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R. C. Batra

Associate Professor,
Department of Engineering Mechanics,
University of Missouri-Rolla,
Rolla, Mo. Mem. ASME

Saint-Venant's Principle for a Helical Spring

We consider a linear elastic helical spring of arbitrary length and of uniform cross section loaded by a self-equilibrated force system at one end only. We show that the elastic energy, stored in the portion of the spring beyond a certain distance from the loaded end, decreases exponentially with the distance. We thus establish an analog of Toupin's version of the Saint-Venant principle for a helical spring.

Introduction

In 1965 Toupin [1][1] gave a precise mathematical formulation of Saint-Venant's principle. He showed that for a linear elastic homogeneous cylindrical body of arbitrary length and cross section loaded on one end only with an arbitrary system of self-equilibrated forces, the elastic energy $U(s)$ stored in that part of the body which is beyond a distance s from the loaded end satisfies the inequality

$$U(s) \leq U(0) \exp\left(-(s - l)/s_c(l)\right). \tag{1}$$

The characteristic decay length $s_c(l)$ depends upon the maximum and the minimum elastic moduli for the material and the smallest nonzero characteristic frequency of free vibration of a slice of the cylinder of length l. For isotropic materials he showed that the inequality (1) implies the exponential decay of the stresses with the distance from the loaded end. An inequality of the type (1) for a homogeneous isotropic micropolar linear elastic cylindrical body has recently been obtained by Berglund [2]. By using an estimate, due to Ericksen [1, p. 88], for the norm of the stress-tensor in terms of the strain-energy density, Berglund showed that $s_c(l)$ depends on the maximum elastic modulus.

Other mathematical versions of Saint-Venant's principle are due to Sternberg [3], Knowles [4], Zanaboni [5], and Robinson [6]. The statements and proofs of these and of Toupin's version of the Saint-Venant principle are also given by Gurtin [7].

In this paper, we prove an inequality similar to (1) for a linear elastic anisotropic helical spring of arbitrary but constant cross section. We assume that the cross sections are materially uniform in the sense that one cross section can be obtained from the other by a rigid body motion. Thus the material properties do not depend upon the axial coordinate of the point. This idea of material uniformity is due to Ericksen [8] who has discussed this concept in more general terms.

Whereas, previous discussions (e.g., Love [9], Shahinpoor [10]) of the deformation of helical bodies use rod-theories, we use the three-dimensional theory. By describing the deformation of the helical body with respect to suitably selected coordinate axes, we keep the analysis close to that of Toupin.

Formulation of the Problem

Consider a linear elastic body B of arbitrary but constant cross section which in the unstressed state is a clockwise helix. Introduce a fixed rectangular Cartesian coordinate system $\mathbf{X}$ with X^3-axis coinciding with the axis of the helix, the plane $X^3 = 0$ containing one end cross section of the helix and $X^3 \geq 0$ for points in the body. Introduce a curvilinear coordinate system $\mathbf{Y}$ by the transformation

$$\begin{bmatrix} Y^1 \\ Y^2 \\ Y^3 \end{bmatrix} = \begin{bmatrix} \cos bX^3 & -\sin bX^3 & 0 \\ \sin bX^3 & \cos bX^3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X^1 \\ X^2 \\ X^3 \end{bmatrix},$$

or

$$\begin{bmatrix} X^1 \\ X^2 \\ X^3 \end{bmatrix} = \begin{bmatrix} \cos bY^3 & \sin bY^3 & 0 \\ -\sin bY^3 & \cos bY^3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y^1 \\ Y^2 \\ Y^3 \end{bmatrix}. \tag{2a}$$

Thus, for each point of the body, $Y^3 = X^3$ and the Y^1, Y^2-coordinate curves are obtained by rotating clockwise the X^1, X^2-coordinate axes through an angle bX^3, the axis of rotation being parallel to the X^3-axis. b equals the angle of twist of the helix. Using the index notation we write (2a) as

$$X^\alpha = R_i^\alpha Y^i. \tag{2b}$$

[1] Numbers in brackets designate References at end of paper.

