

Optimal Design of Functionally Graded Incompressible Linear Elastic Cylinders and Spheres

R. C. Batra*

Virginia Polytechnic Institute and State University,
Blacksburg, Virginia 24061

DOI: 10.2514/1.34937

We find closed-form solutions for axisymmetric plane strain radial deformations of a functionally graded circular hollow cylinder and radial expansion/contraction of a hollow sphere loaded on inner and outer surfaces by uniform hydrostatic pressures. The cylinder and the sphere are presumed to be made of isotropic and incompressible linear elastic materials with the shear modulus a general function of the radius. It is found that the optimal value of the hoop or the circumferential stress in a cylinder and a sphere is a constant and occurs for the linear variation in the radial direction of the shear modulus. The analytical results presented here should serve as benchmarks for verifying numerical solutions of problems.

Nomenclature

$\tilde{b}$	=	constant of integration for radial displacement of a sphere particle
c	=	constant of integration for radial displacement of a cylinder particle
D	=	constant appearing in expressions for stresses in the sphere problem
d	=	constant of integration for pressure in the cylinder problem
$\mathbf{E}$	=	infinitesimal strain tensor
e	=	constant of integration for pressure in the sphere problem
$\hat{g}$	=	constant appearing in expressions for stresses in the cylinder problem
$\mathbf{H}$	=	displacement gradient
$\mathbf{n}$	=	unit outward normal to a bounding surface
n	=	inhomogeneity parameter
p	=	hydrostatic pressure
p_{in}	=	pressure applied on the inner surface
p_{ou}	=	pressure applied on the outer surface
R, Θ, Z	=	coordinates of a point in cylindrical coordinate system
R, Θ, Φ	=	coordinates of a point in spherical coordinate system
R_{in}	=	inner radius
R_{ou}	=	outer radius
u	=	radial displacement of a point
α	=	slope of the shear modulus versus the radius in the cylinder problem
β	=	slope of the shear modulus versus the radius in the sphere problem
μ	=	shear modulus
μ_o	=	reference shear modulus of the material on the inner surface
σ	=	stress tensor
σ_{RR}	=	radial stress at a point
$\sigma_{\Theta\Theta}$	=	hoop stress at a point
σ_{ZZ}	=	normal axial stress at a cylinder particle

$\sigma_{\Phi\Phi}$	=	circumferential stress at a sphere particle
$\sigma_{R\Theta}$	=	shear stress on $R = \text{constant}$ plane in the circumferential direction
$\sigma_{Z\Theta}$	=	shear stress on $Z = \text{constant}$ plane in the circumferential direction
σ_{RZ}	=	shear stress on $R = \text{constant}$ plane in the axial direction
$\sigma_{\Theta\Phi}$	=	shear stress on $\Theta = \text{constant}$ plane in the Φ direction
$\mathbf{1}$	=	3×3 identity tensor

I. Introduction

FUNCTIONALLY graded materials (FGMs) are usually composed of two or more constituents with their volume fractions and, hence, effective material properties varying continuously in one or more spatial directions. The volume fractions of constituents and their materials can be tailored to suit intended applications; e.g., for resisting elevated temperatures, one could design the structure to have pure ceramic on the side exposed to high temperatures and pure metal on the other side to provide strength [1]. Material properties of a polymer can be modified by exposing it to light of different intensity (e.g., see Lambros et al. [2]). For a fiber reinforced composite, one could vary the volume fraction of fibers and their orientations in the thickness direction to obtain suitable gradation in the moduli [3]. Natural materials such as bamboo sticks, tree trunks, and human teeth exhibit continuously varying material properties. There is extensive literature on FGMs and it is almost impossible to review it in a paper other than in a review article. Here, we mention works on cylinders and spheres.

Rubberlike materials are being increasingly used in aerospace, automotive, and biomedical fields; a familiar example is an O-ring in a space shuttle. These materials are usually regarded as incompressible. Their material properties can be varied by vulcanization and using different amounts of additives. In spite of numerous works on FGMs, there is hardly any work related to functionally graded (FG) rubberlike materials. Batra [4] studied numerically, with the finite element method, radial deformations of a cylinder made of a Mooney–Rivlin material, with material parameters varying smoothly in the radial direction, and did not use the phrase FG. Mathematically, these are inhomogeneous materials. In the solution of problems with a numerical method, the procedure to analyze problems for a homogeneous material can be adopted for inhomogeneous materials. For example, Batra [4] used values of material properties at integration points to numerically evaluate integrals defined on an element and obtained results for a nonlinear problem that agreed well with its analytical solution.

Received 2 October 2007; revision received 3 March 2008; accepted for publication 28 March 2008. Copyright © 2008 by Romesh C. Batra. Published by the American Institute of Aeronautics and Astronautics, Inc., with permission. Copies of this paper may be made for personal or internal use, on condition that the copier pay the \$10.00 per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923; include the code 0001-1452/08 \$10.00 in correspondence with the CCC.

*Clifton C. Garvin Professor, Department of Engineering Science and Mechanics, M/C 0219; rbatra@vt.edu

We note that Horgan and Chan [5] have analyzed deformations of FG cylinders composed of compressible isotropic linear elastic materials, with the variation of Young's modulus in the radial direction given by a power law relation and constant Poisson's ratio. Lechnitskii's book [6] has solutions for several problems involving inhomogeneous linear elastic materials. One could also divide the thickness of an FG cylinder into several layers, regard material properties in each layer as uniform, and use the approach outlined in Timoshenko and Goodier's book [7] for composite cylinders. With an increase in the number of layers, the solution for the layered cylinder will approach that for the FG cylinder. However, these books focus on compressible materials. For FG simply supported thermoelastic and magneto-electroelastic plates made of compressible materials, with material properties varying only in the thickness direction, Vel and Batra [8,9], and Pan and Han [10] have given analytical solutions. Tarn [11], and Tarn and Chang [12] have provided exact solutions for FG anisotropic cylinders subjected to thermal and mechanical loads, by assuming that all elastic constants are power law functions of the radius with the same exponent. Alshits and Kirchner [13] have derived Green's functions for hollow and solid cylinders under different boundary conditions and with material properties varying in the radial direction. Oral and Anlas [14] have expressed governing equations for an inhomogeneous cylindrical anisotropic body in terms of stress potentials, and provided closed-form expressions for the potentials when Young's and shear moduli are expressed as power law functions of the radius and Poisson's ratio is constant. Liew et al. [15] computed thermal stresses in an FG cylinder by dividing it into discrete homogeneous subcylinders in the radial direction. Jabbari et al. [16] expanded displacements and temperatures in a Fourier series to study nonaxisymmetric deformations in a thick hollow FG cylinder with temperature and pressure prescribed on its inner and outer surfaces. Obata and Noda [17] found steady-state thermal stresses in FG hollow cylinders and spheres, and Kim and Noda [18] used the Green function to solve the corresponding transient problem. Pan and Roy [19] used the method of separation of variables, expressed the solution in terms of a Fourier series in the circumferential direction, and divided the FG cylinder into multilayers to solve the mechanical problem. Shao and Ma [20] employed the Laplace transform technique and series solution of ordinary differential equations to find stresses in an FG hollow cylinder subjected to mechanical loads and linearly increasing boundary temperatures. All of these works have considered power law variation of the elastic moduli.

