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# Material tailoring and moduli homogenization for finite twisting deformations of functionally graded Mooney-Rivlin hollow cylinders

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**Abstract** We analytically analyze finite plane strain twisting deformations of a hollow cylinder made of an isotropic and inhomogeneous Mooney-Rivlin material with material moduli varying in the radial direction. The cylinder is deformed by applying either tangential tractions on the inner surface and tangential displacements on the outer surface or vice versa. The radial variation of the moduli is found that will minimize the tangential displacement of the bounding surface where tangential traction is specified. Furthermore, the modulus of a homogeneous neo-Hookean cylinder is found that is energetically equivalent to the inhomogeneous cylinder.

## 1 Introduction

Structures made of functionally graded materials (FGMs) are designed to optimize their performance in one or more directions depending upon the loads anticipated to act upon them. There are numerous papers that analyze deformations of such structures made of linear elastic materials. However, very few papers deal with finding the variation of material properties that optimize one or more deformation parameters such as the frequency, the buckling load and the deflection; such problems are usually referred to as material tailoring. Leissa and Vagins [1] assumed that all material moduli of an orthotropic material vary according to the same exponential relation and found the exponent to attain either uniform hoop stress or uniform in-plane shear stress through the cylinder thickness. Bert and Niedenfuhr [2] found the thickness variation for a rotating circular disk to have a uniform hoop stress in the disk. For cylinders and spheres made of isotropic and incompressible Hookean solids, Batra [3] found that the shear modulus must be proportional to the radius $r$ for the hoop stress to be constant in them when they are deformed by applying pressure to their inner surfaces. Batra [4] has generalized these results to cylinders and spheres made of Mooney-Rivlin materials. Tanaka et al. [5] used the finite element method (FEM) to tailor spatial distribution of the volume fraction of constituents in a cylinder subjected to axisymmetric thermal boundary conditions so as to attain stresses within a desirable range. Sun and Hyer [6] have numerically investigated continuously varying the fiber orientation along the circumference of an elliptic cylinder to optimize the axial buckling load. Batra and Jin [7] used the FEM to find through-the-thickness variation of the fiber orientation angle to optimize the fundamental frequency of free vibrations of a rectangular plate. Karandikar and Mistree [8] developed a technique to optimally design fiber volume fraction to meet the design objectives. Naghshineh [9] tailored material properties in a beam to

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minimize the sound radiated from it due to its vibrations. Qian and Batra [10] found the variation of material parameters in two directions for a cantilever plate to have the minimum fundamental frequency. Nie and Batra [11,12] have analyzed material tailoring problems for spheres, cylinders and rotating disks. Batra [13] determined the axial variation of the shear modulus to control the angle of twist in a circular Hookean cylinder deformed by applying torques at the end faces. Except for the work described in [4], these investigations have considered infinitesimal deformations of a structure. Here, we study the material tailoring problem for finite twisting deformations of a hollow cylinder with material properties varying only in the radial direction and deformed by applying tangential displacements on one surface and tangential tractions on the other. The cylinder deformations are similar to those in the Couette flow [14] of a viscous fluid enclosed between two cylinders and one cylinder rotating with respect to the other one.

## 2 Problem formulation

We study finite plane strain deformations of a hollow circular cylinder made of an isotropic and functionally graded Mooney-Rivlin material with values of two material parameters varying continuously in the radial direction. The cylinder is deformed by applying either shearing tractions on the outer surface and tangential displacements on the inner surface or vice versa. We use cylindrical coordinates to describe the position $(r, \theta, z)$ of a material point in the current configuration that in the reference configuration occupied the place $(R, \Theta, Z)$. Deformations of the cylinder are governed by the equilibrium equations

$$\begin{aligned}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0, \\
\frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} &= 0, \\
\frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} &= 0.
\end{aligned} \tag{1}$$

