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ON THE COINCIDENCE OF THE PRINCIPAL AXES OF STRESS AND STRAIN IN ISOTROPIC ELASTIC BODIES

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ABSTRACT

It is shown that for isotropic elastic materials the principal axes of stress are also the principal axes of strain provided the empirical inequalities hold.

1. Introduction

A general form of the constitutive equation for an unconstrained isotropic homogeneous elastic material is ([1], Eqn. (47.9))

$$\tilde{T} = f_0\tilde{1} + f_1\tilde{B} + f_{-1}\tilde{B}^{-1} \qquad (1.1$$

where $\tilde{T}$ is the Cauchy stress tensor, $\tilde{B}$ is the left Cauchy-Green tensor with respect to an undistorted configuration, and the response coefficients f_α ($\alpha = -1, 0, 1$) are functions of the principal invariants of $\tilde{B}$. We note that $\tilde{B}$ is a symmetric positive definite tensor and $\tilde{T}$ is symmetric. It is an immediate consequence of (1.1) that a proper vector of $\tilde{B}$ is also a proper vector of $\tilde{T}$ so that the principal axes of strain are also the principal axes of stress. That the converse need not be true unless some restrictions are imposed upon the response coefficients f_α is clear from the following special case of (1.1):

$$\tilde{T} = f_0\,(\mathrm{III})\,\tilde{1}\ . \qquad (1.2)$$

In (1.2), III is the third principal invariant of $\tilde{B}$. The constitutive relation (1.2) represents an elastic fluid and states that every vector is a proper vector of $\tilde{T}$ but it gives no information about the proper vectors of $\tilde{B}$.

Here we show that

a principal axis of stress is also a principal axis of strain provided that (1.3)

$$f_1 > 0, \quad f_{-1} \leq 0. \tag{1.4}$$

These restrictions on f_1 and f_{-1} and the requirement that $f_0 \leq 0$ were first proposed by Truesdell and Noll ([1], Eqn. 51.27) who named these inequalities as Empirical inequalities.

First we prove the result for unconstrained isotropic elastic materials; then we prove it for incompressible isotropic elastic materials.

2. Proof of (1.3)

Given a Cauchy stress tensor $\tilde{T}$ at any point, with respect to its proper vectors as the bases, we can write $\tilde{T}$ in the form

$$\tilde{T} = \text{diagonal}\ (T_{11}, T_{22}, T_{33} \tag{2.1}$$

The constitutive relation (1.1) requires that $\tilde{T}\,\tilde{B} = \tilde{B}\,\tilde{T}$. This together with (2.1) gives

$$(T_{11} - T_{22}\ \ B_{12} = 0, \quad (T_{22} - T_{33}\ \ B_{23} = 0, \quad (T_{33} - T_{11}\ \ B_{31} = 0. \tag{2.2}$$

When $T_{11} \neq T_{22} \neq T_{33}$, it follows from (2.2) that $B_{12} = B_{13} = B_{23} = 0$ so that $\tilde{B}$ is also diagonal and, therefore, proper vectors of $\tilde{T}$ are also proper vectors of $\tilde{B}$. We now assume that, if at all, atmost two proper values of $\tilde{T}$ are equal, say $T_{11} = T_{22}$. Then in view of (2.2) $\tilde{B}$ has the form

$$\tilde{B} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ & B_{22} & 0 \\ 0 & 0 & B_{33} \end{bmatrix}. \tag{2.3}$$

Since B is positive definite

$$B_{11} > 0, \quad B_{33} > 0, \quad C \equiv B_{11}\,B_{22} - B_{12}^2 > 0 \tag{2.4}$$

Substituting from (2.3) and (2.1) into (1.1), we obtain

$$0 = (f_1 - \frac{1}{C} f_{-1})\ B_{12},$$

$$T_{11} = f_0 + f_1 B_{11} + \frac{1}{C} f_{-1} B_{22}$$

$$T_{22} = f_0 + f_1 B_{22} + \frac{1}{C} f_{-1} B_{11} \qquad (2.5)$$

$$T_{33} = f_0 + f_1 B_{33} + \frac{1}{B_{33}} f_{-1} .$$

When inequalities (1.4) hold, it follows from $(2.5)_1$ that $B_{12} = 0$ so that $\underset{\sim}{B}$ is diagonal even when $T_{11} = T_{22}$.