Whereas an incompressible material can undergo only isochoric deformations, a compressible material can admit both isochoric and nonisochoric deformations. For plane strain problems, results for an incompressible material cannot be derived from those for a compressible material simply by setting Poisson's ratio equal to 0.5.

The constitutive relation for an incompressible material involves a hydrostatic pressure that cannot be determined from the deformation field. Note that only isochoric (volume preserving) deformations are admissible in incompressible materials. The equation corresponding to this condition and three equations expressing the balance of linear momentum are solved for the four unknowns: three components of displacements and the pressure at a point. However, the pressure field can be determined uniquely only if tractions are prescribed on a part of the boundary. Here, we solve analytically the problem of radial expansion/contraction of an FG cylinder and an FG sphere loaded by uniform pressures on the inner and the outer surfaces and the shear modulus varying in the radial direction only. We also find the variation in the shear modulus that optimizes the hoop stress.

The analytical solutions presented herein are for an arbitrary variation of the shear modulus through the plate thickness. The constraint of incompressibility facilitates the solution of the problem. These results should serve as benchmarks for verifying and validating numerical works. For the cylinder problem, results presented herein generalize those included in [21], in which the radial variation of the shear modulus is assumed to be given by a power law function for a second-order elastic material; the problem for the sphere was not studied in [21].

II. Problem Formulation

A. Cylinder

We consider an infinitely long hollow cylinder of inner radius R_{in} and outer radius R_{ou} in the unstressed reference configuration. The cylinder, made of an isotropic and incompressible linear elastic material, is loaded by pressures p_{in} and p_{ou} , respectively, on its inner and outer surfaces as shown in Fig. 1. We assume that values of material parameters of the cylinder vary only in the radial direction. Because the material properties, the cylinder geometry, and the applied loads are independent of the angular position and the axial coordinate of a point, we presume that its deformations are axisymmetric and are independent of the axial coordinate z . Thus, a material point of the cylinder moves only in the radial direction. Let r and R denote radial coordinates of a point in the present and the reference configurations, respectively, and $u(R) = r(R) - R$ its displacement in the radial direction. Note that the radial displacement of a point also induces strain in the circumferential direction, and the state of deformation in the cylinder is that of plane strain in the θz plane where θ is the angular position of a point. Even though the axial strain identically vanishes, the axial stress is not necessarily zero. However, it is independent of the axial coordinate z .

In cylindrical coordinates (r, θ, z) , physical components of the displacement gradient $\mathbf{H}$ and the infinitesimal strain $\mathbf{E} = (\mathbf{H} + \mathbf{H}^T)/2$ are given by

$$[\mathbf{H}] = [\mathbf{E}] = \begin{bmatrix} u' & 0 & 0 \\ 0 & \frac{u}{R} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (1)$$

where $u' = du/dR$, and (R, Θ, Z) are coordinates of a point in the reference configuration.

The constitutive relation for an isotropic incompressible linear elastic material is

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2\mu\mathbf{E} \quad (2)$$

where $\boldsymbol{\sigma}$ is the stress tensor, p the hydrostatic pressure not determined from the deformation field, $\mathbf{1}$ the identity tensor, and $\mu = \mu(R) > 0$ the shear modulus. Note that for an FG cylinder, μ varies with the radius R of a material point.

The FGMs are generally made of two or more constituents and their effective elastic moduli are derived by using a homogenization technique. Here, we do not address the determination of μ for the FGM from that of its constituents. However, because no assumption

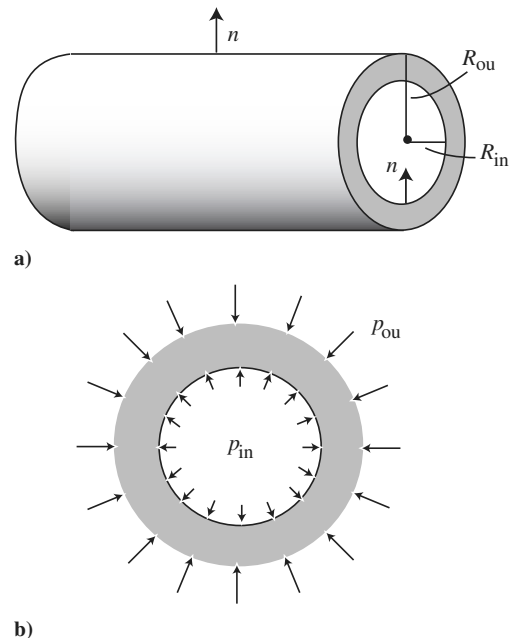


Fig. 1 Schematic sketch of the problem studied.

is made on the specific form of μ , our results are valid for a wide class of distributions of volume fractions of constituents.

Infinitesimal deformations of the cylinder are governed by the following equilibrium equations and boundary conditions:

$$\text{Div } \boldsymbol{\sigma} = \mathbf{0}, \quad R_{\text{in}} < R < R_{\text{ou}} \quad (3a)$$

$$\text{tr } (\mathbf{E}) = 0, \quad R_{\text{in}} < R < R_{\text{ou}} \quad (3b)$$

$$\boldsymbol{\sigma} \mathbf{n} = -p_{\text{in}} \mathbf{n} \quad \text{on } R = R_{\text{in}} \quad (3c)$$

$$\boldsymbol{\sigma} \mathbf{n} = -p_{\text{ou}} \mathbf{n} \quad \text{on } R = R_{\text{ou}} \quad (3d)$$

Here Div is the divergence operator with respect to coordinates in the reference configuration, $\mathbf{n}$ is an outward unit normal to the surface, and p_{in} and p_{ou} are pressures applied, respectively, to the inner and the outer surfaces of the cylinder.