Here $\boldsymbol{\sigma}$ is the Cauchy stress tensor, and $\sigma_{rr}, \sigma_{r\theta} \ldots$ are the physical components of $\boldsymbol{\sigma}$ with respect to cylindrical coordinates $(r, \theta, z)$. The pertinent boundary conditions are

$$\begin{aligned}
\sigma_{r\theta} &= \tau_{\text{in}}, \quad \sigma_{rr} = 0, \quad \sigma_{rz} = 0 \quad \text{on} \quad r = r_{\text{in}}, \\
u_r &= 0, \quad u_\theta = u_{\text{ou}}, \quad u_z = 0 \quad \text{on} \quad r = r_{\text{ou}}.
\end{aligned} \tag{2}$$

In Eq. (2), $\tau_{\text{in}}$ is the prescribed tangential traction on the inner surface $r = r_{\text{in}}$ and $u_{\text{ou}}$ is the prescribed tangential displacement on the outer surface $r = r_{\text{ou}}$ of the cylinder. The radii of the inner and the outer surfaces equal $R_{\text{in}}$ and $R_{\text{ou}}$, respectively, in the undeformed reference configuration.

Equations (1) and (2) are supplemented by the following constitutive relation for the Mooney-Rivlin material [14]:

$$\boldsymbol{\sigma} = -p\mathbf{1} + c_1(R)\mathbf{B} + c_{-1}(R)\mathbf{B}^{-1}. \tag{3}$$

Here $p$ is the hydrostatic pressure not determined from the deformation, $c_1$ and $c_{-1}$ are material parameters that are continuous functions of $R$, and $\mathbf{B}$ is the left Cauchy-Green tensor. For the Mooney-Rivlin material, the strain energy density, $W$, per unit volume is given by

$$2W = c_1(R)(I - 3) - c_{-1}(R)(II - 3), \tag{4}$$

where

$$I = tr(\mathbf{B}), \quad II = tr(\mathbf{B}^{-1}) \tag{5}$$

are the first and the second invariants of $\mathbf{B}$. In writing Eq. (5), we have tacitly assumed that the deformation is isochoric, that is, $\det(\mathbf{B}) = 1$. We note that $\boldsymbol{\sigma} + p\mathbf{1} = (\partial W/\partial \mathbf{B})\mathbf{B}$.

## 3 Problem solution

We use the semi-inverse method and following Ericksen [15] assume the following form for the solution:

$$r = R,\ \theta = \Theta + \varphi(R),\ z = Z. \tag{6}$$

That is, a circle of radius $R$ in the undeformed configuration is simply rotated by an angle $\varphi(R)$ that continuously varies with $R$. For the deformation field (6), *physical components* of the deformation gradient $\mathbf{F}$ and tensors $\mathbf{B} = (\mathbf{FF}^T)$ and $\mathbf{B}^{-1}$ are

$$[F] = \begin{bmatrix} 1 & 0 & 0 \\ r\varphi' & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [B] = \begin{bmatrix} 1 & r\varphi' & 0 \\ r\varphi' & 1 + r^2\varphi'^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [B^{-1}] = \begin{bmatrix} 1 + r^2\varphi'^2 & -r\varphi' & 0 \\ -r\varphi' & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{7}$$

where $\varphi' = d\varphi/dR = d\varphi/dr$.

Substitution of Eq. (7) into Eq. (3) gives $\sigma_{rz} = \sigma_{\theta z} = 0$, and Eq. $(1)_3$ implies that the hydrostatic pressure, $p$, is independent of $z$. Substitution for stresses in Eq. $(1)_2$ gives

$$\frac{1}{r}\left[r^3\mu\varphi'\right]' - \frac{\partial p}{\partial \theta} = 0, \tag{8}$$

where $\mu(r) = c_1(r) - c_{-1}(r)$ is the shear modulus for the cylinder material.

In order for $p$ to be a single-valued function of $\theta$, $\partial p/\partial\theta$ must identically vanish. Thus,

$$\varphi' = \frac{\alpha}{r^3\mu}, \tag{9}$$

where $\alpha$ is a constant, and

$$\varphi = \alpha \int \frac{\mathrm{d}r}{\mu r^3} + \beta, \tag{10}$$

where $\beta$ is a constant.