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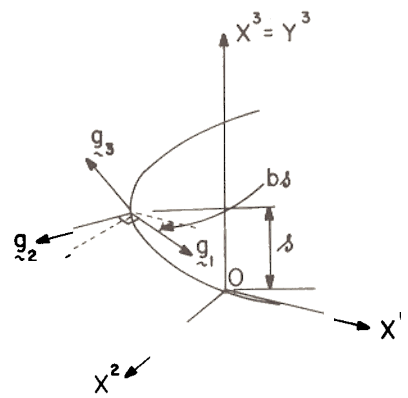

Fig. 1

Throughout this paper we use a mixture of direct and indirect notation. Repeated indices imply summation over the range of indices. $\delta_{ij} = \delta^{ij} = \delta^i_j$ is the Kronecker delta. The Greek indices refer to components with respect to **X**-axes and both the lower case and upper case Latin indices refer to components with respect to **Y**-axes. The upper case Latin indices assume values 1, 2; other indices take values 1, 2, 3. We note that $b = 0$ corresponds to a straight prismatic body. If

$$C_s = \{\mathbf{Y}{:}\mathbf{Y} \in B,\ Y^3 = X^3 = s\},$$

$$= \text{Values of } (Y^1, Y^2, s) \text{ for points of the body which lie in the plane } X^3 = s, \tag{3}$$

then $C_0 = C_s$. That is, the cross section of the helical body is constant. Thus, in the **Y**-coordinate system, the helical body of axial length L occupies the cylindrical region $C_0 \times [0,L]$. Said differently, the helical body is being considered as generated by translating along and uniformly rotating about the X^3-axis a material cross section C_0 contained in the plane $X^3 = 0$.

The covariant base vectors g_i directed tangentially along the Y^i-coordinate curves are given by

$$\mathbf{g}_i = \frac{\partial \mathbf{X}}{\partial Y^i} = (R^\alpha_i + \delta_{i3} R^\alpha_{K,3} Y^K)\mathbf{e}_\alpha,$$

in which a comma followed by an index j indicates partial derivative with respect to Y^j and $\mathbf{e}_\alpha$ are base vectors for the Cartesian coordinate axes **X**. The base vectors $\mathbf{g}_i$ at a typical point are shown in Fig. 1. We note that these do not form an orthogonal set. This fact is also brought out by the explicit expression, given below, for the metric tensor **G** defined as

$$G_{ij} = \frac{\partial X^\alpha}{\partial Y^i}\frac{\partial X^\alpha}{\partial Y^j}, \qquad G^{ij} = \frac{\partial Y^i}{\partial X^\alpha}\frac{\partial Y^j}{\partial X^\alpha}$$

$$[G_{ij}] = \begin{bmatrix} 1 & 0 & -bY^2 \\ 0 & 1 & bY^1 \\ -bY^2 & bY^1 & 1 + b^2((Y^1)^2 + (Y^2)^2) \end{bmatrix},$$

$$[G^{ij}] = \begin{bmatrix} 1 + b^2(Y^2)^2 & -b^2Y^1Y^2 & bY^2 \\ -b^2Y^1Y^2 & 1 + b^2(Y^1)^2 & -bY^1 \\ bY^2 & -bY^1 & 1 \end{bmatrix}$$

One can raise or lower Latin indices by using **G**. Since Greek indices refer to components with respect to rectangular Cartesian **X**-axes, these can be used as subscripts or superscripts. Since det $[G_{ij}] = 1$, the volume element dV given by $dX^1dX^2dX^3$ equals $dY^1dY^2dY^3$.