B. Sphere

For a hollow sphere with material properties varying only in the radial direction, and loaded by hydrostatic pressures on the inner and the outer surfaces, it is reasonable to assume that the displacement of a material point is only in the radial direction. Equations governing deformations of an FG sphere are the same as those for a cylinder, except that the divergence operator has a different expression, and physical components of the infinitesimal strain tensor $\mathbf{E}$ are related to the radial displacement u by

$$[E] = [H] = \begin{bmatrix} u' & 0 & 0 \\ 0 & u/R & 0 \\ 0 & 0 & u/R \end{bmatrix} \quad (4)$$

III. Analytical Solution for Cylinder

Substitution for $\mathbf{E}$ from Eq. (1) into Eq. (3b) gives

$$u' + u/R = 0 \quad (5)$$

whose solution is

$$u = c/R \quad (6)$$

where c is a constant of integration.

We now substitute for u from Eq. (6) into Eq. (1) and the result into Eqs. (2) and (3a) to get three equations, two of which imply that the pressure field does not depend upon the axial coordinate Z and the angular position Θ of a material point. The equation of equilibrium in the radial direction gives

$$p' = -\frac{2\mu'}{R^2} c \quad (7)$$

which upon integration gives

$$p = d - 2c \int_{R_{\text{in}}}^R \frac{\mu'(y)}{y^2} dy \equiv d - 2cf(R) \quad (8)$$

where d is a constant of integration, and $f(R_{\text{in}}) = 0$. For a homogeneous material, $\mu = \text{constant}$ and $f(R) = 0$. For an FGM, values of $f(R)$ depend upon how μ varies with the radius R of a point. The integral in Eq. (8) can be evaluated either analytically or numerically.

Knowing the pressure field p and the radial displacement u , we get the following expressions for the radial stress σ_{RR} , the hoop stress $\sigma_{\Theta\Theta}$, and the axial stress σ_{ZZ} :

$$\sigma_{RR} = -d + 2cf(R) - 2\mu(R) \frac{c}{R^2} \quad (9a)$$

$$\sigma_{\Theta\Theta} = -d + 2cf(R) + 2\mu(R) \frac{c}{R^2} \quad (9b)$$

$$\sigma_{ZZ} = -d + 2cf(R) \quad (9c)$$

Constants c and d , determined from boundary conditions given by Eqs. (3c) and (3d), have the following values:

$$c = (p_{\text{in}} - p_{\text{ou}})/2\hat{g} \quad (10a)$$

$$d = \left[p_{\text{ou}} \frac{\mu(R_{\text{in}})}{R_{\text{in}}^2} + p_{\text{in}} \left(f(R_{\text{ou}}) - \frac{\mu(R_{\text{ou}})}{R_{\text{ou}}^2} \right) \right] / \hat{g} \quad (10b)$$

$$\hat{g} = \left[f(R_{\text{ou}}) - \frac{\mu(R_{\text{ou}})}{R_{\text{ou}}^2} + \frac{\mu(R_{\text{in}})}{R_{\text{in}}^2} \right] \quad (10c)$$

Thus, substitution for c and d from Eqs. (10a) and (10b) into Eqs. (9a–9c) gives

$$\begin{aligned} \sigma_{RR}(R) &= \frac{p_{\text{in}}}{\hat{g}} \left[f(R_{\text{ou}}) - f(R) + \frac{\mu(R)}{R^2} - \frac{\mu(R_{\text{ou}})}{R_{\text{ou}}^2} \right] \\ &\quad - \frac{p_{\text{ou}}}{\hat{g}} \left[f(R) + \frac{\mu(R_{\text{in}})}{R_{\text{in}}^2} - \frac{\mu(R)}{R^2} \right] \end{aligned} \quad (11a)$$

$$\begin{aligned} \sigma_{\Theta\Theta}(R) &= -\frac{p_{\text{in}}}{\hat{g}} \left[f(R_{\text{ou}}) - f(R) - \frac{\mu(R)}{R^2} - \frac{\mu(R_{\text{ou}})}{R_{\text{ou}}^2} \right] \\ &\quad - \frac{p_{\text{ou}}}{\hat{g}} \left[f(R) + \frac{\mu(R_{\text{in}})}{R_{\text{in}}^2} + \frac{\mu(R)}{R^2} \right] \end{aligned} \quad (11b)$$

$$\begin{aligned} \sigma_{ZZ}(R) &= -\frac{p_{\text{in}}}{\hat{g}} \left[f(R_{\text{ou}}) - f(R) - \frac{\mu(R_{\text{ou}})}{R_{\text{ou}}^2} \right] \\ &\quad - \frac{p_{\text{ou}}}{\hat{g}} \left[f(R) + \frac{\mu(R_{\text{in}})}{R_{\text{in}}^2} \right] \end{aligned} \quad (11c)$$

Subtracting each side of Eq. (9a) from the corresponding side of Eq. (9b) gives

$$\sigma_{\Theta\Theta} - \sigma_{RR} = 4\mu(R) \frac{c}{R^2} \quad (12)$$

Thus, the sign of $(\sigma_{\Theta\Theta} - \sigma_{RR})$ depends upon the sign of c since $\mu(R) > 0$.

A. Homogeneous Cylinder

For a homogeneous cylinder, $\mu(R) = \text{constant } \mu_0$, $f(R) \equiv 0$, and Eqs. (11a–11c) simplify to

$$\sigma_{RR} = \frac{p_{\text{in}} R_{\text{in}}^2 - p_{\text{ou}} R_{\text{ou}}^2}{R_{\text{ou}}^2 - R_{\text{in}}^2} + \frac{(p_{\text{ou}} - p_{\text{in}}) R_{\text{ou}}^2 R_{\text{in}}^2}{R^2 (R_{\text{ou}}^2 - R_{\text{in}}^2)} \quad (13a)$$

$$\sigma_{\Theta\Theta} = \frac{-p_{\text{ou}} R_{\text{ou}}^2 + p_{\text{in}} R_{\text{in}}^2}{R_{\text{ou}}^2 - R_{\text{in}}^2} + \frac{(p_{\text{in}} - p_{\text{ou}}) (R_{\text{ou}}^2 R_{\text{in}}^2)}{R^2 (R_{\text{ou}}^2 - R_{\text{in}}^2)} \quad (13b)$$

$$\sigma_{ZZ} = \frac{-p_{\text{ou}} R_{\text{ou}}^2 + p_{\text{in}} R_{\text{in}}^2}{R_{\text{ou}}^2 - R_{\text{in}}^2} \quad (13c)$$

These expressions for stresses are identical to those for a cylinder composed of a compressible isotropic linear elastic material. However, displacement fields in otherwise identical cylinders made of compressible and incompressible materials are different.