The equilibrium equation $(1)_1$ upon integration gives

$$p = -\int \mu r\varphi'^2 \mathrm{d}r + c_1 + c_{-1}(1 + r^2\varphi'^2) + \gamma, \tag{11}$$

where $\gamma$ is a constant.

With $c_1$ and $c_{-1}$ known functions of $r$, Eqs. (10) and (11) can be integrated. The three constants of integration, $\alpha$, $\beta$ and $\gamma$, are determined from boundary conditions $(2)_1$, $(2)_2$ and $(2)_5$, whereas the remaining boundary conditions in (2) are identically satisfied. Boundary condition $(2)_1$ and Eqs. (9), (3) and (7) give

$$\alpha = R_{\mathrm{in}}^2\tau_{\mathrm{in}}, \tag{12}$$

where $R_{\mathrm{in}}$ is the inner radius of the undeformed hollow cylinder. For boundary condition $(2)_5$ to be satisfied,

$$\beta + \alpha\left[\int \frac{\mathrm{d}r}{\mu r^3}\right]\Bigg|_{r=r_{\mathrm{ou}}} = u_{\mathrm{ou}}/R_{\mathrm{ou}}, \tag{13}$$

which determines $\beta$. Boundary condition $(2)_2$ gives

$$\gamma = \left[\int \mu r\varphi'^2 \mathrm{d}r\right]\Bigg|_{r=r_{\mathrm{in}}}. \tag{14}$$

Knowing the function $\varphi$, we can find the stored energy density, $W$, from Eqs. (4), (5) and (7). Thus,

$$2W = (c_1 - c_{-1})r^2\varphi'^2 = (c_1 - c_{-1})\frac{R_{\mathrm{in}}^4\tau_{\mathrm{in}}^2}{r^4\mu^2}. \tag{15}$$

In order for $W$ to be nonnegative everywhere, $(c_1 - c_{-1})$ must be nonnegative for every value of $r = R$ in the range $[R_{\text{in}}, R_{\text{ou}}]$. Equations (7), (9) and (3) result in

$$\sigma_{r\theta} = \mu r \varphi' = \frac{\tau_{\text{in}} R_{\text{in}}^2}{r^2}. \tag{16}$$

In order for the shear stress $\sigma_{r\theta}$ and the shear strain $B_{r\theta}$ to have the same sign, $\mu$ and equivalently $(c_1 - c_{-1})$ must be positive everywhere in $[R_{\text{in}}, R_{\text{ou}}]$. Equation (16) implies that for the deformation (6), the shear stress $\sigma_{r\theta}$ in every inhomogeneous isotropic Mooney-Rivlin material is the same and it depends only upon the tangential traction applied on the inner surface and the inner radius of the cylinder. That is, Eq. (16) *is a universal relation*. A similar result will hold if the shear traction were prescribed on the outer surface of the cylinder. It follows from the equilibrium condition requiring that the torque on each thin circular cylinder of radius $r$ be the same. In particular, Eq. (16) holds for a cylinder made of a homogeneous isotropic Mooney-Rivlin material. We note that even though the shear stress $\sigma_{r\theta}$ at a point is independent of the value of $\mu$, the tangential displacement $u_\theta = r\varphi$ depends upon $\mu$. Thus, for the same boundary conditions applied on the inner and the outer surfaces of the hollow cylinder, the radial distribution of $u_\theta$ will be different in cylinders made of different materials.

For a homogeneous cylinder with $\mu = \mu_h$, the total strain energy, $E_h$, per unit length of the cylinder can be readily found from Eq. (15) and is given by

$$E_h = \frac{\pi (c_{1h} - c_{-1h})}{2\mu_h^2} \tau_{\text{in}}^2 R_{\text{in}}^2 \left(1 - \frac{R_{\text{in}}^2}{R_{\text{ou}}^2}\right), \tag{17}$$

where $c_{1h}$ and $c_{-1h}$ are values of $c_1$ and $c_{-1}$, respectively, for the homogeneous cylinder.