Due to the application of loads to the body, let the points of the body undergo a displacement $\hat{\mathbf{u}}$. Then

$$x^\alpha = X^\alpha + \hat{u}^\alpha \tag{4}$$

gives the present position of material particles with respect to the fixed rectangular Cartesian **X**-axes. However, in the following, we will work with the ordered triplet u^1, u^2, u^3 denoted by **u** and defined as

$$\hat{u}^\alpha = R^\alpha_i u^i \quad \text{or} \quad u^i = R^i_\alpha \hat{u}^\alpha. \tag{5}$$

In terms of **u**, (4) becomes

$$x^\alpha = R^\alpha_i(Y^i + u^i).$$

We note that u^3 equals the displacement of a point along the axis of the helix and, u^1 and u^2 equal components of $\hat{u}$ along Y^1 and Y^2 coordinate curves. Thus u^3 is not a component of **u** along the Y^3-coordinate curve. Note that (5) is a linear relationship between **u** and $\hat{\mathbf{u}}$ and also it is a one to one correspondence between **u** and the displacement vector $\hat{\mathbf{u}}$. The use of **u** rather than of $\hat{\mathbf{u}}$ simplifies considerably the algebraic work involved. Calculating

$$2E_{ij} = g_{ij} - G_{ij},$$

where

$$g_{ij} = x^\alpha_{,i} x^\alpha_{,j},$$

and retaining terms linear in **u**, we obtain for the infinitesimal strain tensor **e** the following;

$$\begin{aligned} 2e_{KL} &= \delta_{Kj}u^j_{,L} + \delta_{Lj}u^j_{,K}, \\ 2e_{K3} &= \delta_{Kj}u^j_{,3} + \delta_{3j}u^j_{,K} + b\epsilon_{K3j}u^j + bu^j_{,K}\,\epsilon_{j3N}Y^N, \\ e_{33} &= \delta_{3j}\,u^j_{,3} - bu^j_{,3}\,\epsilon_{j3N}Y^N + b^2u^MY^N\delta_{MN}. \end{aligned} \tag{6}$$

In (6) ϵ_{ijk} is the permutation symbol assuming values 1 or −1 according as i, j, k form an even or an odd permutation of 1, 2, 3 and zero otherwise. e_{ij} are components of the infinitesimal strain tensor with respect to **Y**-coordinate axes. That their expressions in terms of **u** and its gradients involve **u** and **Y** should not be surprising since such is also the case in cylindrical coordinates (r, θ, z).

Since in the unstressed reference configuration, various cross sections are assumed to be materially uniform, the strain energy density W per unit volume $(dV = dY^1dY^2dY^3)$ is a function of g_{ij} and at most of Y^K. That is

$$\begin{aligned} W &= W(g_{ij}, Y^K), \\ &= \hat{W}(E_{ij}, G_{ij}, Y^K). \end{aligned}$$

In the linear theory, $\hat{W}$ is approximated by

$$\hat{W} \simeq \hat{W}(\mathbf{0}, \mathbf{G}, Y^K) + 2\chi(e_{ij}), \tag{7}$$

where

$$2\chi = \left.\frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}}\right|_{\mathbf{E}=\mathbf{0}} e_{ij}\,e_{kl} \equiv A^{ijkl}\,e_{ij}\,e_{kl}$$

$$\equiv 2\overline{\chi}(u^i_{,j}, u^k, Y^K), \tag{8}$$

$$A^{ijkl} = A^{jikl} = A^{ijlk} = A^{klij} \tag{9}$$

In (7) the term linear in e_{ij} vanishes because the reference configuration is unstressed. χ is assumed to be positive-definite. The elasticity **A** is a function of Y^K. Note that even when the body is homogeneous in the reference configuration in the sense that $\hat{W}$ does not depend upon **X** explicitly, the elasticity **A** will still depend upon Y^K because of the dependence of G_{ij} upon Y^K. The strain energy density χ is invariant with respect to the superimposed infinitesimal rigid body motion. An infinitesimal rigid body motion in rectangular Cartesian coordinates is given by

$$\omega^\alpha = c^\alpha + \Omega^\alpha_\beta X^\beta,\ \Omega^\alpha_\beta = -\Omega^\beta_\alpha,$$

where c^α and Ω^α_β are constants. By using the coordinate transformation (2), we obtain the following:

$$\omega^i = R^i_\alpha(c^\alpha + \Omega^\alpha_\beta R^\beta_j Y^j),$$

$$\begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix} = \begin{bmatrix} c^1 \cos\phi + c^2 \sin\phi + d^1 Y^2 + Y^3(d^2 \cos\phi + d^3 \sin\phi) \\ -c^1 \sin\phi + c^2 \cos\phi - d^1 Y^1 + Y^3(-d^2 \sin\phi + d^3 \cos\phi) \\ c^3 - d^2(Y^1 \cos\phi - Y^2 \sin\phi) - d^3(Y^1 \sin\phi + Y^2 \cos\phi) \end{bmatrix}, \tag{10}$$

$\phi \equiv bY^3$.

