COMMUNICATIONS IN APPLIED NUMERICAL METHODS, Vol. 1, 169–176 (1985)

FINITE DEFORMATIONS OF A VISCOELASTIC ROLL COVER CONTACTING A RIGID PLANE SURFACE

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SUMMARY

Steady-state finite plane strain deformations of a uniformly rotating rubber-covered roll contacting a rigid plane surface have been studied by the finite element method. The rubberlike material has been modelled by a constitutive relation recently proposed by Christensen. Both the hydrostatic pressure within an element and the nodal displacements are taken as independent variables. For a constant indentation of the roll cover, the effect of the change in the thickness of the rubberlike layer, the speed of rotation and the values of the material moduli on the total load required to cause the indentation are presented.

INTRODUCTION

Contact problems involving the indentation of a rubberlike layer by a relatively hard body find technical applications in the paper, textile and printing industry. In most of these applications the deformations of the rubber are large, and one needs to study a geometrically and materially nonlinear problem. An interesting aspect of these problems is that both the contact width and the pressure distribution at the contact surface are not known *a priori*. For linear elastic problems involving smooth contact between two bodies, one can use Barenblatt's theorem[1,2] to find one or more such unknowns and thereby make the problem tractable. Linear viscoelastic contact problems have been studied by analytical[3–5] and numerical methods.[6,7] Spengos[8] has investigated experimentally the contact of a rubberlike roll cover with another hard cylinder. We note that in his experimental set-up, the ratio of the diameter to the length of cylinders was essentially one. However, the strains induced for loads generally expected to occur in practical situations are large enough to be outside the range of applicability of the linear theory.

Recently, Bapat and Batra[9] studied the steady-state finite deformations of a uniformly rotating rubber-covered roll contacting another cylinder. Herein we extend the previous work to the case when the indentor is a flat rigid surface. We investigate the effect of the change in the thickness of the rubberlike layer, the speed of rotation and the values of the material moduli on the load required to cause the same amount of indentation of the rubberlike layer. Results presented graphically include the pressure profile at the contact surface, the distribution of the radial and shear stresses near the core and the stress distribution through the thickness of the rubberlike layer.

FORMULATION OF THE PROBLEM

The system being studied is shown in Figure 1. We use rectangular Cartesian co-ordinates, with origin at the centre of the undeformed position of the roll cover, to describe the deformations of the rubberlike material. Neglecting the effect of body forces and inertia forces, equations governing the steady-state deformations of the rubber are

$$\det [F_{iA}] = 1 \tag{1}$$

$$S_{iA,A} = 0 \tag{2}$$

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0748–8025/85/040169–08$01.00

Received 17 September 1984

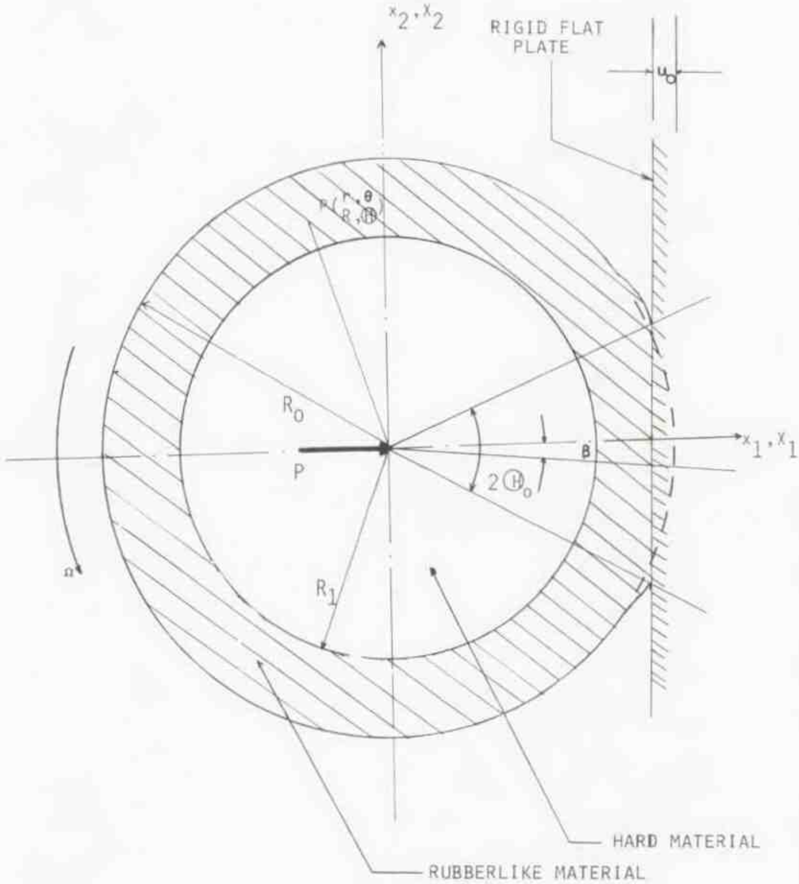

Figure 1. Problem to be studied

$$u_i = 0 \text{ on the surface } X_1^2 + X_2^2 = R_1^2 \tag{3}$$

$$S_{iA}N_A = 0 \text{ on } \Gamma_{01}, \tag{4}$$

$$\left.\begin{aligned} x_1 &= R_0 - u_0 \\ e_i S_{iA} N_A &= 0 \end{aligned}\right\} \text{ on } \Gamma_{02} \tag{5}$$