## 4 Material tailoring

We study the aforestated problem for boundary conditions $u_\theta = u_{\text{in}}$ at $r = r_{\text{in}} = R_{\text{in}}$ and $\sigma_{r\theta} = \tau_{\text{ou}}$ at $r = r_{\text{ou}} = R_{\text{ou}}$. The remaining boundary conditions listed in Eq. (2) are identically satisfied. It follows from Eqs. (9) and (12) that

$$\varphi' = \frac{\tau_{\text{ou}} R_{\text{ou}}^2}{\mu r^3} \tag{18}$$

and

$$u_\theta = r\varphi = \tau_{\text{ou}} R_{\text{ou}}^2 r \int_{r_{\text{in}}}^{r} \frac{\mathrm{d}r}{\mu r^3} + u_{\text{in}}. \tag{19}$$

We wish to find $\mu(r)$ so that $u_\theta^0 \equiv u_\theta(r = r_{\text{ou}})$ is a minimum and

$$\int_{r_{\text{in}}}^{r_{\text{ou}}} \mu r \, \mathrm{d}r = \frac{\mu_a}{2}(R_{\text{ou}}^2 - R_{\text{in}}^2), \tag{20}$$

is a constant. That is, among all radial variations of $\mu(r)$ that have the same average value, $\mu_a$, find the one that minimizes $u_\theta^0$. Thus, the material tailoring problem can be stated as follows: find $\mu(r)$ such that

$$I = \int_{R_{\text{in}}}^{R_{\text{ou}}} \frac{\mathrm{d}r}{\mu r^3} + \lambda^2 \int_{R_{\text{in}}}^{R_{\text{ou}}} (\mu - \mu_a) r \mathrm{d}r \tag{21}$$

takes an extreme value. In Eq. (21), $\lambda^2$ is a constant Lagrange multiplier. Using standard techniques in variational calculus, we obtain

$$\mu = \frac{1}{\lambda r^2}, \ \lambda = \frac{2 \ln \eta}{\mu_a (\eta^2 - 1) R_{\text{in}}^2}, \ \eta = R_{\text{ou}}/R_{\text{in}}. \tag{22}$$

Substitution for $\mu$ from Eq. (22) into Eq. (19) and evaluation of the integral gives

$$\left(u_\theta^0\right)_{\min} = \frac{2\tau_{\text{ou}} R_{\text{ou}} (\eta \ln \eta)^2}{\mu_a (\eta^2 - 1)} + u_{\text{in}}. \tag{23}$$

For $\mu > 0$, Eq. (21) gives $\partial^2 I / \partial \mu^2 > 0$ and indeed Eq. (23) gives the minimum value of $u_\theta^0$. Thus, for the circumferential displacement prescribed on the inner surface, the modulus $\mu$ must be inversely proportional to $r^2$ for the circumferential displacement on the outer surface to be the minimum. It follows from Eqs. (7), (18) and $(22)_1$ that the shear strain, $B_{r\theta}$, is constant for this cylinder, and equals

$$\lambda \tau_{\text{ou}} R_{\text{ou}}^2 \quad \text{or} \quad 2\tau_{\text{ou}} \eta^2 \ln \eta / \left(\mu_a (\eta^2 - 1)\right).$$

Recalling that the strain energy stored in the elastic body equals work done by external forces, therefore, the total strain energy per unit length, $E$, for the cylinder with $\mu$ given by Eq. (22) will be the minimum, and will equal $2\pi R_{\text{ou}} \tau_{\text{ou}} (u_\theta^0)_{\min}$.