Here ω^1 and ω^2 are measured along Y^1 and Y^2 coordinate curves and ω^3 is measured along the axis of the helix. The expressions for the components of the infinitesimal rigid body motion involve six arbitrary constants $c^1, c^2, c^3, d^1, d^2, d^3$, corresponding to the six degrees of freedom. These are not linear in $\mathbf{Y}$ because $\mathbf{R}$ in (2) depends upon Y^3. The situation here is somewhat similar to that in cylindrical (r, θ, z) coordinates in which an infinitesimal rigid body motion is given by

$$\begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} = \begin{bmatrix} c^1 \cos\theta + c^2 \sin\theta + (d^1 \sin\theta + d^2 \cos\theta)z \\ -c^1 \sin\theta + c^2 \cos\theta + d^3 r + (d^1 \cos\theta - d^2 \sin\theta)z \\ c^3 - (d^1 \sin\theta + d^2 \cos\theta)r \end{bmatrix};$$

$c^1, c^2, c^3, d^1, d^2, d^3$ being constants. (10) can also be obtained by solving the system of differential equations $e_{ij} = 0$. Thus, for any given $\mathbf{u}$, if

$$= \mathbf{u} + \boldsymbol{\omega}, \tag{11$_1$}$$

then

$$e_{ij}(\mathbf{v}) = e_{ij}(\mathbf{u}),$$
$$\bar{\chi}(v^i_{,j}, v^i, Y^K) = \bar{\chi}(u^i_{,j}, u^i, Y^K). \tag{11$_{2,3}$}$$

Equilibrium equations governing the static deformations of a helical body B in the absence of body forces obtained by taking the extremum of

$$\int_B \bar{\chi} dV - \int_{\partial B} f_i u^i \, dU^1 dU^2$$

$$\left(\frac{\partial\bar{\chi}}{\partial u^i_{,j}}\right)_{,j} - \frac{\partial\bar{\chi}}{\partial u^i} = 0 \quad \text{in} \quad B,$$
$$\frac{\partial\bar{\chi}}{\partial u^i_{,j}} n_j = f_i \quad \text{on} \quad \partial B, \tag{12}$$

$$n_i \equiv \pm \epsilon_{ijk} \frac{\partial Y^j}{\partial U^1} \frac{\partial Y^k}{\partial U^2}. \tag{13}$$

Here ∂B, the boundary of the body B, is assumed to be given parametrically by $\mathbf{Y} = \mathbf{Y}(U^1, U^2)$ and the sign is selected so that $\mathbf{n}$ points out of B. $\mathbf{f}$ is the applied force per unit coordinate area $dU^1 dU^2$. We are interested in the case when the loads are applied at the end $X^3 = 0$ and the remainder of the boundary is traction free. In order that there exist a solution of $(12)_{1,2}$ under these conditions, the applied loads must be self-equilibrated and hence should satisfy

$$\int_{C_0} \frac{\partial\bar{\chi}}{\partial u^i_{,3}} dY^1 dY^2 = \int_{C_0} f_i dY^1 dY^2 = 0,$$
$$\int_{C_0} \delta^{kl} \epsilon_{ijk} Y^j f_l dY^1 dY^2 = 0. \tag{14}$$

Here moments are taken about the origin and C_0 is defined by (3). With the definitions

$$a \equiv b \sup_{\mathbf{Y} \in C_0} (|Y^1|, |Y^2|), \tag{15}$$
$$C_{s,l} \equiv \{\mathbf{Y}: \mathbf{Y} \in B, s \le Y^3 \le s + l\},$$
$$= \text{portion of the body between the planes } Y^3 = s \text{ and } Y^3 = s + l, \tag{16}$$

$$U(s) = \int_{X^3 \ge s} \chi dY^1 dY^2 dX^3, \tag{17}$$

we state the theorem which we prove below.

