table 2 Frequencies of an EB beam

Frequency	Cantilever		Simply supported		Clamped-simply supported	
	Numerical	Analytical	Numerical	Analytical	Numerical	Analytical
Ω_1	3.51602	3.5160	9.8696	9.8696	15.4182	15.4182
Ω_2	22.0345	22.0345	39.4784	39.45	49.9649	49.9649
Ω_3	61.6972	61.6972	88.8264	88.8264	104.248	104.2427
Ω_4	120.902	120.902	157.914	157.914	178.270	178.2695
Ω_5	199.86	199.860	246.742	246.74	272.037	272.0310
Ω_6	298.569	298.569	355.357	335.306	385.6263	385.5314

3 Euler–Bernoulli beam made of strain-gradient-dependent linear elastic material

3.1 Governing equations

We now study vibrations of an EB beam made of a linear elastic, isotropic, and homogeneous material in which there is a higher-order axial (or a couple) stress $\tau = -zE\gamma^2\bar{v}'''$ in addition to the usual axial stress $\sigma = -zE\bar{v}''$. Here, γ has dimensions of length, z is the vertical distance of a point from the neutral axis, and $\bar{v}' = \partial\bar{v}/\partial x$; γ is usually called the material characteristic length. For a beam with a distributed force q per unit length acting on it, the Hamiltonian, H, is given by

$$H = \int_0^{t_1} \mathrm{d}t \int_0^{L} \left[\frac{1}{2}\left(\rho\dot{\bar{v}}^2\right) - \frac{EI}{2}\left(\left(\bar{v}''\right)^2 + \gamma^2\left(\bar{v}'''\right)^2\right) + q\bar{v}\right] \mathrm{d}x, \tag{37}$$

where $\dot{\bar{v}} = \partial\bar{v}/\partial t$, and the positive q and positive $\bar{v}$ are in opposite directions along the z-axis. Here, we have assumed that boundary conditions are such that there is no work done by forces acting on the boundaries. As is often assumed for an EBB, there is no contribution to H from the shear deformations. Furthermore, the contribution to H from the quadratic term $\tau\sigma$ has been neglected in Eq. (37). The Hamilton principle gives the following equations governing deformations of the beam:

$$-\rho\ddot{\bar{v}} = EI\left(\bar{v}^{IV} - \gamma^2\bar{v}^{VI}\right) + q, \quad 0 < x < L; \tag{38.1}$$

$$\delta\bar{v}(-\bar{v}''' + \gamma^2\bar{v}^{V}) = 0, \quad \text{at } x = 0, L; \tag{38.2}$$

$$\delta\bar{v}'(\bar{v}'' - \gamma^2\bar{v}^{IV}) = 0, \quad \text{at } x = 0, L; \tag{38.3}$$

$$\delta\bar{v}''(\bar{v}''') = 0, \quad \text{at } x = 0, L; \tag{38.4}$$

$$\bar{v}(x, 0) = \bar{v}_0(x), \quad \dot{\bar{v}}(x, 0) = \dot{\bar{v}}_0(x). \tag{38.5}$$

Here, $\bar{v}_0(x)$ and $\dot{\bar{v}}_0(x)$ equal the initial transverse displacement and the initial transverse velocity, respectively, of a point x of the beam.

For studying free harmonic vibrations, we set $q(x) = 0$ and do not need initial conditions. Assuming a solution of the form given by Eq. (8), we get the following equations for the determination of the frequency ω:

$$\mu^2 v = v^{IV} - \gamma^2 v^{VI}, \quad 0 < x < L; \tag{39.1}$$

$$\delta v(v''' - \gamma^2 v^{V}) = 0, \quad x = 0, L; \tag{39.2}$$

$$\delta v'(v'' - \gamma^2 v^{IV}) = 0, \quad x = 0, L; \tag{39.3}$$

$$\delta v'' v''' = 0, \quad x = 0, L. \tag{39.4}$$

We reduce the sixth-order ordinary differential Eq. (39.1) into six first-order ordinary differential equations by introducing five additional variables as follows:

$$v' = \varphi, \quad \varphi' = \eta, \quad \eta' = \beta, \quad M = -EI(\eta - \gamma^2\beta'), \quad T = M'. \tag{40}$$

Thus, Eqs. (39.1–4) become

$$-\rho\omega^2 v = T', \tag{41.1}$$

$$\delta v T = 0,\ \delta v' M = 0,\ \delta\eta\beta = 0 \quad \text{at } x = 0, L. \tag{41.2}$$

In this theory, the bending moment, M, and the shear force, T, depend upon v^{IV} and v^{V}, respectively. For $\gamma = 0$, Eqs. (39.1–3) and expressions for M and T reduce to those for a classical EBB.

In terms of the six-dimensional vector

$$\mathbf{y} = (v, \varphi, \eta, \beta, M, T), \tag{42}$$

Eqs. (40) and (41.1) can be written as

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \mathbf{A}\mathbf{y}, \tag{43}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma^2} & 0 & \frac{1}{\gamma^2 EI} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\rho\omega^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{44}$$

3.2 Matricant for the gradient EB theory

Elements of the matricant matrix, $Y_{ab}(t, t_1)$, $(a, b = 1, 2, 3, 4)$, evaluated by using 8 terms in Eq. (4) are listed below.