B. Homogeneous Cylinder Subjected Only to Internal Pressure

When the cylinder is subjected to internal pressure only, $p_{\text{ou}} = 0$, and

$$\sigma_{RR} = \frac{p_{in} R_{in}^2}{R_{ou}^2 - R_{in}^2} \left(1 - \frac{R_{ou}^2}{R^2} \right) \leq 0 \quad (14a)$$

$$\sigma_{\Theta\Theta} = \frac{p_{in} R_{in}^2}{R_{ou}^2 - R_{in}^2} \left(1 + \frac{R_{ou}^2}{R^2} \right) > 0 \quad (14b)$$

$$\sigma_{ZZ} = \frac{p_{in} R_{in}^2}{R_{ou}^2 - R_{in}^2} \quad (14c)$$

For $R_{ou} \gg R_{in}$, Eqs. (14a–14c) yield

$$\sigma_{RR} = -p_{in} \left(\frac{R_{in}}{R} \right)^2 \quad (15a)$$

$$\sigma_{\Theta\Theta} = p_{in} \left(\frac{R_{in}}{R} \right)^2 \quad (15b)$$

$$\sigma_{ZZ} = 0 \quad (15c)$$

Equations (15a–15c) represent the stress field in an infinitely thick homogeneous isotropic body with a uniformly pressurized cylindrical cavity with zero traction at the remote surfaces.

C. Functionally Graded Cylinder Subjected to Internal Pressure

Setting $p_{ou} = 0$ in Eqs. (10a–10c) and (11a–11c), we find that the radial displacement, the radial stress, and the hoop stress are given, respectively, by the following equations:

$$u(R) = \frac{p_{in}}{2\hat{g}} \frac{1}{R} \quad (16a)$$

$$\sigma_{RR}(R) = \frac{p_{in}}{\hat{g}} \left[f(R_{ou}) - f(R) + \frac{\mu(R)}{R^2} - \frac{\mu(R_{ou})}{R_{ou}^2} \right] \quad (16b)$$

$$\sigma_{\Theta\Theta}(R) = \frac{p_{in}}{\hat{g}} \left[-f(R_{ou}) + f(R) + \frac{\mu(R)}{R^2} + \frac{\mu(R_{ou})}{R_{ou}^2} \right] \quad (16c)$$

We note that results derived from the preceding equations are valid for an arbitrary but continuous variation of $\mu(R)$. In an FGM, $\mu(R)$ is expected to be a continuous function of the radius R .

Intuitively, one expects that in a hollow cylinder with pressure applied on the inner surface, all points will move radially outward.

Requiring that every point of the cylinder move radially outward, we assume that $\mu = \mu(R)$ is such that

$$\hat{g} > 0 \text{ or, equivalently, } \int_{R_{in}}^{R_{ou}} \frac{\mu'(y)}{y^2} dy > \frac{\mu(R_{ou})}{R_{ou}^2} - \frac{\mu(R_{in})}{R_{in}^2} \quad (17)$$

We note that $\sigma_{\Theta\Theta}$ is tensile at the outer surface, and for it to be tensile at the inner surface,

$$\int_{R_{in}}^{R_{ou}} \frac{\mu'(y)}{y^2} dy < \frac{\mu(R_{ou})}{R_{ou}^2} + \frac{\mu(R_{in})}{R_{in}^2} \quad (18)$$

To see if inequalities (17) and (18) can be simultaneously satisfied, we assume that

$$\mu(R) = \mu_0 (R/R_{in})^n \quad (19)$$

where μ_0 and n are constants, and $n \neq 2$. Then, the two inequalities (17) and (18) are satisfied, provided that

$$\frac{2}{n-2} A > 0, \quad \frac{4-n}{n-2} A < 2, \quad A \equiv \left(\frac{R_{ou}}{R_{in}} \right)^{n-2} - 1 \quad (20)$$

We now consider the case of a very thick cylinder, i.e., $R_{ou} \gg R_{in}$. For $n > 4$, the two inequalities in Eq. (20) are satisfied and $\sigma_{\Theta\Theta}$ is tensile at the inner surface. For certain combinations of n and

R_{ou}/R_{in} , $\sigma_{\Theta\Theta}$ may be compressive or zero at some points in the cylinder.

For $n = 2$,

$$f(R) = \frac{4\mu_0}{R_{in}^2} \ln(R/R_{in})$$

and $\sigma_{\Theta\Theta}$ is tensile at the outer surface when

$$\left(1 - R_{in}^2/R_{ou}^2 \right) < 4 \ln(R_{ou}/R_{in}) < \left(1 + R_{in}^2/R_{ou}^2 \right) \quad (21)$$

For $n > 0$, the material defined by Eq. (19) hardens (i.e., its shear modulus increases) with an increase in R , and the hardening increases with an increase in n . However, the material softens with an increase in R for $n < 0$.

Knowing the shear moduli μ_1 and μ_2 of the two constituents, their volume fractions V_1 and $V_2 = 1 - V_1$ can be determined from the rule of mixtures

$$\mu(R) = \mu_1 V_1(R) + \mu_2 [1 - V_1(R)] \quad (22)$$

Equation (22) can be readily solved for $V_1(R)$ in terms of $\mu(R)$, μ_1 , and μ_2 .

D. Optimum Value of Hoop Stress in FG Cylinder

Assuming that $p_{in} \neq p_{ou}$, the point inside the cylinder where $\sigma_{\Theta\Theta}$ has the extreme value is given by

$$0 = \sigma'_{\Theta\Theta} = f'(R) + \frac{\mu'(R)}{R^2} - \frac{2\mu(R)}{R^3} \quad (23a)$$

$$= \frac{\mu'(R)}{R^2} - \frac{\mu(R)}{R^3} \quad (23b)$$

$$= \frac{1}{R} \left(\frac{\mu(R)}{R} \right)' \quad (23c)$$

where the constant c has been assumed to be nonzero, which is reasonable because, otherwise, $u = 0$ everywhere in the cylinder. Thus, $\sigma_{\Theta\Theta}$ has the extreme value at either $R = R_{in}$ or $R = R_{ou}$, or at

$$\mu(R) = \alpha R \quad (24)$$

where α is a positive constant.