Theorem: If for a linear elastic body which in the unstressed state is helical, the loads applied at the end $X^3 = 0$ satisfy (14) and

$$\frac{\partial\bar{\chi}}{\partial u^i_{,j}} n_j = 0 \text{ on } \partial B - C_0$$

then

$$U(s) \le U(0) \exp\left(-(s - l)/s_c(l)\right) \tag{18}$$

where

$$s_c(l) = 2\sqrt{\frac{\mu}{\lambda(l)}}(1 + 2a), \tag{19}$$

μ = the supremum of the eigenvalues of $\mathbf{A}$ regarded as a linear transformation on the space of symmetric tensors, (20)

$\lambda(l)$ = the smallest nonzero characteristic value of free vibration of a slice of the helical spring of axial length l and mass density per unit volume equal to one. (21)

Proof of the Theorem. Since $\bar{\chi}$ is a homogeneous quadratic form in $u^i_{,j}$ and u^i, by Euler's theorem, we have

$$U(s) = \int_{Y^3 \ge s} \bar{\chi} dV = \frac{1}{2} \int_{Y^3 \ge s} \left(\frac{\partial\bar{\chi}}{\partial u^i_{,j}} u^i_{,j} + \frac{\partial\bar{\chi}}{\partial u^i} u^i\right) dV$$
$$= -\frac{1}{2}\int_{C_s} \frac{\partial\bar{\chi}}{\partial u^i_{,3}} u^i dA. \tag{22}$$

To obtain $(22)_3$ from $(22)_2$, we used the divergence theorem, equilibrium equations $(12)_1$, and $ds_k = -dY^1 dY^2 \delta_{3k} = -dA\,\delta_{3k}$ on C_s. Because of (11) we can replace $\mathbf{u}$ in (22) by

$$\mathbf{v} = \mathbf{u} + \bar{\boldsymbol{\omega}} \tag{23}$$

where $\bar{\boldsymbol{\omega}}$ is an infinitesimal rigid body motion given by (10). Thus

$$U(s) = -\frac{1}{2}\int_{C_s} \frac{\partial\bar{\chi}}{\partial v^i_{,3}} v^i dA. \tag{24}$$

Physically this expresses the fact that any self-equilibrated force system does no work during a rigid motion of the body. From (6) and (8) we obtain

$$\frac{\partial\bar{\chi}}{\partial v^K_{,3}} = \frac{1}{2} \frac{\partial\chi}{e_{m3}} \qquad \frac{\partial\chi}{e_{33}} b\epsilon_{K3n} Y^n,$$
$$\frac{\partial\bar{\chi}}{\partial v^3_{,3}} = \frac{\partial\chi}{\partial e_{j3}} \delta_{j3}.$$

The Schwarz and geometric-arithmetic mean inequality give (e.g., see Toupin [1, p. 93])

$$2\int_B fh dV \le \nu \int_B f^2 dV + \frac{1}{\nu} \int_B h^2 dV$$

for all $\nu > 0$ and all scalar fields f and h defined on B. Thus

$$\int_{C_s} \frac{\partial\bar{\chi}}{\partial v^1_{,3}} v^1 dA = \int_{C_s} \frac{1}{2} \frac{\partial\chi}{\partial e_{13}} v^1 dA - b \int \frac{\partial\chi}{\partial e_{33}} Y^2 v^1 dA,$$
$$\le \int_{C_s} \frac{1}{2} \frac{\partial\chi}{\partial e_{13}} v^1 dA + a \int_{C_s} \left|\frac{\partial\chi}{\partial e_{33}} v^1\right| dA,$$
$$\le \frac{1}{2}\left[\nu \int_{C_s} \frac{1}{4} \frac{\partial\chi}{\partial e_{13}} \frac{\partial\chi}{\partial e_{13}} dA + \frac{1}{\nu} \int_{C_s} (v^1)^2 dA\right.$$
$$\left. + a\left\{\nu \int_{C_s} \frac{\partial\chi}{\partial e_{33}} \frac{\partial\chi}{\partial e_{33}} dA + \frac{1}{\nu} \int_{C_s} (v^1)^2 dA\right\}\right].$$