## 5 Equivalent homogeneous cylinder

We now investigate the problem of finding the shear modulus, $\mu_h^{\min}$, for the homogeneous cylinder for which the circumferential displacement on the outermost surface equals the minimum value given by Eq. (23) for the inhomogeneous cylinder. For the homogeneous cylinder, Eq. (19) gives

$$\left(u_\theta^0\right)_h = \frac{\tau_{\text{ou}} R_{\text{ou}}}{2\mu_h} \left(\eta^2 - 1\right) + u_{\text{in}}. \tag{24}$$

Setting $\left(u_\theta^0\right)_{\min}$ given by Eq. (23) equal to $\left(u_\theta^0\right)_h$ given by Eq. (24), we get

$$\frac{\mu_h^{\min}}{\mu_a} = \left(\frac{\sinh \psi}{\psi}\right)^2, \quad \psi = \ln \eta. \tag{25}$$

We now consider an inhomogeneous cylinder with power law variation of the shear modulus. Recalling that $\mu$ for the cylinder with minimum value of $u_\theta^0$ is inversely proportional to $r^2$, we find $\mu_h$ for the inhomogeneous cylinder for which

$$\mu_n = \frac{\mu_a}{k_n \rho^{2(1-n)}}, \quad \rho = r / R_{\text{in}}, \tag{26}$$

where $n \neq 0$. The value of $k_n$ found from the relation

$$\int_1^\eta \left(\frac{1}{k_n \rho^{2(1-n)}} - 1\right) \rho d\rho = 0 \tag{27}$$

is given by

$$k_n = \frac{\eta^{2n} - 1}{n(\eta^2 - 1)}. \tag{28}$$

Thus, the average modulus of a cylinder of modulus $\mu_n$ equals $\mu_a$. Equating $u_\theta^0$ for homogeneous and inhomogeneous cylinders, we get

$$\frac{\mu_h}{\mu_a} = \left(\frac{\sinh \psi}{\psi}\right)^2 \left(\frac{n\psi}{\sinh n\psi}\right)^2. \tag{29}$$

The expression (25) for $\mu_h^{\min}$ is obtained from Eq. (29) by letting $n \to 0$ and recalling that

$$\lim_{n \to 0} \frac{n\psi}{\sinh n\psi} = 1. \tag{30}$$

Setting $n = 1$ gives results for a homogeneous cylinder. We note that expression (29) for $\mu_h$ is an even function of $n$. The graph of $\mu_h / \mu_a$ versus $n$ is depicted in Fig. 1 for $\eta = 2$, 3 and 4. For a given $n$, the value of $\mu_h / \mu_a$ strongly depends upon $\eta = R_{\text{ou}} / R_{\text{in}}$. For a given $\eta$, $\mu_h / \mu_a$ varies continuously with $n$, and results for $n = 0$ can be obtained by taking the limit as $n \to 0$.

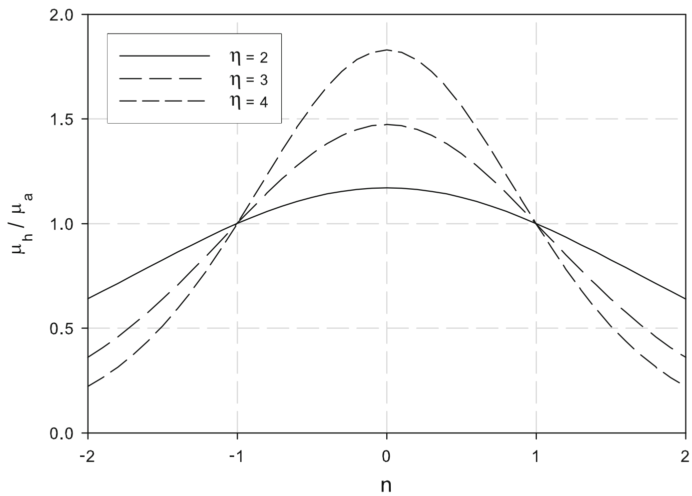


**Fig. 1** Graph of $\mu_h/\mu_a$ versus $n$ obtained from Eq. (29) for the three values of $\eta$ indicated on the graph. Homogeneous solid corresponds to $n = 1$ so that $\mu_h = \mu_a$, and as shown this also occurs when $n = -1$

## 6 Energetically equivalent homogeneous neo-Hookean cylinder

The constitutive relation for a neo-Hookean material is obtained from Eq. (3) by setting $c_{-1} = 0$; thus $\mu = c_1, c_{1h} = \mu_h$, and $\mu_h$ is given by Eq. (29).