Γ_{01} = part of the surface $X_1^2 + X_2^2 = R_0^2$, where $|\Theta - \beta| \geq \Theta_0$;

Γ_{02} = part of the surface $X_1^2 + X_2^2 = R_0^2$, where $|\Theta - \beta| < \Theta_0$;

$\Theta = \arctan(X_2/X_1)$;

$F_{iA} = \partial x_i/\partial X_A$,

$$\bar{S}_{AB} = F_{iB}S_{iA} = -p(C^{-1})_{AB} + g_0\delta_{AB} + \int_{-\infty}^{t} g_1(t-\tau)\,\frac{\partial E_{AB}}{\partial \tau}\,\mathrm{d}\tau \tag{6}$$

$$g_1(t) = g_1(0)\exp(-t/\tau_r) \tag{7}$$

$$C_{AB} = F_{iA}F_{iB},\ 2E_{AB} = C_{AB} - \delta_{AB} \tag{8}$$

Here, S_{iA} and $\bar{S}_{AB}$ are, respectively, the first and second Piola–Kirchhoff stress tensors, F_{iA} is the deformation gradient, g_0 and g_1 are the material moduli, N_A is a unit outward normal to the outer surface in the reference configuration, and e_i is a unit tangent vector at a point on the outer surface in the current configuration. Throughout the paper, we use lower (upper) case indices to signify components of a quantity with respect to co-ordinates in the present (reference) configuration. Equation (1) is the continuity equation in the referential description and implies that the deformations are isochoric. The boundary condition (3) signifies that the rubber is perfectly bonded to the core. The boundary conditions (4) and (5) signify that the part Γ_{01} of the outer surface that is not in contact with the rigid plane surface is traction free, and the remainder Γ_{02} of the outer surface that is abutting the indentor is free of tangential tractions and, in the deformed position, is a plane surface. Said differently, the contact between the roll cover and the indentor has been taken to be smooth. Thus, the resultant of forces exerted by the indentor on the plane deformed surface will be equal and opposite to the force P but will not pass through the centre of the roll cover.

In the constitutive relation (6) recently proposed by Christensen,[10] p is a hydrostatic pressure that is not determined by the deformation history at the particle. Because of the presence of $(C^{-1})_{AB}$ in equation (6), the dependence of the stress upon the deformation measure C_{AB} is nonlinear. Even though it has not been shown whether equation (6) models rubberlike materials well or not, it is perhaps the simplest one to use in a nonlinear theory. In the expression (7) for the shear modulus, τ_r is the relaxation time of the material of the roll cover. This form for $g_1(t)$ signifies that the material does not remember much of its past deformations. We note that Θ_0, u_0 and β are unknown and are to be determined as a part of the solution of the problem. Θ_0 is the semicontact angle in the reference configuration, u_0 is the indentation and β is the angle of asymmetry in the angle of contact measured in the reference configuration. We assume that the length of the roll cover is considerably large as compared to its diameter, and that plane strain state of deformation prevails.

As was done in earlier studies,[6,7,9] we presume that the speed Ω, the load P and the relaxation time τ_r are such that stresses and strains decay to zero at points for which $|\Theta| > L\Theta_0$, L being a suitable number such that $L\Theta_0 < \pi$. If such were not the case then the stresses at a material particle will depend upon its deformations during the previous cycles of deformation, and our solution technique, although valid in principle, will have to be modified considerably.

We divide the spatial region of space enclosed by the radial lines $\theta = \pm L\theta_0$ into uniform rectangular elements as shown in Figure 2. For any specific material point, we start the clock when it passes the line AB. Then, from equation (6), the stress at this material point when it is at the centroid of the nth element is

$$\tilde{S}_{AB} = -p(C^{-1})_{AB} + g_0\delta_{AB} + \sum_{m=1}^{n} g_1\left(\frac{\psi-\psi^*_m}{\Omega}\right)\{E_{AB}(\psi_m) - E_{AB}(\psi_{m-1})\} \tag{9}$$

