

## EXTENSION OF AN ENERGY THEOREM OF BARENBLATT'S TO ELASTODYNAMICS

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### Introduction

In some problems of the mechanics of continuous media one encounters the situation that the solution of the differential equations of the problem, satisfying initial and boundary conditions, is determined only to within one or more arbitrary parameters. In order to determine the appropriate value of these parameters for a given such problem one usually makes use of subsidiary conditions such as the finiteness of velocities, stresses, etc. For elastostatic problems, Barenblatt [1] has given a theorem to establish these additional conditions which is an extension of the principle of minimum potential energy. These conditions provide the equations to be appended to the boundary value problem in order to obtain a complete solution to the problem. Barenblatt illustrates the use of his theorem to rederive such conditions for a contact problem and a crack problem; these results previously had been established by requiring stresses to remain finite.

In the present note we generalize Barenblatt's result to cover a class of elastodynamic problems. The problems included are those for which the system is in a steady state in the sense that the total kinetic energy, the total strain energy and the work done by the external forces are independent of time.

### Extension of Barenblatt's Theorem to the Steady State Elasto-Dynamic Case

Consider a dynamic, linear, elastic system which is in a steady state in the sense that the total kinetic energy, the total strain energy and the total work done by the external forces are finite constants\*. Note that if a system is in a steady state according to the usual definition which requires that the velocity of every point of the system be time-independent, then the system

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\*The surface tractions and the body forces at a material point are assumed to be functions of its position only.

would be in a steady state according to the present definition. However, the converse need not be true. Now, assume that the solution of the field equations and boundary conditions of the problem contains some arbitrary parameters which remain undetermined. We denote the set of undetermined elements of the solution by  $M$ . According to Hamilton's principle

$$\delta K - \delta W + \delta A = 0 \quad (1)$$

where  $\delta K$ ,  $\delta W$  and  $\delta A$  are, respectively, the variations of the kinetic energy  $K$ , the strain energy  $W$  and the work  $A$  of the external forces for a given virtual state of the elastic system. Let  $\underline{u}$  represent the displacement field of the system which corresponds to some fixed  $M$ ;  $\delta_1 \underline{u}$  is a variation of this field satisfying the geometric constraints imposed upon the system and corresponding to the same fixed  $M$ , and  $\delta_2 \underline{u}$  is a variation of the displacement field corresponding to the variation  $\delta M$  of the set of undetermined elements. We denote the variations in  $K$ ,  $W$  and  $A$  corresponding to these variations in  $\underline{u}$  by  $\delta_1 K$ ,  $\delta_1 W$ ,  $\delta_1 A$  and  $\delta_2 K$ ,  $\delta_2 W$ ,  $\delta_2 A$ . In view of the independence of the variations  $\delta_1 \underline{u}$  and  $\delta M$ , we obtain from (1)

$$\begin{aligned} \delta_1 K - \delta_1 W + \delta_1 A &= 0 \\ \delta_2 K - \delta_2 W + \delta_2 A &= 0 \end{aligned} \quad (2)$$

From relation (2)<sub>1</sub> we obtain in the usual manner the differential equations

$$\rho \ddot{u}_i - \left( \frac{\partial W}{\partial u_{i,j}} \right)_{,j} + b_i = 0 \quad \text{in } R \quad (3)$$

and the boundary conditions

$$\begin{aligned} \left( \frac{\partial W}{\partial u_{i,j}} \right) n_j &= f_i \quad \text{on } \partial_1 R \\ \underline{u} &= \underline{u}_0 \quad \text{on } \partial R - \partial_1 R \end{aligned} \quad (4)$$