We now consider an FGM for which μ is given by Eq. (24). It then follows from Eq. (8) that

$$f(R) = \alpha \left(\frac{1}{R_{in}} - \frac{1}{R} \right) \quad (25)$$

which together with Eq. (9b) gives

$$\sigma_{\Theta\Theta} = -\hat{d} + 2\hat{c}\alpha/R_{in} = \text{constant} \quad (26)$$

That is, for a linear variation of the shear modulus, the hoop stress is constant throughout the cylinder. However, the radial stress is not a constant, and its variation is given by

$$\sigma_{RR} = -\hat{d} + 2\hat{c}\alpha \left(\frac{1}{R_{in}} - \frac{2}{R} \right) \quad (27)$$

and has the extreme value either on the inner surface or on the outer surface. Values of constants $\hat{c}$ and $\hat{d}$ cannot be found from Eqs. (10a–10c) because the denominator vanishes. Their values computed from Eq. (27) and boundary conditions (3c) and (3d) are

$$\hat{c} = \frac{(p_{in} - p_{ou})}{2\alpha[(1/R_{in}) - (1/R_{ou})]} \quad (28a)$$

$$\hat{d} = \frac{p_{\text{in}}[1 - 2(R_{\text{in}}/R_{\text{ou}})] + p_{\text{ou}}}{2[1 - (R_{\text{in}}/R_{\text{ou}})]} \quad (28b)$$

Substitution for $\hat{c}$ and $\hat{d}$ from Eqs. (28a) and (28b) into Eq. (26) gives

$$\sigma_{\Theta\Theta} = \frac{p_{\text{in}}[(R_{\text{in}}/R_{\text{ou}})] - p_{\text{ou}}}{[1 - (R_{\text{in}}/R_{\text{ou}})]} \quad (29)$$

For a cylinder loaded with internal pressure only, $\hat{c} > 0$, and the constant hoop stress is tensile everywhere. The hoop stress identically vanishes if $p_{\text{in}}R_{\text{in}} = p_{\text{ou}}R_{\text{ou}}$; otherwise, it is constant throughout the cylinder thickness. The hoop stress is tensile so long as $p_{\text{in}}R_{\text{in}} > p_{\text{ou}}R_{\text{ou}}$; otherwise, it is compressive.

E. Homogeneous Cylinder Subjected Only to External Pressure

Setting $p_{\text{in}} = 0$ in Eqs. (13a–13c) gives

$$\sigma_{RR} = \frac{p_{\text{ou}}R_{\text{ou}}^2}{R_{\text{ou}}^2 - R_{\text{in}}^2} \left(-1 + \frac{R_{\text{in}}^2}{R^2} \right) \quad (30a)$$

$$\sigma_{\Theta\Theta} = -\frac{p_{\text{ou}}R_{\text{ou}}^2}{R_{\text{ou}}^2 - R_{\text{in}}^2} \left(1 + \frac{R_{\text{in}}^2}{R^2} \right) \quad (30b)$$

$$\sigma_{ZZ} = -\frac{p_{\text{ou}}R_{\text{ou}}^2}{R_{\text{ou}}^2 - R_{\text{in}}^2} \quad (30c)$$

For $R_{\text{ou}} \gg R_{\text{in}}$, Eqs. (30a–30c) reduce to

$$\sigma_{RR} = p_{\text{ou}} \left(-1 + \frac{R_{\text{in}}^2}{R^2} \right) \quad (31a)$$

$$\sigma_{\Theta\Theta} = -p_{\text{ou}} \left(1 + \frac{R_{\text{in}}^2}{R^2} \right) \quad (31b)$$

$$\sigma_{ZZ} = -p_{\text{ou}} \quad (31c)$$

Thus, the magnitude of the compressive circumferential or the hoop stress at the inner surface of the cylinder equals twice the uniform pressure applied on the outermost surface which is far from the inner surface.

F. Functionally Graded Cylinder Subjected to External Pressure

For μ given by Eq. (19), $p_{\text{in}} = 0$, and $n \neq 2$, Eqs. (11a) and (11b) simplify to

$$\sigma_{RR}(R) = -\frac{p_{\text{ou}}}{R_{\text{ou}}^{n-2} - R_{\text{in}}^{n-2}} \left(R^{n-2} - R_{\text{in}}^{n-2} \right) \quad (32a)$$

$$\sigma_{\Theta\Theta}(R) = -\frac{p_{\text{ou}}}{R_{\text{ou}}^{n-2} - R_{\text{in}}^{n-2}} \left[(n-1)R^{n-2} - R_{\text{in}}^{n-2} \right] \quad (32b)$$

As pointed out earlier, $\sigma_{\Theta\Theta}$ is constant for $n = 1$. For $n > 2$ and $R_{\text{ou}} \gg R_{\text{in}}$, $(R_{\text{in}}/R_{\text{ou}})^{n-2} \simeq 0$, and Eqs. (32a) and (32b) give

$$\sigma_{RR}(R) \simeq -p_{\text{ou}} \left(\frac{R}{R_{\text{ou}}} \right)^{n-2} \quad (33a)$$

$$\sigma_{\Theta\Theta}(R) \simeq -(n-1)p_{\text{ou}} \left(\frac{R}{R_{\text{ou}}} \right)^{n-2} \quad (33b)$$

Thus, the sign of $\sigma_{\Theta\Theta}$ depends upon the value of n in Eq. (19). For $n < 2$ and $R_{\text{ou}} \gg R_{\text{in}}$,

$$(R_{\text{ou}}/R_{\text{in}})^{n-2} = (R_{\text{in}}/R_{\text{ou}})^{2-n} \simeq 0 \quad (34)$$

and Eqs. (32a) and (32b) can be approximated by

$$\begin{aligned} \sigma_{RR}(R) &\simeq p_{\text{ou}} \left[\left(\frac{R}{R_{\text{in}}} \right)^{n-2} - 1 \right] \\ \sigma_{\Theta\Theta}(R) &\simeq p_{\text{ou}} \left[(n-1) \left(\frac{R}{R_{\text{in}}} \right)^{n-2} - 1 \right] \end{aligned} \quad (35)$$

Hence, the inhomogeneity of the cylinder material significantly affects the stress near the hole in a very thick cylinder loaded by a uniform pressure on the outer surface. For $n = 0$, stresses computed from Eqs. (32a) and (32b) agree with those given by Eqs. (31a) and (31b).

For $n = 2$,

$$\sigma_{RR}(R) = \frac{-p_{\text{ou}} \ell_n(R/R_{\text{in}})}{\ell_n(R_{\text{ou}}/R_{\text{in}})} \quad (36a)$$

$$\sigma_{\Theta\Theta}(R) = -p_{\text{ou}} \frac{[\ell_n(R/R_{\text{in}}) + 1]}{\ell_n(R_{\text{ou}}/R_{\text{in}})} \quad (36b)$$

G. Functionally Graded Hollow Cylinder with Affine Variation of Shear Modulus

It is shown in [21] that, for $\mu(R) = \mu_0[1 + m(R/R_{\text{in}})]$, the hoop stress at $R = \sqrt{R_{\text{in}}R_{\text{ou}}}$ is independent of the constant m , and is the same as in a homogeneous cylinder.