$$Y_{11} = Y_{22} = 1 - \frac{\mu^2(t-t_1)^6\left((t-t_1)^2 + 56\gamma^2\right)}{40{,}320\gamma^4}, \tag{45.1}$$

$$Y_{12} = -(1/(EI\mu^2))Y_{61} = Y_{56} = -\frac{\mu^2(t-t_1)^7}{5{,}040\gamma^2} + t - t_1, \tag{45.2}$$

$$\begin{aligned} Y_{13} = Y_{24} = {} & \frac{t^2}{2} - tt_1 + \frac{{t_1}^2}{2} + (t-t_1)^4\left(t^4 - 4t^3 t_1 + 6t^2{t_1}^2 + 56t^2\gamma^2 - 4t{t_1}^3 - \gamma^4\mu^2(t-t_1)^4\right. \\ & \left. -112tt_1\gamma^2 + {t_1}^4 + 56{t_1}^2\gamma^2 + 1{,}680\gamma^4\right)/(40{,}320\gamma^6), \end{aligned} \tag{45.3}$$

$$Y_{14} = \frac{(t-t_1)^3\left(\frac{(t-t_1)^4}{\gamma^4} + \frac{42(t-t_1)^2}{\gamma^2} + 840\right)}{5{,}040}, \tag{45.4}$$

$$Y_{15} = Y_{26} = \frac{\gamma^4(t-t_1)^4 + \frac{1}{30}\gamma^2(t-t_1)^6 + \frac{(t-t_1)^8}{1{,}680}}{24\mathrm{EI}\gamma^6}, \tag{45.5}$$

$$\mu^2 EIY_{16} = -Y_{21} = -Y_{32} = -Y_{54}/(\gamma^2 EI) = \frac{(t-t_1)^5\left((t-t_1)^2 + 42\gamma^2\right)}{5{,}040\mathrm{EI}\gamma^4}, \tag{45.6}$$

$$Y_{23} = Y_{34} = EI\gamma^2 Y_{45} = t - t_1 + \frac{(t-t_1)^3\left(-\gamma^4\mu^2(t-t_1)^4 + 42\gamma^2(t-t_1)^2 + (t-t_1)^4 + 840\gamma^4\right)}{5{,}040\gamma^6}, \tag{45.7}$$

$$Y_{25} = \frac{4\gamma^4(t-t_1)^3 + \frac{1}{5}\gamma^2(t-t_1)^5 + \frac{1}{210}(t-t_1)^7}{24\mathrm{EI}\gamma^6}, \tag{45.8}$$

$$\begin{aligned} Y_{53} &= EI\gamma^2 Y_{31} = Y_{64} = \gamma^2 EIY_{42} \\ &= -\frac{\mathrm{EI}\mu^2(t-t_1)^4\left(56\gamma^2(t-t_1)^2 + (t-t_1)^4 + 1{,}680\gamma^4\right)}{40{,}320\gamma^4}, \end{aligned} \tag{45.9}$$

$$\begin{aligned} Y_{33} = Y_{44} = 1 &- \frac{\mu^2(t-t_1)^6\left((t-t_1)^2 + 56\gamma^2\right)}{40{,}320\gamma^4} + \frac{1}{\gamma^2}\left(\frac{t^2}{2} - tt_1 + \frac{t_1{}^2}{2}\right) \\ &+ \frac{1}{\gamma^2}\left((t-t_1)^4\left(t^4 - 4t^3t_1 + 6t^2t_1{}^2 + 56t^2\gamma^2 - 4tt_1{}^3 - \gamma^4\mu^2(t-t_1)^4\right.\right. \\ &\left.\left. -112tt_1\gamma^2 + t_1{}^4 + 56t_1{}^2\gamma^2 + 1{,}680\gamma^4\right)/(40{,}320\gamma^6)\right), \end{aligned} \tag{45.10}$$

$$\begin{aligned} Y_{35} = \left(-\frac{1}{40{,}320\mathrm{EI}\gamma^8}\right)(t-t_1)^2 &\left(-t^6 + 6t^5t_1 - 15t^4t_1{}^2 - 56t^4\gamma^2\right. \\ &+20t^3t_1{}^3 + 224t^3t_1\gamma^2 - 15t^2t_1{}^4 - 336t^2t_1{}^2\gamma^2 - 1{,}680t^2\gamma^4 + 6tt_1{}^5 \\ &+ 224tt_1{}^3\gamma^2 + \gamma^4\mu^2(t-t_1)^6 + 3{,}360tt_1\gamma^4 - t_1{}^6 - 56t_1{}^4\gamma^2 \\ &\left. - 1{,}680t_1{}^2\gamma^4 - 20160\gamma^6\right), \end{aligned} \tag{45.11}$$

$$Y_{36} = \frac{4\gamma^4(t-\mathrm{t}_1)^3 + \frac{1}{5}\gamma^2(t-\mathrm{t}_1)^5 + \frac{1}{210}(t-\mathrm{t}_1)^7}{24\mathrm{EI}\gamma^6}, \tag{45.12}$$

$$Y_{41} = \frac{1}{\gamma^2 EI}Y_{63} = -\frac{\mu^2(t-\mathrm{t}_1)^3\left(42\gamma^2(t-\mathrm{t}_1)^2 + (t-\mathrm{t}_1)^4 + 840\gamma^4\right)}{5{,}040\gamma^6}, \tag{45.13}$$

$$\begin{aligned} Y_{43} = &-\frac{\mu^2(t-t_1)^5\left((t-t_1)^2 + 42\gamma^2\right)}{5{,}040\gamma^4} \\ &+\frac{(t-t_1)^3\left(-\gamma^4\mu^2(t-t_1)^4 + 42\gamma^2(t-t_1)^2 + (t-t_1)^4 + 840\gamma^4\right)}{5{,}040\gamma^8} + \frac{(t-t_1)}{\gamma^2}, \end{aligned} \tag{45.14}$$