which corresponds to an arbitrary fixed set of undetermined elements  $M$ . Here  $R = E^3$  denotes the bounded region of the Euclidean space occupied by the body in the reference configuration,  $\underline{n}$  is an outward unit normal to  $\partial R$ , the boundary of  $R$ , and  $\rho$  is the mass density per unit volume. Note that we have referred the deformation to a fixed rectangular Cartesian co-ordinate system. A comma followed by a subscript  $i$  denotes partial differential with respect to  $x_i$  and a superposed dot stands for the material time derivative. Furthermore,  $\underline{f}$  denotes the surface tractions applied on the part  $\partial_1 R$  of the boundary of  $R$  and  $\underline{b}$  gives the body force per unit volume. Now if the field  $\underline{u}$  satisfies the differential equations of motion and the boundary conditions, then

$$\begin{aligned}
A &= \int_{\partial R} \underline{f} \cdot \underline{u} dA + \int_R \underline{b} \cdot \underline{u} dV, \\
&= \int_{\partial R} \left( \frac{\partial W}{\partial u_{i,j}} \right) n_j u_i dA + \int_R \underline{b} \cdot \underline{u} dV, \\
&= \frac{d}{dt} \int_R \rho \dot{\underline{u}} \cdot \underline{u} dV - \int_R \rho \dot{u}_i \dot{u}_i dV + \int_R \frac{\partial W}{\partial u_{i,j}} u_{i,j} dV, \\
&= 2(W-K) + \frac{d}{dt} \int_R \dot{\underline{u}} \cdot \underline{u} dV \quad (5)
\end{aligned}$$

In order to obtain  $(5)_3$  from  $(5)_2$  we have used the divergence theorem and (3). By our definition of a generalized steady state system and  $(5)_4$

$$\frac{d}{dt} \int_R \rho \dot{\underline{u}} \cdot \underline{u} dV = C \text{ (constant).}$$

We claim that  $C$  is zero. Indeed, if  $C \neq 0$ , then  $\left| \int_R \rho \dot{\underline{u}} \cdot \underline{u} dV \right| \rightarrow \infty$  as  $t \rightarrow \infty$ . But by the Cauchy-Schwarz inequality,

$$\left| \int_R \rho \dot{\underline{u}} \cdot \underline{u} dV \right|^2 \leq \int_R \rho \dot{u}^2 dV \int_R \rho u^2 dV.$$

Since the right-hand side of this inequality is bounded, because of the assumptions of linear elasticity,  $\int_R \rho \dot{\underline{u}} \cdot \underline{u} dV$  is bounded which is a contradiction. Hence  $C = 0$  and, therefore,

$$A = 2(W - K).$$

Since this equation holds for arbitrary  $M$  we have

$$\delta_2 A = 2(\delta_2 W - \delta_2 K). \quad (6)$$

Upon substitution of  $(2)_2$  in (6), we obtain

$$\delta_2 (K-W) = 0 \quad (7)$$

This relation is a general condition which determines the set of undetermined elements  $M$  of the problem. In particular, if the solution of (3) and (4) is determined to within a finite number of parameters,  $C_1, C_2, \dots, C_n$ , then from (7)

$$\frac{\partial (K-W)}{\partial C_i} = 0, \quad i = 1, 2, \dots, n. \quad (8)$$

In general, (8) is a system of  $n$  equations for  $n$  unknowns. If (8) has a unique solution, then the solution of the problem (3) and (4) would also be determined uniquely. We summarize the preceding discussion in the following

Theorem: If, for a dynamic, linear, elastic system

$$\dot{K} = 0, \quad \dot{W} = 0, \quad \dot{A} = 0 \quad (9)$$

and the solution of the field equations (3) and boundary conditions (4) is determined to within a set of undetermined parameters, then these parameters are solutions of (7).

We remark that, in practice, it may be easier to determine the unknown parameters from

$$\delta_2 A = 0. \quad (10)$$

Clearly, in view of (6), (10) is equivalent to (7).

Batra, Levinson and Hahn [2] have applied the above theorem to the problem of the indentation, by a rigid cylinder, of an elastic layer bonded to a uniformly rotating rigid cylinder. They used the complete equations of motion including the Coriolis acceleration terms and were able to rigorously reduce that the deformations of the elastic layer for this steady state elastodynamic problem were symmetric with respect to the line joining the centers of the two contacting cylinders; this result was not obvious.

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#### References

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