IV. Analytical Solution for Sphere

Substitution from Eq. (4) into Eq. (3b), expressing the requirement that deformations be isochoric, gives

$$u' + 2\frac{u}{R} = 0 \quad (37a)$$

whose solution is

$$u = \frac{\tilde{b}}{R^2} \quad (37b)$$

where $\tilde{b}$ is a constant of integration. We now substitute for u from Eq. (37b) into Eq. (4), and the result into the constitutive relation listed as Eq. (2) to obtain

$$\sigma_{RR} = -p - 4\mu(R) \frac{\tilde{b}}{R^3} \quad (38a)$$

$$\sigma_{\Theta\Theta} = \sigma_{\phi\phi} = -p + 2\mu(R) \frac{\tilde{b}}{R^3} \quad (38b)$$

$$\sigma_{R\Theta} = \sigma_{\Theta\phi} = \sigma_{\phi R} = 0 \quad (38c)$$

Equations (38a–38c) when substituted into Eq. (3a) imply that the pressure p is independent of Θ and ϕ . The equation of equilibrium in the radial direction gives

$$p' = -\frac{4\mu'\tilde{b}}{R^3} \quad (39)$$

whose integral is

$$p(R) = e - 4\tilde{b} \int_{R_{\text{in}}}^R \frac{\mu'(y)}{y^3} dy \equiv e - 4\tilde{b}g(R) \quad (40)$$

where e is a constant of integration. Substitution for p from Eq. (40) into Eqs. (38a–38c) gives the following expressions for the stresses:

$$\sigma_{RR} = -e + 4\tilde{b}g(R) - 4\mu(R) \frac{\tilde{b}}{R^3} \quad (41a)$$

$$\sigma_{\Theta\Theta} = \sigma_{\phi\phi} = -e + 4\tilde{b}g(R) + 2\mu(R)\frac{\tilde{b}}{R^3} \quad (41b)$$

Boundary conditions (3c) and (3d) require that

$$-e - 4\frac{\mu(R_{in})}{R_{in}^3}\tilde{b} = -p_{in} \quad (42a)$$

$$-e + 4\tilde{b}g(R_{ou}) - 4\frac{\mu(R_{ou})}{R_{ou}^3}\tilde{b} = -p_{ou} \quad (42b)$$

Subtraction of each side of Eq. (42b) from the respective side of Eq. (42a) gives an equation for the determination of $\tilde{b}$ with the following result:

$$\tilde{b} = (p_{in} - p_{ou})/4D \quad (43)$$

where

$$D = \left[\frac{\mu(R_{in})}{R_{in}^3} - \frac{\mu(R_{ou})}{R_{ou}^3} + g(R_{ou}) \right] \quad (44)$$

Thus,

$$e = \frac{1}{D} \left[p_{in} \left(g(R_{ou}) - \frac{\mu(R_{ou})}{R_{ou}^3} \right) + \frac{\mu(R_{in})}{R_{in}^3} p_{ou} \right] \quad (45)$$

and stresses are given by

$$\begin{aligned} \sigma_{RR}(R) = & -\frac{p_{in}}{D} \left[g(R_{ou}) - \frac{\mu(R_{ou})}{R_{ou}^3} - g(R) + \frac{\mu(R)}{R^3} \right] \\ & - \frac{p_{ou}}{D} \left[\frac{\mu(R_{in})}{R_{in}^3} + g(R) - \frac{\mu(R)}{R^3} \right] \end{aligned} \quad (46a)$$

$$\begin{aligned} \sigma_{\Theta\Theta}(R) = & -\frac{p_{in}}{D} \left[g(R_{ou}) - \frac{\mu(R_{ou})}{R_{ou}^3} - g(R) - \frac{1}{2} \frac{\mu(R)}{R^3} \right] \\ & - \frac{p_{ou}}{D} \left[\frac{\mu(R_{in})}{R_{in}^3} + g(R) + \frac{\mu(R)}{2R^3} \right] \end{aligned} \quad (46b)$$

Subtraction of Eq. (41a) from Eq. (41b) gives

$$\sigma_{\Theta\Theta} - \sigma_{RR} = \frac{6\mu(R)}{R^3}\tilde{b} \quad (47)$$

and has the same sign as the constant $\tilde{b}$.

A. Homogeneous Hollow Sphere

For a homogeneous sphere, $\mu(R) = \text{constant } \mu_0$, $g(R) \equiv 0$, and Eqs. (46a) and (46b) simplify to

$$\sigma_{RR}(R) = \frac{p_{in}R_{in}^3}{R_{ou}^3 - R_{in}^3} \left(1 - \frac{R_{ou}^3}{R^3} \right) - \frac{p_{ou}R_{ou}^3}{R_{ou}^3 - R_{in}^3} \left(1 - \frac{R_{in}^3}{R^3} \right) \quad (48a)$$

$$\sigma_{\Theta\Theta}(R) = \frac{p_{in}R_{in}^3}{R_{ou}^3 - R_{in}^3} \left(1 + \frac{R_{ou}^3}{2R^3} \right) - \frac{p_{ou}R_{ou}^3}{R_{ou}^3 - R_{in}^3} \left(1 + \frac{R_{in}^3}{2R^3} \right) \quad (48b)$$

For a homogeneous hollow sphere loaded internally by a hydrostatic pressure, $p_{ou} = 0$, Eqs. (48a) and (48b) give

$$\sigma_{RR}(R) = \frac{p_{in}R_{in}^3}{R_{ou}^3 - R_{in}^3} \left(1 - \frac{R_{ou}^3}{R^3} \right) < 0 \quad (49a)$$

$$\sigma_{\Theta\Theta}(R) = \frac{p_{in}R_{in}^3}{R_{ou}^3 - R_{in}^3} \left(1 + \frac{R_{ou}^3}{2R^3} \right) > 0 \quad (49b)$$

Thus, σ_{RR} is compressive and $\sigma_{\Theta\Theta}$ is tensile everywhere. Whereas

expressions for stresses in a sphere made of a homogeneous either compressible or incompressible material are the same, the displacement fields are different. For a very thick hollow sphere, $R_{ou} \gg R_{in}$, and Eqs. (49a) and (49b) give

$$\sigma_{RR}(R) = -p_{in} \frac{R_{in}^3}{R^3} \quad (50a)$$

$$\sigma_{\Theta\Theta}(R) = \frac{p_{in}R_{in}^3}{2R^3} \quad (50b)$$

Thus, the magnitude of the hoop stress at points on the inner surface of a very thick sphere is one-half of the applied pressure.