$$\begin{aligned} Y_{46} = \left(\frac{1}{24\mathrm{EI}\gamma^8}\right)&\left(12\gamma^6(t-t_1)^2 - \frac{\gamma^4\mu^2(t-t_1)^8}{1{,}680} + \gamma^4(t-t_1)^4\right. \\ &\left. + \frac{1}{30}\gamma^2(t-t_1)^6 + \frac{(t-t_1)^8}{1{,}680}\right), \end{aligned} \tag{45.15}$$

$$Y_{51} = Y_{62} = \frac{\mathrm{EI}\mu^4(t-t_1)^8}{40{,}320\gamma^2} - \frac{1}{2}\mathrm{EI}\mu^2(t-t_1)^2, \tag{45.16}$$

$$Y_{55} = Y_{66} = 1 - \frac{\mu^2(t-t_1)^6\left((t-t_1)^2 + 56\gamma^2\right)}{40{,}320\gamma^4}, \tag{45.17}$$

$$Y_{52} = -\frac{1}{6}\mathrm{EI}\mu^2(t-\mathrm{t}_1)^3, \tag{45.18}$$

$$Y_{65} = -\frac{\mu^2(t-\mathrm{t}_1)^5\left((t-\mathrm{t}_1)^2 + 42\gamma^2\right)}{5{,}040\gamma^4}. \tag{45.19}$$

3.3 Frequencies of a strain gradient EBB

Boundary condition (39.4) implies that at the ends $x = 0$ and $x = L$ of the beam either v''' is prescribed as zero or v'' is prescribed. In the absence of a clear picture of which one of these two boundary conditions to use, we solve the problem for each boundary condition. Even though there are four combinations of these two boundary conditions, we consider only two, that is, either $v''' = 0$ at $x = 0$ and $x = L$ or $v'' = 0$ at $x = 0$ and L. Henceforth, we call these boundary conditions higher order. At a free end, the shear force T and the bending moment M vanish; requiring v'' to also vanish at a free end implies that v^{IV} vanishes there. However, $v''' = \gamma^2 v^V$ at the free end and need not vanish there. We note that three boundary conditions are needed at each end point because the differential equation (39.1) is 6th order in v. Kong et al. [11] used $v = v' = v'' = 0$ at a clamped end, and $M = T = v''' = 0$ at a free end. Frequencies for any combination of higher-order boundary conditions can be computed by adopting the procedure used here.

3.4 Cantilever beam fixed at the end $x = 0$

We first find frequencies of a cantilever beam under the following boundary conditions:

$$v(0) = 0, \varphi(0) = 0, \eta(0) = 0, M(L) = 0, T(L) = 0, \eta(L) = 0. \tag{46.1}$$

Thus, Eq. (2) becomes

$$\begin{bmatrix} v(L) \\ \varphi(L) \\ 0 \\ \beta(L) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Y_{11}(L) & Y_{12}(L) & Y_{13}(L) & Y_{14}(L) & Y_{15}(L) & Y_{16}(L) \\ Y_{21}(L) & Y_{22}(L) & Y_{23}(L) & Y_{24}(L) & Y_{25}(L) & Y_{26}(L) \\ Y_{31}(L) & Y_{32}(L) & Y_{33}(L) & Y_{34}(L) & Y_{35}(L) & Y_{36}(L) \\ Y_{41}(L) & Y_{42}(L) & Y_{43}(L) & Y_{44}(L) & Y_{45}(L) & Y_{46}(L) \\ Y_{51}(L) & Y_{52}(L) & Y_{53}(L) & Y_{54}(L) & Y_{55}(L) & Y_{56}(L) \\ Y_{61}(L) & Y_{62}(L) & Y_{63}(L) & Y_{64}(L) & Y_{65}(L) & Y_{66}(L) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \beta(0) \\ M(0) \\ T(0) \end{bmatrix}, \tag{46.2}$$

where we have written $Y_{ab}(L)$ for $Y_{ab}(L, 0)$, $(a, b = 1, 2, 3, 4, 5, 6)$. Equation (46.2) gives the following equations:

$$\begin{aligned} Y_{34}(L)\beta(0) + Y_{35}(L)M(0) + Y_{36}(L)T(0) &= 0, \\ Y_{54}(L)\beta(0) + Y_{55}(L)M(0) + Y_{56}(L)T(0) &= 0, \\ Y_{64}(L)\beta(0) + Y_{65}(L)M(0) + Y_{66}(L)T(0) &= 0. \end{aligned} \tag{47}$$

Thus, the natural frequencies are roots of the equation

$$\det \begin{bmatrix} Y_{34}(L) & Y_{35}(L) & Y_{36}(L) \\ Y_{54}(L) & Y_{55}(L) & Y_{56}(L) \\ Y_{64}(L) & Y_{65}(L) & Y_{66}(L) \end{bmatrix} = 0, \tag{48}$$

and the mode shapes are given by

$$v(x) = Y_{14}(x, 0)\beta(0) + Y_{15}(x, 0)M(0) + Y_{16}(x, 0)T(0), \tag{49}$$

where

$$\beta(0) = -\frac{Y_{35}(L, 0)M(0) + Y_{36}(L, 0)T(0)}{Y_{34}(L, 0)}, \tag{50.1}$$

$$M(0) = -\frac{Y_{64}(L, 0)Y_{56}(L, 0) - Y_{54}(L, 0)Y_{66}(L, 0)}{Y_{64}(L, 0)Y_{55}(L, 0) - Y_{54}(L, 0)Y_{65}(L, 0)} T(0). \tag{50.2}$$