Bounding the other two terms in a similar way, we obtain

$$U(s) \le \frac{\Gamma}{2} \delta_{ij} \left[\nu \int_{C_s} \frac{\partial\chi}{\partial e_{i3}} \frac{\partial\chi}{\partial e_{j3}} dA + \frac{1}{\nu} \int_{C_s} v^i v^j dA\right], \tag{25}$$

with $\Gamma = \frac{1}{2}(1 + 2a)$. Now

$$\delta_{ij} \frac{\partial\chi}{\partial e_{i3}} \frac{\partial\chi}{\partial e_{j3}} \le \delta_{ij}\,\delta_{kl} \frac{\partial\chi}{\partial e_{ik}} \frac{\partial\chi}{\partial e_{jl}} = \delta_{ij}\,\delta_{kl} A^{ikpq} e_{pq} A^{jlmn} e_{mn}, \tag{26}$$

in which we have used (8). $\mathbf{A}$ regarded as a linear transformation on the space of symmetric tensors has six positive eigenvalues $\bar{\mu}$'s given by

$$A^{ijkl}\,\bar{e}_{kl} = \bar{\mu}\,\bar{e}_{pq}\,\delta^{ip}\,\delta^{jq}.$$

Here $\bar{e}_{kl}$ are components in the six dimensional space of symmetric tensors of an eigenvector corresponding to the eigenvalue $\bar{\mu}$. Since $\mathbf{A}$ is a function of Y^K, $\bar{\mu}$ will also be a function of Y^K. Denoting by μ the supremum taken for all $\mathbf{Y} \in C_0$ of the eigenvalues of $\mathbf{A}$, we conclude that

$$\delta_{ij}\,\delta_{kl}\,e_{mn}\,A^{ikpq}\,A^{jlmn}\,e_{pq} \leq 2\mu\chi. \tag{27}$$

Details of obtaining Ericksen's estimate (27) are given by Berglund [2]. Substituting from (27) and (26) into (25) we arrive at

$$U(s) \leq \frac{\Gamma}{2}\left[2\nu\,\mu \int_{C_s} \chi\, dA + \frac{\delta_{ij}}{\nu}\int_{C_s} v^i v^j dA\right]. \tag{28}$$

Integrating both sides of this inequality with respect to Y^3 from $Y^3 = s$ to $Y^3 = s + l$ for some $l > 0$ and setting

$$\frac{1}{l}\int_s^{l+s} U(s')ds' = Q(s,l), \tag{29}$$

we obtain

$$Q(s,l) \leq \frac{\Gamma}{2l}\left[2\nu\mu \int_{C_{s,l}} \chi dV + \frac{\delta_{ij}}{\nu}\int_{C_{s,l}} v^i v^j dV\right]. \tag{30}$$

We now proceed to bound the second integral on the right-hand side of (30) by an integral of χ. Consider free-vibration problem of the helical spring and define a characteristic solution (e.g., see Gurtin [7, Section 75]) as the ordered pair $[\lambda, \mathbf{u}]$ such that λ is a scalar and $\mathbf{u}$ is a smooth field on B, and

$$\left(\frac{\partial \bar{\chi}}{\partial u^i_{,k}}\right)_{,k} - \frac{\partial \bar{\chi}}{\partial u^i} + \lambda u^i = 0 \quad \text{in} \quad B,$$

$$\delta_{ij}\int u^i u^j dV = 1, \qquad u^i \frac{\partial \bar{\chi}}{\partial u^i_{,k}} dS_k = 0 \quad \text{on} \quad \partial B.$$