We now consider the case when all principal stresses are equal *i.e.* $\underset{\sim}{T} = q \underset{\sim}{1}$, where q is a constant. Thus (1.1) becomes

$$q \underset{\sim}{1} = f_0 \underset{\sim}{1} + f_1 \underset{\sim}{B} + f_{-1} \underset{\sim}{B}^{-1}.$$

Assuming that this equation can be solved, not necessarily uniquely, for $\underset{\sim}{B}$, we take proper vectors of B as the bases. With respect to these bases $\underset{\sim}{B}$ is a diagonal matrix and therefore

$$q = f_0 + f_1 B_{11} + f_{-1} B_{11}^{-1}$$

$$q = f_0 + f_1 B_{22} + f_{-1} B_{22}^{-1}$$

$$q = f_0 + f_1 B_{33} + f_{-1} B_{33}^{-1} .$$

Subtraction of $(2.6)_1$ from $(2.6)_2$ and $(2.6)_3$ gives equations which imply that $B_{11} = B_{22} = B_{33} = b$ provided that (1.4) holds. Thus for a spherical stress tensor, $\underset{\sim}{B}$ is also a spherical tensor.

The constitutive relation for an incompressible homogeneous isotropic elastic material is

$$\underset{\sim}{T} = -p \underset{\sim}{1} + f_1 \underset{\sim}{B} + f_{-1} \underset{\sim}{B}^{-1}$$

where p is an arbitrary hydrostatic pressure and, f_1 and f_{-1} are functions of the first and second invariant of $\underset{\sim}{B}$; the third invariant of $\underset{\sim}{B}$ equals 1. The E inequalities suggested as plausible by Truesdell ([2], Eqn. 41.24) are

$$f_1 > 0 , \quad f_{-1} \leq 0$$

Since we used restrictions of this type on f_1 and f_{-1} in proving (1.3) for unconstrained materials, it follows that E inequalities imply (1.3) for incompressible homogeneous isotropic materials

3. Remarks

Subtraction of $(2.5)_3$ from $(2.5)_2$ and of $(2.5)_4$ from $(2.5)_3$ gives equations which together with the inequalities (1.4) imply that

$$B_{11} \geq B_{22} \geq B_{33} \text{ whenever } T_{11} \geq T_{22} \geq T_{33} \ . \tag{3.1}$$

That is, the principal stretches are ordered in the same way as are the principal stresses. That the greater principal stress occur in the direction of the greater principal stretch has been proposed by Baker and Ericksen [3] who also showed that this is equivalent to the inequalities

$$f_1 \quad \frac{1}{b_i b_j} f_{-1} \geq 0. \quad (i,j = 1, 2, 3, i \neq j) \tag{3.2}$$

In (3.2) b_1, b_2, b_3 are proper values of $\underset{\sim}{B}$ and the equality sign holds only if $b_i = b_j$. We note that (3.2) is also sufficient to conclude from $(2.5)_1$ that $B_{12} = 0$ and from the equations obtained by subtracting $(2.6)_1$ from $(2.6)_2$ and $(2.6)_3$ that $B_{11} = B_{22} = B_{33}$. Hence *Baker - Ericksen inequalities* (3.2) *also imply* (1.3).

Batra [4] proved that a simple tensile load produces a simple extension in an isotropic elastic material provided (1.4) holds. This is a special case of (3.1) and is also implied by (3.2).

We remark that (3.1) holds for both unconstrained and incompressible isotropic elastic materials. For unconstrained materials, B_{11}, B_{22}, and B_{33} are solutions of $(2.5)_{2,3,4}$ whereas for incompressible materials, B_{11}, B_{22} and B_{33} are solutions of equations obtained by subtracting $(2.5)_3$ and $(2.5)_4$ from $(2.5)_2$ and the equation $B_{11}\ B_{22}\ B_{33} = 1$. As is shown in Section 2, $\underset{\sim}{B} = b\ \underset{\sim}{1}$ whenever $\underset{\sim}{T} = q\ \underset{\sim}{1}$. For unconstrained isotropic elastic materials, b is a solution of

$$q = f_0 + f_1\ b + f_{-1}\ b^{-1} \tag{3.3}$$

where f_0, f_1 and f_{-1} are functions of $(3b, 3b^2, b^3)$. For incompressible materials $b = 1$. We remark that we have neither shown that (3.3) has a real solution for real q nor that a solution of (3.3) even if it is assumed to exist lies between 0 and 1 for negative q and is greater than 1 for positive q However, when f_1 and f_{-1} are constants which satisfy (2.8) then (2.7) can be solved for $\underset{\sim}{B}$ uniquely ([1], p. 351).

We note that the results obtained above hold locally, that is, at a material point.

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