We now use the boundary conditions

$$v(0) = 0, \quad \varphi(0) = 0, \quad \beta(0) = 0, \quad M(L) = 0, \quad T(L) = 0, \quad \beta(L) = 0, \tag{51.1}$$

to find frequencies of a cantilever beam. Proceeding in the same way as for the boundary condition (46.1), we conclude that the natural frequencies are roots of the equation

$$\det \begin{bmatrix} Y_{43}(L) & Y_{45}(L) & Y_{46}(L) \\ Y_{53}(L) & Y_{55}(L) & Y_{56}(L) \\ Y_{63}(L) & Y_{65}(L) & Y_{66}(L) \end{bmatrix} = 0, \tag{51.2}$$

and the mode shapes are given by

$$v(x) = Y_{13}(x, 0)\eta(0) + Y_{15}(x, 0)M(0) + Y_{16}(x, 0)T(0), \tag{52}$$

where

$$\eta(0) = -\frac{Y_{45}(L, 0)M(0) + Y_{46}(L, 0)T(0)}{Y_{43}(L, 0)}, \tag{53.1}$$

$$M(0) = -\frac{Y_{63}(L, 0)Y_{56}(L, 0) - Y_{53}(L, 0)Y_{66}(L, 0)}{Y_{63}(L, 0)Y_{55}(L, 0) - Y_{53}(L, 0)Y_{65}(L, 0)} T(0). \tag{53.2}$$

Table 3 For $\gamma = L/10$, frequencies of the cantilever strain-gradient-dependent EBB with higher-order boundary conditions $\eta(0) = 0$, $\eta(L) = 0$

Frequencies\number of terms	5	6	8	10	Classical EBB
Ω_1	4.3087	4.3087	4.3087	4.3087	3.5160
Ω_2	28.5958	28.5902	28.5909	28.5909	22.0345
Ω_3	89.5499	89.2359	89.2691	89.2686	61.6972
Ω_4	204.4314	199.8840	200.1633	200.1562	120.902
Ω_5	414.7899	379.4458	379.1888	379.2162	199.860
Ω_6	7,874.4228	661.1975	643.7202	644.9710	298.569

Table 4 For $\gamma = L/10$, frequencies of the cantilever strain-gradient-dependent EBB with higher-order boundary conditions $\beta(0) = 0$, $\beta(L) = 0$

Frequencies\number of terms	5	6	8	10	Classical EBB
Ω_1	3.58434	3.58434	3.58434	3.58434	3.5160
Ω_2	24.7183	24.7151	24.7156	24.7156	22.0345
Ω_3	78.3248	78.1334	78.1545	78.1543	61.6972
Ω_4	178.889	175.983	176.189	176.185	120.902
Ω_5	358.146	335.263	335.459	335.460	199.860
Ω_6	746.158	582.499	573.455	574.085	298.569

For $\frac{\gamma}{L} = \frac{1}{10}$, we have listed in Tables 3 and 4 the first six natural frequencies of the strain-gradient-dependent EBB computed by using 5, 6, 8, and 10 terms in the expression (4.1) of the matricant. We have also listed frequencies for the EBB with $\gamma = 0$ to delineate the effect of the material characteristic length γ on the computed frequencies. It is clear that retaining ten terms in the expression for $\mathbf{Y}$ yields converged values of the first six frequencies.

3.5 Beam clamped at both ends

For the boundary conditions

$$v(0) = \varphi(0) = \eta(0) = 0, \quad v(L) = \varphi(L) = \eta(L) = 0, \tag{54}$$

the characteristic equation for the determination of ω^2 is

$$\det \begin{bmatrix} Y_{14}(L) & Y_{15}(L) & Y_{16}(L) \\ Y_{24}(L) & Y_{25}(L) & Y_{26}(L) \\ Y_{34}(L) & Y_{35}(L) & Y_{36}(L) \end{bmatrix} = 0. \tag{55}$$

The corresponding mode shapes are given by

$$v(x) = Y_{14}(x,0)\beta(0) + Y_{15}(x,0)M(0) + Y_{16}(x,0)T(0), \tag{56}$$

where

$$\beta(0) = -\frac{Y_{15}(L,0)M(0) + Y_{16}(L,0)T(0)}{Y_{14}(L,0)}, \tag{57.1}$$

$$M(0) = -\frac{Y_{34}(L,0)Y_{26}(L,0) - Y_{24}(L,0)Y_{36}(L,0)}{Y_{34}(L,0)Y_{25}(L,0) - Y_{24}(L,0)Y_{35}(L,0)}T(0). \tag{57.2}$$

When the boundary conditions are taken to be

$$v(0) = \varphi(0) = \beta(0) = 0, \quad v(L) = \varphi(L) = \beta(L) = 0, \tag{58}$$

we get the characteristic equation

$$\det \begin{bmatrix} Y_{13}(L) & Y_{15}(L) & Y_{16}(L) \\ Y_{23}(L) & Y_{25}(L) & Y_{26}(L) \\ Y_{43}(L) & Y_{45}(L) & Y_{66}(L) \end{bmatrix} = 0, \tag{59}$$

for the determination of ω^2. The mode shapes are given by

Table 5 For $\gamma = L/10$, frequencies of a clamped–clamped strain-gradient-dependent EBB for two higher-order boundary conditions

Frequency\boundary conditions	$\eta(0) = \eta(L) = 0$	$\beta(0) = \beta(L) = 0$	Classical EBB
Ω_1	35.8935	26.6861	22.3733
Ω_2	108.9315	85.6811	61.6728
Ω_3	239.1780	195.6366	120.9034
Ω_4	444.2329	374.1191	199.859
Ω_5	742.5377	639.5018	298.555
Ω_6	1,153.9843	998.9380	416.9907

$$v(x) = Y_{13}(x,0)\eta(0) + Y_{15}(x,0)M(0) + Y_{16}(x,0)T(0), \tag{60}$$

where

$$\eta(0) = -\frac{Y_{15}(L,0)M(0) + Y_{16}(L,0)T(0)}{Y_{13}(L,0)}, \tag{61.1}$$

$$M(0) = -\frac{Y_{43}(L,0)Y_{26}(L,0) - Y_{23}(L,0)Y_{26}(L,0)}{Y_{43}(L,0)Y_{25}(L,0) - Y_{23}(L,0)Y_{45}(L,0)}T(0). \tag{61.2}$$

As for the cantilever strain gradient beam, 8 terms in Eq. (4.1) provide converged values of the first 6 frequencies. For $\gamma = L/10$, these frequencies for the two sets of boundary conditions, (54) and (58), are listed in Table 5.