We cannot find $c_{1h}$ and $c_{-1h}$ for the Mooney-Rivlin cylinder since we will need to find values of two parameters from one equation (29). We realize that the homogenization problem for composites involves finding values of several material parameters. However, there we consider a variety of deformations but here we are considering only twisting deformations.

For the radial expansion of a hollow cylinder made of a compressible Hookean solid, Dryden and Batra [16] found upper and lower bounds on Young's modulus of an energetically equivalent homogeneous cylinder. Here, we have found the exact value of the shear modulus for the energetically equivalent homogeneous cylinder.

## 7 Cylinder made of inhomogeneous and incompressible Hookean material

We note that the assumption of a material being Hookean is valid only when deformations are infinitesimal. Analogous to Eq. (6) we postulate that

$$u_r = 0, \quad u_\theta = u_\theta(R), \quad u_z = 0. \tag{31}$$

Corresponding components of the infinitesimal strain tensor, $\mathbf{e}$, are

$$e_{rr} = e_{\theta\theta} = e_{zz} = e_{\theta z} = e_{zr} = 0, \; 2e_{r\theta} = \left(\frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right). \tag{32}$$

The constitutive relation for an isotropic, incompressible and inhomogeneous Hookean material is

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2\bar{\mu}(R)\mathbf{e}, \tag{33}$$

where $\bar{\mu}$ is the shear modulus. Equations (32) and (33) give

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = -p, \quad \sigma_{rz} = \sigma_{\theta z} = 0, \quad \sigma_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right). \tag{34}$$

Following the same reasoning as that used in Sect. 3, we get

$$p = \text{constant}, \quad \sigma_{r\theta} = \tau_{\text{in}}\frac{R_{\text{in}}^2}{R^2} = \mu R \frac{\partial}{\partial R}\left(\frac{u_\theta}{R}\right). \tag{35}$$

Thus,

$$u_\theta = \tau_{\text{in}} R_{\text{in}}^2 R \int_{R_{\text{in}}}^{R} \frac{dR}{\mu R^3} + u_\theta(R_{\text{in}}). \tag{36}$$

Equation (36) is the same as Eq. (19) for the finite deformation problem since $\tau_{\text{in}} R_{\text{in}}^2 = \tau_{\text{ou}} R_{\text{ou}}^2$. Thus, the displacement fields and the shear stresses for the linear and the nonlinear problems are identical but the hydrostatic pressure $p$ and hence normal stresses $\sigma_{rr}$, $\sigma_{\theta\theta}$ and $\sigma_{zz}$ have different values. One can thus analyze the material tailoring and the energetically equivalent homogeneous cylinder problem in the same way as for the nonlinear problem discussed earlier.

We recall that for simple shearing deformations of a block of elastic material enclosed between two rigid flat infinite plates, the solution for the nonlinear problem has non-zero normal stresses on the top and the bottom plates but these stresses vanish for the linear problem; for example, see [14].

## Conclusions

We have analytically studied finite plane strain deformations of an inhomogeneous Mooney-Rivlin hollow cylinder deformed by applying tangential displacements on one surface and tangential tractions on the other surface. It is found that for the tangential displacements of points on the surface where tangential tractions are prescribed to be the minimum, the shear modulus of the material must be proportional to $r^{-2}$ where $r$ is the radial coordinate of a point. For the neo-Hookean cylinder, we have found the modulus of a homogeneous cylinder whose total strain energy of deformation equals that of the inhomogeneous cylinder and the two cylinders subjected to same boundary conditions. For the Mooney-Rivlin cylinder, a similar result holds for the shear modulus.

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