For a very thin sphere (e.g., a balloon) of thickness t and radius R , Eq. (49b) reduces to

$$\sigma_{\Theta\Theta} = p_{in}R/(2t) \quad (51)$$

For a very thick homogeneous hollow sphere loaded externally by a uniform pressure, i.e., $p_{in} = 0$, $R_{ou} \gg R_{in}$, Eqs. (48a) and (48b) give

$$\sigma_{RR}(R) = -p_{ou} \left(1 - \frac{R_{in}^3}{R^3} \right) \quad (52a)$$

$$\sigma_{\Theta\Theta}(R) = -p_{ou} \left(1 + \frac{R_{in}^3}{2R^3} \right) \quad (52b)$$

Thus, the compressive hoop stress at a point on the inner surface of an externally loaded thick homogeneous cylinder equals one and a half times the uniform pressure applied on the outer surface.

B. Functionally Graded Hollow Sphere Subjected to Internal Pressure

Substitution for $\tilde{b}$ from Eq. (43) into Eqs. (37b), (38a), and (38b) gives the following expressions for the radial displacement, the radial stress, and the circumferential stress:

$$u(R) = \frac{p_{in}}{4D} \frac{1}{R^2} \quad (53a)$$

$$\sigma_{RR}(R) = -\frac{p_{in}}{D} \left[g(R_{ou}) - \frac{\mu(R_{ou})}{R_{ou}^3} - g(R) + \frac{\mu(R)}{R^3} \right] \quad (53b)$$

$$\sigma_{\Theta\Theta}(R) = \frac{p_{in}}{D} \left[-g(R_{ou}) + \frac{\mu(R_{ou})}{R_{ou}^3} + g(R) + \frac{1}{2} \frac{\mu(R)}{R^3} \right] \quad (53c)$$

where D is given by Eq. (44).

So that every point of the sphere moves radially outward, we assume that

$$D > 0 \text{ or, equivalently, } \frac{\mu(R_{in})}{R_{in}^3} - \frac{\mu(R_{ou})}{R_{ou}^3} + \int_{R_{in}}^{R_{ou}} \frac{\mu'(y)}{y^3} dy > 0 \quad (54)$$

Equation (53c) implies that $\sigma_{\Theta\Theta}$ is tensile on the outer surface. For $\sigma_{\Theta\Theta}$ to be tensile on the inner surface,

$$\frac{\mu(R_{ou})}{R_{ou}^3} + \frac{\mu(R_{in})}{2R_{in}^3} - \int_{R_{in}}^{R_{ou}} \frac{\mu'(y)}{y^3} dy > 0 \quad (55)$$

To check whether or not inequalities (54) and (55) can be simultaneously satisfied, we assume that

$$\mu(R) = \mu_0(R/R_{in})^n \quad (56)$$

where μ_0 and n are constants, and $n \neq 3$. Inequality (54) holds and inequality (55) is satisfied, provided that

$$\left(\frac{n-1}{2(n-3)} \right) R_{in}^{n-3} - \left(\frac{1}{n-3} \right) R_{ou}^{n-3} > 0 \quad (57)$$

Thus, the sign of the circumferential stress strongly depends upon the value of n and the ratio of the outer to the inner radii of the sphere.

For a very thick sphere, $R_{ou}/R_{in} \gg 1$. When $n > 3$ we get

$$\sigma_{\Theta\Theta}(R) \simeq -p_{in} \left[1 - \frac{n-1}{2} \left(\frac{R}{R_{ou}} \right)^{n-3} \right] \quad (58)$$

and for $n < 3$,

$$\sigma_{\Theta\Theta}(R) \simeq p_{in} \left[-\frac{n-1}{2} \left(\frac{R}{R_{in}} \right)^{n-3} \right] \quad (59)$$

That is, the distribution of the circumferential stress in a very thick sphere depends upon the value of the index n in Eq. (56). However, for a very thin sphere, the hoop stress is unaffected by the through-the-thickness inhomogeneity of the material.

C. Functionally Graded Thick Hollow Sphere Subjected to External Pressure

For $n > 3$, we have

$$\sigma_{\Theta\Theta}(R) \simeq -p_{ou}(n-1) \left(\frac{R}{R_{ou}} \right)^{n-3} \quad (60)$$

and for $n < 3$,

$$\sigma_{\Theta\Theta}(R) \simeq -p_{ou} \left[1 - \frac{(n-1)}{2} \left(\frac{R}{R_{in}} \right)^{n-3} \right] \quad (61)$$

For $R = R_{in}$, Eq. (61) gives $\sigma_{\Theta\Theta}(R_{in}) = -(3-n)p_{ou}/2$.

D. Optimum Value of the Circumferential Stress

Assuming that $p_{in} \neq p_{ou}$, points interior to the sphere where $\sigma_{\Theta\Theta}$ has extreme values are roots of the equation

$$\sigma'_{\Theta\Theta} = 0 \quad (62)$$

or, equivalently,

$$R\mu'(R) - \mu(R) = 0 \quad (63)$$

where we have used Eq. (41b). The solution of Eq. (63) is

$$\mu(R) = \beta R \quad (64)$$

where β is a positive constant.

We now consider an FG sphere for which μ is given by Eq. (64). For this variation of μ ,

$$g(R) = \frac{\beta}{2} \left(-\frac{1}{R^2} + \frac{1}{R_{in}^2} \right) \geq 0 \quad (65)$$

Thus,

$$\sigma_{\Theta\Theta} = \frac{p_{in}R_{in}^2 - p_{ou}R_{ou}^2}{(R_{ou}^2 - R_{in}^2)} \quad (66)$$

and is a constant throughout the sphere. However, σ_{RR} varies with R . The stress $\sigma_{\Theta\Theta}$ identically vanishes when $p_{in}R_{in}^2 = p_{ou}R_{ou}^2$.

V. Conclusions

We have studied analytically radial deformations of functionally graded cylinders and spheres made of incompressible isotropic linear elastic materials, with shear moduli varying continuously in the radial direction, and their inner and outer surfaces loaded by uniform pressures. For each case, the variation of the shear modulus that optimizes the hoop stress is found to be linear in the radial coordinate. The optimum hoop stress is constant throughout the cylinder/sphere and vanishes when pressures on the inner and outer surfaces are inversely proportional to their radii for a cylinder, and square of the radii for a sphere. The closed-form solutions provided herein will

help in validating numerical algorithms developed to solve problems for incompressible materials.