3.6 Simply supported beam

Henceforth, we list the boundary conditions, the characteristic equation for finding the frequencies, the equation for the mode shapes, and tabulate converged values of frequencies.

Boundary conditions:

$$v(0) = 0,\ \eta(0) = 0,\ M(0) = 0,\ \ v(L) = 0,\ \eta(L) = 0,\ M(L) = 0. \tag{62.1}$$

Characteristic equation:

$$\det\begin{bmatrix} Y_{12}(L) & Y_{14}(L) & Y_{16}(L) \\ Y_{32}(L) & Y_{34}(L) & Y_{36}(L) \\ Y_{52}(L) & Y_{54}(L) & Y_{56}(L) \end{bmatrix} = 0. \tag{62.2}$$

Equation for mode shapes:

$$v(x) = Y_{12}(x,0)\varphi(0) + Y_{14}(x,0)\beta(0) + Y_{16}(x,0)T(0), \tag{63}$$

where

$$\varphi(0) = -\frac{Y_{14}(L,0)\beta(0) + Y_{16}(L,0)T(0)}{Y_{12}(L,0)}, \tag{64.1}$$

$$\beta(0) = \frac{Y_{32}(L,0)Y_{56}(L,0) - Y_{52}(L,0)Y_{36}(L,0)}{Y_{52}(L,0)Y_{34}(L,0) - Y_{32}(L,0)Y_{54}(L,0)}T(0). \tag{64.2}$$

Boundary conditions:

$$v(0) = 0,\ \beta(0) = 0,\ M(0) = 0,\ \ v(L) = 0,\ \beta(L) = 0,\ M(L) = 0. \tag{65.1}$$

Characteristic equation:

$$\det\begin{bmatrix} Y_{12}(L) & Y_{13}(L) & Y_{16}(L) \\ Y_{42}(L) & Y_{43}(L) & Y_{46}(L) \\ Y_{52}(L) & Y_{53}(L) & Y_{56}(L) \end{bmatrix} = 0. \tag{65.2}$$

Equation for mode shapes:

$$u(x) = Y_{12}(x,0)\varphi(0) + Y_{13}(x,0)\eta(0) + Y_{16}(x,0)T(0), \tag{66}$$

where

$$\varphi(0) = -\frac{Y_{13}(L,0)\eta(0) + Y_{16}(L,0)T(0)}{Y_{12}(L,0)}, \tag{67.1}$$

$$\eta(0) = \frac{Y_{42}(L,0)Y_{56}(L,0) - Y_{52}(L,0)Y_{46}(L,0)}{Y_{52}(L,0)Y_{43}(L,0) - Y_{42}(L,0)Y_{53}(L,0)}T_0. \tag{67.2}$$

For $\gamma/L = 10$, the lowest six frequencies for the two sets of boundary conditions, Eqs. (62.1) and (65.1), are listed in Table 6.

3.7 Clamped-simply supported beam

(a) Boundary conditions:

$$v(0) = \varphi(0) = \eta(0) = 0, \quad v(L) = \eta(L) = M(L) = 0. \tag{68}$$

Characteristic equation:

$$\det\begin{bmatrix} Y_{14}(L) & Y_{15}(L) & Y_{16}(L) \\ Y_{34}(L) & Y_{35}(L) & Y_{36}(L) \\ Y_{54}(L) & Y_{55}(L) & Y_{56}(L) \end{bmatrix} = 0. \tag{69}$$

Equation for the mode shapes:

$$v(x) = Y_{14}(x,0)\beta(0) + Y_{15}(x,0)M(0) + Y_{16}(x,0)T(0), \tag{70}$$

where

$$\beta(0) = -\frac{Y_{15}(L,0)M(0) + Y_{16}(L,0)T(0)}{Y_{14}(L,0)}, \tag{71.1}$$

$$M(0) = -\frac{Y_{54}(L,0)Y_{36}(L,0) - Y_{34}(L,0)Y_{56}(L,0)}{Y_{54}(L,0)Y_{35}(L,0) - Y_{34}(L,0)Y_{55}(L,0)}T(0). \tag{71.2}$$

(b) Boundary conditions:

$$v(0) = \varphi(0) = \beta(0) = 0, \quad v(L) = \beta(L) = M(L) = 0. \tag{72}$$

Characteristic equation:

$$\det\begin{bmatrix} Y_{13}(L) & Y_{15}(L) & Y_{16}(L) \\ Y_{23}(L) & Y_{25}(L) & Y_{26}(L) \\ Y_{43}(L) & Y_{45}(L) & Y_{66}(L) \end{bmatrix} = 0. \tag{73}$$

