ON THE FADING MEMORY OF INITIAL CONDITIONS*

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1. Introduction. In [1], I attempted to derive the two-dimensional balance laws for loading devices from their three-dimensional counterparts. Loosely speaking, the loading device L was defined to be a three-dimensional deformable continuum which occupies a region R in the exterior of the region R_b occupied by the body and is such that $\partial \bar{R} \cap \partial \bar{R}_b \neq \phi$. Taking a half space as a loading device, I derived, by mechanistic calculations, the linear representations for some of the two-dimensional constitutive quantities from their known three-dimensional counterparts. These calculations suggest that, in a purely thermal problem, a possible choice of the independent variables for the surface constitutive quantities would be the fields of temperature θ or heat flux q defined on the boundary ∂R of R. In a purely mechanical problem, one could take the fields of displacement $\mathbf{u}$ or surface tractions $\mathbf{f}$ defined on ∂R as the independent variables. It should become clear from the details of these calculations [1, Sects. 3, 4] that these surface constitutive quantities would also depend upon the initial state of the loading device, i.e. the deformation of the loading device at the instant of glueing to the body. Here we show that such is not the case when L is linearly heat-conducting and the history of either the temperature or the heat flux at the boundary points of L is known. Rather, we prove such a result for a general three-dimensional continuum. In particular, we show that for an inhomogeneous, anisotropic, linearly heat-conducting continuum, the memory of the initial conditions fades away exponentially. The rate of the fading of the memory depends upon the shape of the body, the specific heat, and the thermal conductivity. Said differently, one can determine, merely from a knowledge of the history of the boundary conditions, a unique solution of the heat equation.

In [2], Meizel and Seidman studied a somewhat similar problem. For a homogeneous, isotropic, linear heat conductor of special geometry, they established the following result. For a thermally insulated continuum occupying a region $D_* = (0, 1) \times D \subset R^n$, the mapping A from $L^2(D \times (0, T))$ to $L^2(D_*)$ defined by

$$A : g = g(\mathbf{Y}, t) = g(0, \mathbf{Y}, t) \mapsto \theta(\mathbf{X}, \mathbf{Y}, T), 0 < t < T, (\mathbf{X}, \mathbf{Y}) \in D_* ,$$

is a well-defined, bounded (using L^2 norms) linear map for the solutions of the heat equation. It may be remarked that no information about the initial temperature distribution is required. However, on the portion D of the boundary of D_*, one knows both the heat flux and the temperature $g(\mathbf{Y}, t)$; the latter is assumed to satisfy certain consistency conditions. Meizel and Seidman [3] have proved a result similar to the above for more general regions. It seems that the result proved below is slightly different in spirit.

Notation: We refer the deformation of the deformable continuum to a fixed set of rectangular cartesian set of axes. The vector $\mathbf{X}$ denotes the position of a material particle in the reference configuration which we take to be the one occupied by the continuum at time $t = -\infty$. A comma followed by an index i designates differentiation with respect to X_i. The superposed dot stands for the partial differentiation with respect to time t. We adopt the summation convention. E^n denotes Euclidean n-space; c_1, c_2, etc. denote positive constants. The vector $\mathbf{n}(\mathbf{X}, t)$ designates the outer unit normal to the current configuration of the boundary ∂R of R at the point $\mathbf{X} \in \partial R$. ∂R_1 and ∂R_2 denote complementary parts of the boundary ∂R of R, i.e. $\partial R = \partial R_1 \cup \partial R_2$ and $\partial R_1 \cap \partial R_2 = \phi$.

Thermal Problem.

THEOREM 2.1:   Let $R \subset E^n$ be a bounded open region with a smooth boundary. the solution of

$$c(\mathbf{X})\dot\theta(\mathbf{X}, t) = (K_{ij}(\mathbf{X})\theta_{,j}(\mathbf{X}, t))_{,i}, \qquad (\mathbf{X}, t) \in R \times (-\infty, t],$$

$$\theta(\mathbf{X}, t) = \theta_0(\mathbf{X}, t), \qquad (\mathbf{X}, t) \in \partial R_1 \times (-\infty, t],$$

$$q(\mathbf{X}, t) = q_0(\mathbf{X}, t), \qquad (\mathbf{X}, t) \in \partial R_2 \times (-\infty, t],$$

is unique provided

$$\partial R_1 \neq \phi, \qquad 0 < c(\mathbf{X}) \leq c_2,$$

$$\int_R K_{ij}\theta_{,j}\theta_{,i} \, dv \geq c_1 \int_R \theta_{,i}\theta_{,i} \, dv.$$

Remark 2.1.   In stating the above theorem, the existence of a solution is presumed. Here c denotes the specific heat and K_{ij} designates the conductivity tensor. For a thermoelastic body, $c > 0$ was shown by Ericksen [4] to be a necessary condition for stability. The positive semidefiniteness of K_{ij} can be established by thermodynamic arguments, see e.g. Day [5]. The requirement $(2.2)_1$ is a necessary condition for (2.6) to hold.

Proof of Theorem 2.1:   We first note that the whole problem is invariant with respect to the translation of the time axis. Hence it suffices to show that the solution of

$$c(\mathbf{X})\dot\theta(\mathbf{X}, t) = (K_{ij}(\mathbf{X})\theta_{,j}(\mathbf{X}, t))_{,i}, \quad (\mathbf{X}, t) \in R \times (0, t] \qquad (2.3)$$

under the null boundary conditions approaches the null solution as $t \to \infty$.