Acknowledgments

This work was partially supported by the Office of Naval Research grant N00014-06-1-0567 to Virginia Polytechnic Institute and State University with Y. D. S. Rajapakse as the program manager. Views expressed herein are those of the author and not of the funding agency nor of Virginia Polytechnic Institute and State University.

References

- [1] Choy, K.-L., and Felix, E., "Functionally Graded Diamond-Like Carbon Coatings on Metallic Substrates," *Material Science and Engineering: A*, Vol. 278, Nos. 1–2, 2000, pp. 162–169.
- [2] Lambros, A., Narayanaswamy, A., Santare, M. H., and Anlao, G., "Manufacturing and Testing of a Functionally Graded Material," *Journal of Engineering Materials and Technology*, Vol. 121, No. 4, 1999, pp. 488–493. doi:10.1115/1.2812406
- [3] Batra, R. C., and Jin, J., "Natural Frequencies of a Functionally Graded Anisotropic Rectangular Plate," *Journal of Sound and Vibration*, Vol. 282, Nos. 1–2, 2005, pp. 509–516. doi:10.1016/j.jsv.2004.03.068
- [4] Batra, R. C., "Finite Plane Strain Deformations of Rubberlike Materials," *International Journal for Numerical Methods in Engineering*, Vol. 15, No. 1, 1980, pp. 145–160. doi:10.1002/nme.1620150112
- [5] Horgan, C. O., and Chan, A. M., "Pressurized Hollow Cylinder or Disk Problem for Functionally Graded Isotropic Linear Elastic Materials," *Journal of Elasticity*, Vol. 55, No. 1, 1999, pp. 43–59. doi:10.1023/A:1007625401963
- [6] Lechnitskii, S. G., *Theory of Elasticity of an Anisotropic Body*, Mir Publishers, Moscow, 1981.
- [7] Timoshenko, S. P., and Goodier, J. N., *Theory of Elasticity*, McGraw-Hill, New York, 1969.
- [8] Vel, S. S., and Batra, R. C., "Exact Solutions for Thermoelastic Deformations of Functionally Graded Thick Rectangular Plates," *AIAA Journal*, Vol. 40, No. 7, 2002, pp. 1421–1433.
- [9] Vel, S. S., and Batra, R. C., "Three-Dimensional Exact Solution for the Vibration of Functionally Graded Rectangular Plates," *Journal of Sound and Vibration*, Vol. 272, Nos. 3–5, 2004, pp. 703–730. doi:10.1016/S0022-460X(03)00412-7
- [10] Pan, E., and Han, F., "Exact Solution for Functionally Graded and Layered Magneto-electroelastic Plates," *International Journal of Engineering Science*, Vol. 43, Nos. 3–4, 2005, pp. 321–339. doi:10.1016/j.ijengsci.2004.09.006
- [11] Tam, J. Q., "Exact Solutions for Functionally Graded Anisotropic Cylinders Subjected to Thermal and Mechanical Loads," *International Journal of Solids and Structures*, Vol. 38, Nos. 46–47, 2001, pp. 8189–8206. doi:10.1016/S0020-7683(01)00182-2
- [12] Tam, J. Q., and Chang, H. H., "Torsion of Cylindrical Orthotropic Elastic Circular Bars with Radial Inhomogeneity: Some Exact Solutions and End Effects," *International Journal of Solids and Structures*, Vol. 45, No. 1, 2008, pp. 303–319. doi:10.1016/j.jsolstr.2007.08.012
- [13] Alshits, V. I., and Kirchner, H. O. K., "Cylindrically Anisotropic, Radially Inhomogeneous Elastic Materials," *Proceedings of the Royal Society of London, Series A: Mathematical, Physical and Engineering Sciences*, Vol. 457, No. 2007, 2001, pp. 671–693.
- [14] Oral, A., and Anlas, G., "Effects of Radially Varying Moduli on Stress Distribution of Nonhomogeneous Anisotropic Cylindrical Bodies," *International Journal of Solids and Structures*, Vol. 42, No. 20, 2005, pp. 5568–5588. doi:10.1016/j.jsolstr.2005.02.044
- [15] Liew, K. M., Kitipornchai, S., Zhang, X. Z., and Lim, C. W., "Analysis of the Thermal Stress Behavior of Functionally Graded Hollow Circular Cylinders," *International Journal of Solids and Structures*, Vol. 40, No. 10, 2003, pp. 2355–2380. doi:10.1016/S0020-7683(03)00061-1
- [16] Jabbari, M., Sohrabpour, S., and Eslami, M. R., "General Solution for Mechanical and Thermal Stresses in a Functionally Graded Hollow Cylinder Due to Nonaxisymmetric Steady State Loads," *Journal of Applied Mechanics*, Vol. 70, No. 1, 2003, pp. 111–118. doi:10.1115/1.1509484

- [17] Obata, Y., and Noda, N., "Steady Thermal Stresses in a Hollow Circular Cylinder and a Hollow Sphere of a Functionally Gradient Material," *Journal of Thermal Stresses*, Vol. 17, No. 3, 1994, pp. 471–488.
doi:10.1080/01495739408946273
- [18] Kim, K. S., and Noda, N., "Green's Function Approach to Unsteady Thermal Stresses in an Infinite Hollow Cylinder of Functionally Graded Material," *Acta Mechanica*, Vol. 156, Nos. 3–4, 2002, pp. 145–161.
doi:10.1007/BF01176753
- [19] Pan, E., and Roy, A. K., "Simple Plane-Strain Solution for Functionally Graded Multilayered Isotropic Cylinders," *Structural Engineering and Mechanics*, Vol. 24, No. 6, 2006, pp. 727–740.
- [20] Shao, Z. S., and Ma, G. W., "Thermo-Mechanical Stresses in Functionally Graded Circular Hollow Cylinder with Linearly Increasing Boundary Temperature," *Composite Structures*, Vol. 83, No. 3, 2008, pp. 259–265.
doi:10.1016/j.compstruct.2007.04.011
- [21] Batra, R. C., and Iaccarino, G. L., "Exact Solution for Radial Deformations of a Functionally Graded Isotropic and Incompressible Second-Order Elastic Cylinder," *International Journal of Non-Linear Mechanics*, Vol. 43, No. 5, 2008, pp. 383–398.

C. Cesnik
Associate Editor