Equation for the natural frequencies:

$$v(x) = Y_{13}(x,0)\eta(0) + Y_{15}(x,0)M(0) + Y_{16}(x,0)T(0), \tag{74}$$

where

$$\eta(0) = -\frac{Y_{15}(L,0)M(0) + Y_{16}(L,0)T(0)}{Y_{13}(L,0)}, \tag{75.1}$$

$$M(0) = -\frac{Y_{63}(L,0)Y_{46}(L,0) - Y_{43}(L,0)Y_{66}(L,0)}{Y_{63}(L,0)Y_{45}(L,0) - Y_{43}(L,0)Y_{65}(L,0)}T(0). \tag{75.2}$$

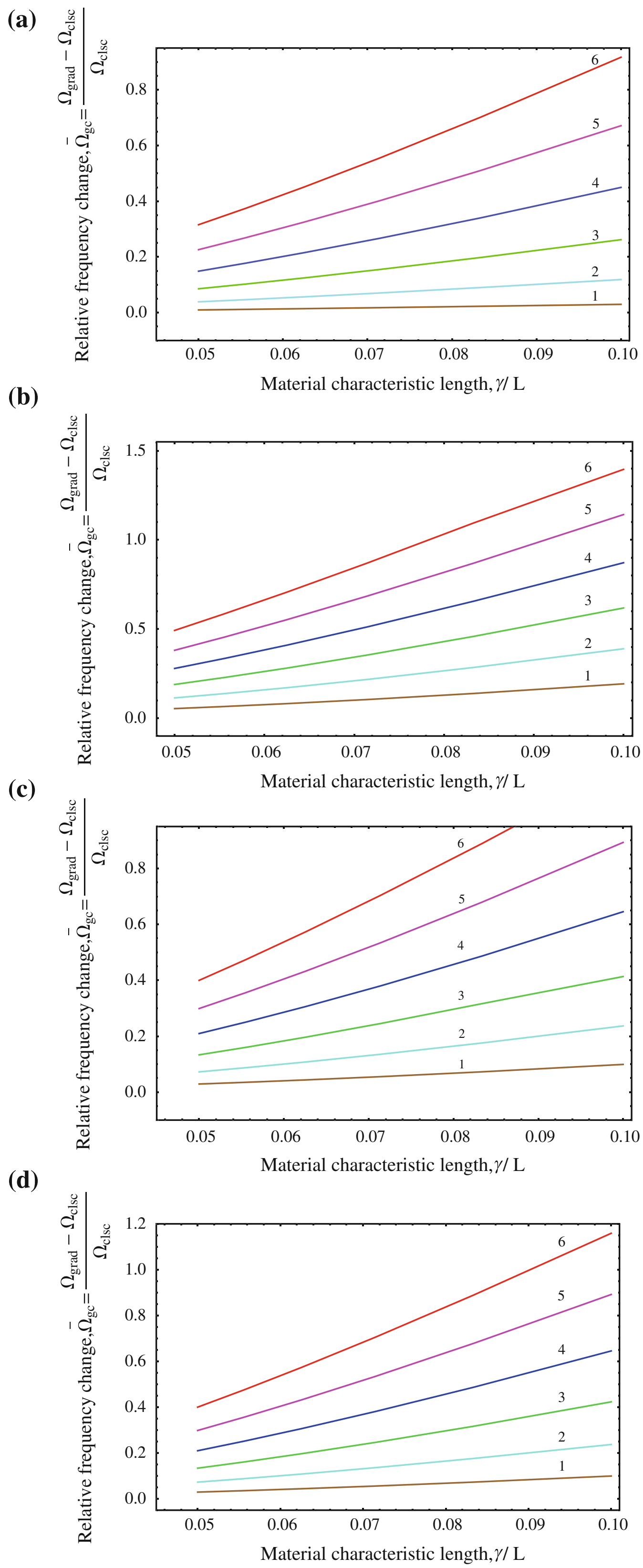

Fig. 1 Dependence of the relative frequency change upon the non-dimensional characteristic length for a beam with **a** both ends clamped, **b** cantilever, **c** clamped-simply supported, and **d** simply supported. A number on a curve represents the frequency (1st, 2nd, . . .) for which it is plotted

Table 6 For $\gamma = L/10$, frequencies of a simply supported strain-gradient-dependent EBB for two higher-order boundary conditions

Frequency\boundary conditions	$\eta(0) = \eta(L) = 0$	$\beta(0) = \beta(L) = 0$	Classical EBB
Ω_1	10.3452	10.1639	9.8696
Ω_2	46.6244	44.1696	39.45
Ω_3	122.0600	112.0949	88.8264
Ω_4	253.6043	228.9214	157.914
Ω_5	459.4597	412.3388	246.74
Ω_6	758.3004	681.1816	335.306

Table 7 For $\gamma = L/10$, frequencies of a clamped-simply supported strain-gradient-dependent EBB for two higher-order boundary conditions

Frequency\boundary conditions	$\eta(0) = \eta(L) = 0$	$\beta(0) = \beta(L) = 0$	Classical EBB
Ω_1	19.9926	16.9473	15.4182
Ω_2	72.4153	61.7966	49.9649
Ω_3	172.8228	148.3390	104.2427
Ω_4	338.6917	293.2958	178.2695
Ω_5	588.3368	514.9730	272.0310
Ω_6	940.6942	832.3242	385.5314

For $\gamma/L = 10$, the lowest six frequencies for the two sets of boundary conditions, Eqs. (68) and (72), are listed in Table 7.