By proceeding in the same way as that given in [7] we can show that

$$\lambda = \frac{2\int \bar{\chi} dV}{\delta_{ij}\int u^i u^j dV},$$

and that if $[\lambda, \mathbf{u}]$ and $[\bar{\lambda}, \bar{\mathbf{u}}]$ are two characteristic solutions, then

$$(\lambda - \bar{\lambda})\delta_{ij}\int u^i \bar{u}^j dV = 0. \tag{31}$$

Hence $\mathbf{u}$'s corresponding to different λ's are orthogonal in the sense made precise by (31). If $\lambda(l)$ denotes the lowest nonzero characteristic value corresponding to the free vibration of $C_{s,l}$, then

$$\lambda(l) \leq 2\frac{\int_{C_{s,l}} \bar{\chi}\,(v^i_{,k}, v^i, Y^K)dV}{\delta_{ij}\int_{C_{s,l}} v^i v^j dV} \tag{32}$$

for every smooth displacement field $\mathbf{v}$ on $C_{s,l}$ that satisfies

$$\delta_{ij}\int_{C_{s,l}} v^i v^j dV \neq 0, \qquad \int_{C_{s,l}} v^i dV = \int_{C_{s,l}} \epsilon_{ijk}\, Y^j v^k\, dV = 0. \tag{33}$$

Following Toupin [1], the rigid displacement $\bar{\omega}$ in (23) can be chosen so as to satisfy (33). Substituting from (32) into (30) and using $(11)_3$ we get

$$Q(s,l) \leq \bar{s}_c(l)\frac{1}{l}\int_{C_{s,l}} \chi dV, \tag{34}$$

in which

$$\bar{s}_c(l) = \Gamma\left[\mu\nu + \frac{1}{\lambda\nu}\right].$$

The minimum value

$$s_c(l) = 2\Gamma\sqrt{\frac{\mu}{\lambda(l)}} \tag{35}$$

occurs for $\nu = 1/\sqrt{\mu\lambda}$. Henceforth, we assume that ν has this value. Differentiation of (29) with respect to s yields

$$\frac{dQ}{ds} = [U(s+l) - U(s)]/l = -\frac{1}{l}\int_{C_{s,l}} \chi dV,$$

and this together with (34) and (35) results in

$$s_c(l)\frac{dQ}{ds} + Q \leq 0.$$

Integrating this and using

$$U(s+l) \leq Q(s,l) \leq U(s)$$

which follows from the fact that $U(s)$ is a nonincreasing function of s, we obtain

$$\frac{U(s_2 + l)}{U(s_1)} \leq \exp\left((s_2 - s_1)/s_c(l)\right).$$

The choice $s_1 = 0$ and $s_2 + l = s$ gives the desired inequality (18).

Remarks

When $b = 0$ the helical spring becomes a straight prismatic body and the $\mathbf{Y}$-coordinate curves coincide with the $\mathbf{X}$-coordinate curves. From (6) one sees that the expressions for strains reduce to the familiar ones. Also from (15) one gets $a = 0$ and the value of Γ given immediately after equation (25) becomes 1. Thus the characteristic decay length $s_c(l)$ reduces to essentially that given by Toupin, the remaining difference being due to the sharper estimate (27) used in our work.

If one had worked with the displacement vector $\hat{\mathbf{u}}$, then calculations similar to that carried out in the foregoing would show that the elasticity $\mathbf{A}$ and the strain energy density $\bar{\chi}$ depend upon X^3 also. This would require taking supremum of the eigenvalues of $\mathbf{A}$ over the entire body. Also the choice of the length l and the lowest characteristic frequency of the free vibration $\lambda(l)$ in (32) would be more involved.

It does not seem obvious that the present analysis will apply, without any major modification, to other theories in which the governing differential equations are similar to (12) with $\bar{\chi}$ a homogeneous positive-definite quadratic function of the $\mathbf{u}_{,k}$ and $\mathbf{u}$. This is because we make explicit use of the strain-displacement relations (6) and the alternate form $(8)_1$ of the strain-energy density.

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