Multiplying (2.3) by θ, then integrating over the region R and using the divergence theorem, we obtain

$$\int_R c\dot\theta\theta \, dv = \int_{\partial R} K_{ij}\theta_{,j}\theta n_i \, dA - \int_R K_{ij}\theta_{,j}\theta_{,i} \, dv. \qquad (2.4)$$

Since the boundary data are the null data, the first term on the right-hand side of (2.4) vanishes. Use of $(2.2)_3$, (2.4) yields

$$\int_R c\dot\theta\theta \, dv \leq -c_1 \int_R \theta_{,i}\theta_{,i} \, dv.$$

For functions $\theta \in C^1(R)$, $\theta = 0$ on ∂R_1, we have Poincaré's inequality [6, p. 355]

$$\int_R \theta^2 \, dv \le p \int_R \theta_{,i} \theta_{,i} \, dv$$

where p is a constant which depends on R and ∂R_1. Combining (2.5) and (2.6), we obtain

$$\frac{d}{dt} \int c\theta^2 \, dv \le -\frac{2c_1}{p} \int_R \theta^2 \, dv,$$

$$= -\frac{2c_1}{pc_2} \int_R c_2 \theta^2 \, dv$$

Now, using (2.2)$_2$, we get

$$\frac{d}{dt} \int_R c\theta^2 \, dv \le \frac{2c_1}{c_2 p} \int_R c\theta^2 \, dv,$$

which upon integration gives

$$\int_R c\theta^2(\mathbf{X}, t) \, dv \le \left(\int_R c\theta^2(\mathbf{X}, 0) \, dv \right) \exp\left(-2c_1 t / c_2 p \right).$$

Thus

$$\theta(\mathbf{X}, t) \xrightarrow{L^2} 0 \quad \text{as} \quad t \to \infty.$$

When $\partial R_1 = \phi$, the solution of (2.1) can be expected to be unique only to within an arbitrary constant. In order to rule out this trivial non-uniqueness, we normalize the initial data by setting

$$\int_R \theta(\mathbf{X}, 0) = 0.$$

This entails no loss of generality. Integrating (2.3) over the region R, using the divergence theorem and the boundary condition

$$q(\mathbf{X}, t) = 0, \quad (\mathbf{X}, t) \in \partial R \times (0, t],$$

we obtain

$$\int_R c(\mathbf{X}) \theta(\mathbf{X}, t) = 0, \qquad \forall t \in [0, t].$$

If the specific heat is an absolute constant, (2.9) gives

$$\frac{d}{dt} \int_R \theta(\mathbf{X}, t) = 0,$$

and therefore

$$\int_R \theta(\mathbf{X}, t) = 0, \qquad \forall t \in [0, t]. \tag{2.10}$$

In order to obtain (2.10), we used (2.8). For continuously differentiable functions which also satisfy (2.10), Poincaré's inequality is [8, p. 284]

$$\int_R \theta^2 \, dv \le \frac{n \, d^2}{2} \int_R \theta_{,i} \theta_{,i} \, dv, \tag{2.11}$$

where d is the diameter of R. Now, by following a procedure similar to the one used in getting (2.7) from (2.5), we obtain

$$\int_R \theta^2(\mathbf{X}, t) \, dv \leq \left(\int_R \theta^2(\mathbf{X}, 0) \, dv\right) \exp\left(-4c_1 t/n \, d^2 c\right). \qquad (2.12)$$

Thus we have proved the following

THEOREM 2.2:   Let $R \subset E^n$ be a bounded open region with a smooth boundary. Then the solution of

$$c\dot{\theta}(\mathbf{X}, t) = (K_{ij}(\mathbf{X})\theta_{,j}(\mathbf{X}, t))_{,i}$$

under the boundary conditions $(2.1)_{2,3}$ is unique provided $(2.2)_3$ holds and the specific heat c is a constant.

An immediate corollary of Theorem 2.2 is the following result:

THEOREM 2.3:   Let $R \subset E^n$ be a bounded open region with a smooth boundary. Then

$$\dot{\theta}(\mathbf{X}, t) = \theta_{,ii}(\mathbf{X}, t), \qquad (\mathbf{X}, t) \in R \times (-\infty, t],$$

$$\theta(\mathbf{X}, t) = \theta_0(\mathbf{X}, t), \qquad (\mathbf{X}, t) \in \partial R_1 \times (-\infty, t],$$

$$q(\mathbf{X}, t) = q_0(\mathbf{X}, t), \qquad (\mathbf{X}, t) \in \partial R_2 \times (-\infty, t],$$

has a unique solution.

Proof: Theorem 2.2 implies the uniqueness of the solution. Take any smooth field, say

$$\theta(\mathbf{X}, -\infty) = 0$$

as the initial temperature distribution. Then the existence of the solution follows from the known theorems [7, p. 320].

Remark 2.2:   The inequality (2.12) loosely confirms the intuitive idea that the larger the region R, more is the time required for the fading away of the memory of the initial state.

The physical idea underlying the above result is the following: whatever energy is initially imparted to the continuum would be dissipated because of the thermal conduction. In a purely mechanical problem, the source of energy dissipation is the viscosity and a simple example is provided by a linearly viscous material. For these materials I can prove, by following a method essentially similar to the one used in the thermal problem, that the history of the boundary conditions uniquely determines the stress field in the body. Also, the memory of the initial conditions fades away exponentially and the rate of the fading of the memory depends upon the shape of the body, the density and the viscosity. This technique of proving the uniqueness of solutions seems to work for linear thermoelastic and linear viscoelastic materials. But at present I have not been able to obtain sharp estimates for these materials.

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