For the EBB with four sets of end conditions, we have plotted in Fig. 1a–d the relative deviation $\bar{\Omega}_{gc} = \left(\Omega_{\text{grad}} - \Omega_{\text{clsc}}\right)/\Omega_{\text{clsc}}$, in the frequencies of a strain gradient beam with $\beta(0) = \beta(L) = 0$ from those of a classical beam as a function of the material characteristic length parameter, γ/L. Here, Ω_{grad} and Ω_{clsc} denote, respectively, the frequency of a strain gradient and the corresponding classical EBB. These plots evince that for each one of the first six frequencies this deviation increases with an increase in the value of γ/L. For a cantilever beam with $\gamma/L = 0.05$ (e.g., see Fig. 1b), this difference equals 2 % for the first frequency but nearly 40 % for the 6th frequency. For a clamped–clamped beam (Fig. 1a), this difference increases from 3 % for the first frequency to 35 % for the 6th frequency. Thus, one way to delineate whether a beam exhibits length scale effects is to find how much its natural frequencies differ from those of a classical EBB. One can overcome the difficulty of realizing in a laboratory the ideal clamped and simply supported edge conditions assumed herein by studying frequencies of a beam free at both ends.

In an attempt to help decide which one of the two higher-order boundary conditions is applicable, we have plotted in Fig. 2a–d the difference, $\bar{\Omega}_{\eta\beta} = \left(\Omega_{\text{grad}\eta} - \Omega_{\text{grad}\beta}\right)/\Omega_{\text{grad}\beta}$, between the frequencies of the strain gradient EB beam for the two higher-order boundary conditions for different values of γ/L. Here, $\Omega_{\text{grad}\eta}$ and $\Omega_{\text{grad}\beta}$ equal, respectively, the frequency of a strain-gradient-dependent EBB with boundary conditions $\eta(0) = \eta(L) = 0$ and $\beta(0) = \beta(L) = 0$, respectively. Depending upon the end conditions and the value of γ/L, these two types of boundary conditions can affect the fundamental frequency by as much as 15 % and the sixth lowest frequency by 35 %. Thus, test results can help determine which higher-order boundary conditions are pertinent for a strain-gradient-dependent EBB. As was also reported by Kong et al. [11], natural frequencies of the strain gradient beam depend upon which boundary conditions out of Eq. (39.4) are used; this difference equals ~17 % for Ω_1 and ~11 % for Ω_6. For an EBB, the thickness, h, may vary from $L/10$ to $L/100$. Thus, for $\gamma/L = 1/20$, γ/h will vary from 1/2 to 5. Assuming that the thickness of a nanobeam equals 2 nm, then for $\gamma = 1$ nm, one will see the difference between the natural frequencies of a classical EBB and those of a strain-gradient-dependent EBB.

3.8 Comparison of results with those of Kong et al.

Kong et al. [11] studied free vibrations of a gradient-dependent EBB under the following boundary conditions:

$$v(0) = \varphi(0) = \eta(0) = 0, \quad M(L) = T(L) = \beta(L) = 0. \tag{76}$$

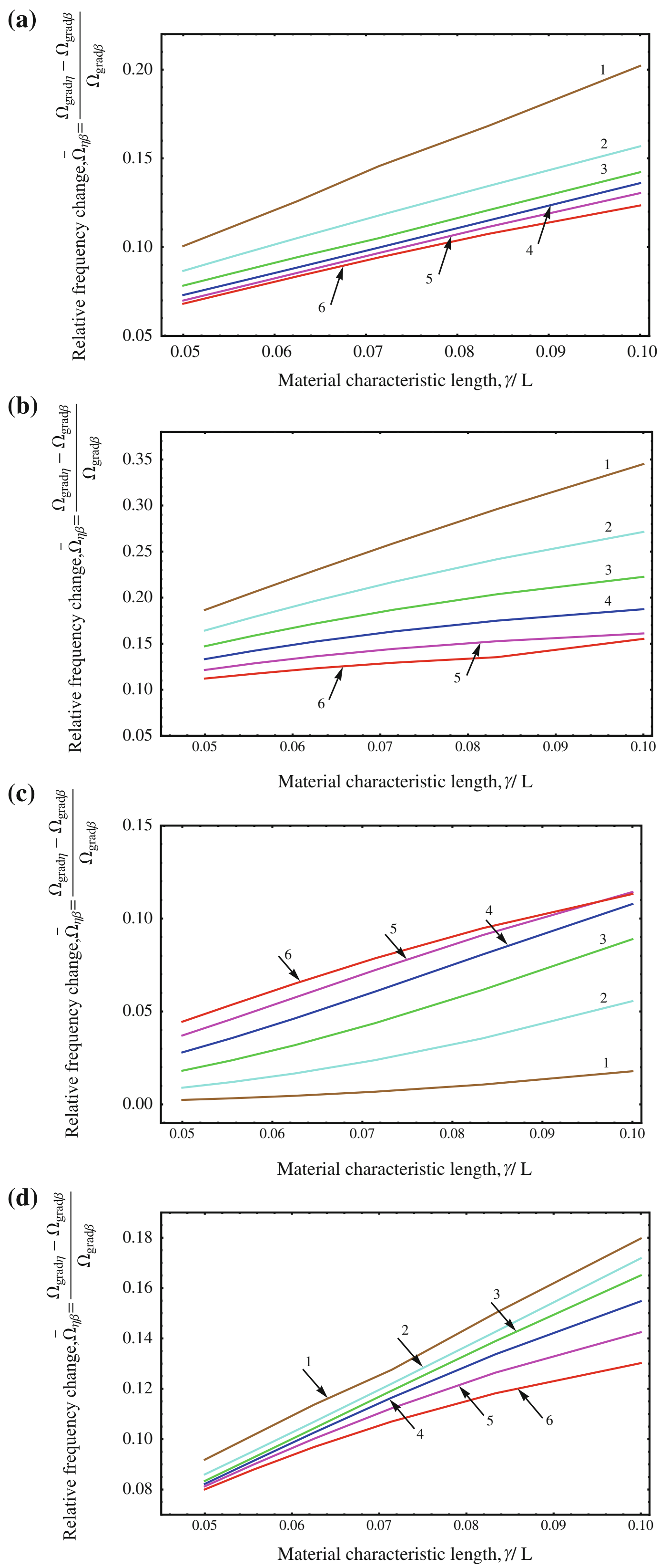

Fig. 2 Dependence of the relative frequency change for two types of higher-order boundary conditions; **a** clamped–clamped beam, **b** cantilever, **c** clamped-simply supported, and **d** simply supported. A number on a curve corresponds to the frequency for which the plot is shown

Following the procedure of Sect. 3.4, the characteristic equation for the determination of natural frequencies is

$$\det \begin{bmatrix} Y_{44}(L) & Y_{45}(L) & Y_{46}(L) \\ Y_{54}(L) & Y_{55}(L) & Y_{56}(L) \\ Y_{64}(L) & Y_{65}(L) & Y_{66}(L) \end{bmatrix} = 0. \tag{77}$$

For $L = 400\ \mu$m and values of other material and geometric parameters for which results are plotted in Fig. 4 of Kong et al. [11], we get

$$\Omega_{1\text{clsc}} = 3.5160, \quad \Omega_{2\text{clsc}} = 22.0345, \quad \Omega_{3\text{clsc}} = 61.6972, \quad \Omega_{4\text{clsc}} = 120.902, \tag{78.1}$$

and

$$\Omega_{1\text{grad}} = 4.1101, \quad \Omega_{2\text{grad}} = 26.6733, \quad \Omega_{3\text{grad}} = 80.0041, \quad \Omega_{4\text{grad}} = 171.8570. \tag{78.2}$$

Here, $\Omega_{1\text{clsc}}$ and $\Omega_{2\text{grad}}$ equal, respectively, the first frequency of the classical and the gradient-dependent EBB. Our results agree with those of Kong et al. after the coefficient 43/225 in Eq. (23) of their paper is corrected to 8/15, and the units of frequency in Fig. 4 of their paper are changed from Hz to rad/s.

4 Conclusions

We have successfully extended the method of initial values (or the transfer matrix method) to find natural frequencies of a classical Euler–Bernoulli (EB) beam to that for studying vibrations of a strain-gradient-dependent EB beam under different end conditions, for example, clamped, free, simply supported. For the classical EB beam, the analytical expression for the matricant (or the carry-over matrix) is found and closed-form expressions for the natural frequencies and the corresponding mode shapes are derived, which agree with those found by directly solving the fourth-order ordinary differential equation. For the strain gradient EBB, the matricant is found numerically.

For the strain gradient EB in addition to the end conditions used for the classical EB beam, there are two types of boundary conditions that need to be imposed at the ends. It is shown that the first six natural frequencies found by using the two types of boundary conditions differ by as much as 35 % depending upon the value of the material characteristic length. Furthermore, the 6th frequency of a strain-gradient-dependent EB beam may differ from that of the classical EB beam by 30 % for the material characteristic length equal to 0.05 L where L equals the beam length. Thus, by carefully measuring the natural frequencies of a beam, one can delineate whether or not it is made of a strain-gradient-dependent material. This difference in frequencies is prevalent even when both edges of a beam are free which facilitates conducting experiments.

References

1. Toupin, R.A.: Elastic materials with couple stresses. Arch. Ration. Mech. **11**, 385–414 (1962)
2. Mindlin, R.D.: Second gradient of strain and surface-tension in linear elasticity. Int. J. Solids Struct. **1**, 415–448 (1962)
3. Gurtin, M.E.: Thermodynamics and the possibility of spatial interaction on elastic materials. Arch. Ration. Mech. Anal. **19**, 339–352 (1965)
4. Eringen, A.C.: Nonlocal polar elastic continua. Int. J. Eng. Sci. **10**, 1–16 (1972)
5. Batra, R.C.: Thermodynamics of non-simple materials. J. Elasticity **6**, 451–456 (1976)
6. Green, A.E., Laws, N.: On the entropy production inequality. Arch. Rat. Mech. Anal. **45**, 47–53 (1972)
7. Dillon, O.D. Jr., Kratochvil, J.: A strain gradient theory of plasticity. Int. J. Solids Struct. **6**, 1513–1533 (1970)
8. Batra, R.C.: The initiation and growth of, and the interaction among adiabatic shear bands in simple and dipolar materials. Int. J. Plasticity **3**, 75–89 (1987)
9. Peddieson, J., Buchanan, G.R., McNitt, R.P.: Application of nonlocal continuum models to nanotechnology. Int. J. Eng. Sci. **41**, 305–312 (2003)
10. Lam, D.C.C., Fang, Y., Chong, A.C.M., Wang, J., Tong, P.: Experiments and theory in strain gradient elasticity. J. Mech. Phys. Solids **51**, 1477–1508 (2003)
11. Kong, S., Zhou, S.J., Nie, Z.F., Wang, K.: Static and dynamic analysis of microbeams based on strain gradient elasticity theory. Int. J. Eng. Sci. **47**, 487–498 (2009)
12. Ashgari, M., Kahrobaiyan, M.H., Nikfar, M., Ahmadian, M.T.: A size-dependent nonlinear Timoshenko microbeam model based on the strain gradient theory. Acta Mech. **223**, 1233–1249 (2012)
13. Marguerre, K. Wölfel: Mechanics of Vibrations. Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands (1979)
14. Magrab, E.B.: Vibrations of Elastic Structural Members. Mechanics of Structural Systems 3. Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands (1979)