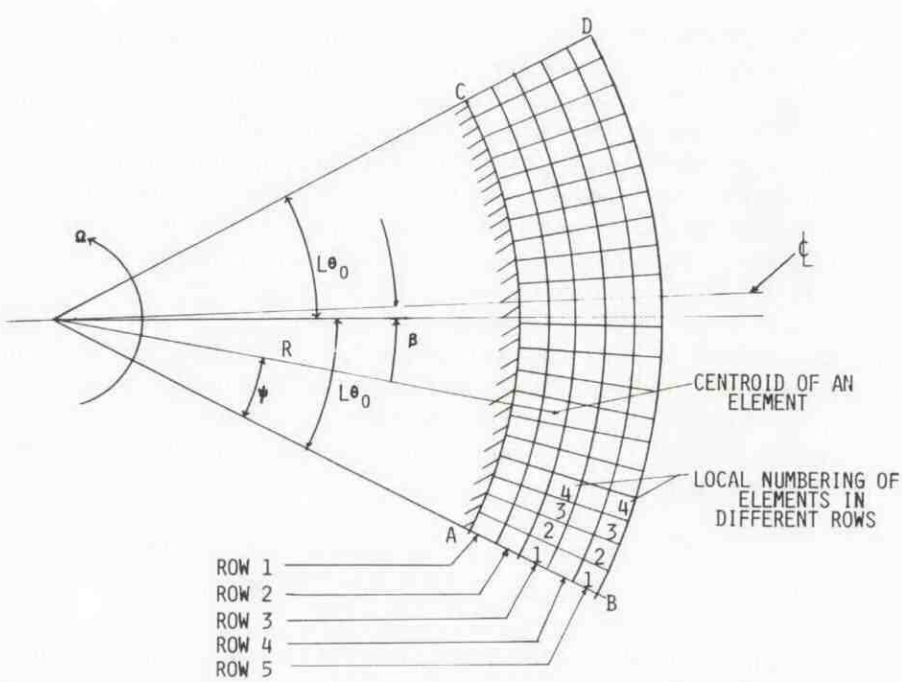

Figure 2. A representative division of the region into finite elements

where $\psi_{m-1} < \psi_m^* < \psi_m$; ψ_m is the angular position of the centroid of the mth element with respect to line AB. Thus the stress at the material point when it is at the centroid of the mth element in any row depends upon the strains it experienced in going from the line AB to its present position. Said differently, at any given instant, the stress at the material point that currently is at the centroid of the mth element in any row depends upon the strains experienced by material points currently at the centroids of elements 1 through m in that particular row. The constitutive relation (9) is similar to that of an elastic body exhibiting non-local response.

We refer the reader to References 9, 11 and 12 for details of the finite element formulation of the problem. Below we outline the procedure used to solve the problem.

SOLUTION PROCEDURE

We start by presuming $2\Theta_0$ and β, and replace the essential boundary condition $(5)_1$ by the traction boundary condition. We estimate a pressure profile on the contact surface. Now the problem is well defined and therefore can be solved numerically. In the solution of the problem we use 4-node isoparametric quadrilateral elements with nodal displacements and uniform hydrostatic pressure within an element as independent variables.

After having solved the problem we find the deformed position of the outer surface of the roll cover. Now we ensure that nodal points outside the presumed contact width have not penetrated into the indentor. If even one of these nodes has penetrated into the indentor, either the presumed contact width or the estimated pressure profile is changed and the problem is solved again. However, if none of the nodes outside the assumed contact width penetrates into the indentor, the estimated pressure profile is iterated upon until the deformed shape of the outer surface of the roll cover within the assumed contact area is, within an acceptable tolerance, a plane surface. When such is the case, the problem has been solved. However, in the results presented below, the indentation u_0 has been kept fixed. In order to do so, the pressure distribution on the contact width is modified till the computed u_0 equals the desired one.

COMPUTATION AND DISCUSSION OF RESULTS

The region of the roll cover within nearly three times the estimated contact width is divided into uniform rectangular elements as shown in Figure 2. We realize that the region considered is perhaps inadequate, but the rather tight computer resources prevented us from experimenting with different grids. In the results presented herein, the ends AB and CD were taken to be traction free. Equilibrium iterations[13] used to satisfy the balance of linear momentum and the continuity equation within a given tolerance were stopped when the increment in the x_1-component of the displacement of each node was less than 0.25 per cent of its total x_1-displacement up to that load increment. Also, the deformed surface of the roll cover was taken to match the plane profile of the indentor if the distance from the indentor of each nodal point on the assumed contact width was less than 1·5 per cent of the indentation u_0. The pressure distribution that results in the acceptable deformed surface is plotted, and the values of the contact width angle $2\Theta_0$ and the angle β of asymmetry are read from this graph. A precise but considerably more expensive way of finding Θ_0 and β will be to use a very fine grid so that the extremities of the contact width are nodes.

To solve a specific problem we have taken, somewhat arbitrarily, the following values of various material and geometric parameters:

$$R_0 = 2{\cdot}39 \text{ in.}, \; R_1 = 1{\cdot}83 \text{ in.}, \; \Omega = 12{\cdot}5 \text{ rad/sec}, \quad g_0 = 25 \text{ psi}, \; g_1(t) = 25\exp(-t/0{\cdot}002) \text{ psi} \tag{10}$$

The choice $(10)_5$ implies that the instantaneous value $g_1(0)$ of the relaxation modulus is 25 psi and the relaxation time of the material is 0·002 sec. In order to present concisely the results when one of the parameters R_1, Ω and $g_1(0)$ is changed at a time, we number the problems as shown in Table 1.

Table I. Problem labelling†

Parameter(s) changed	Its new value	Problem no.
None		1
$g_0, g_1(0)$	$g_0 = 50$, $g_1(0) = 0$	2
R_1	1·97 in.	3
Ω	40 rad/sec	4

†In problem 2 the rubber is modelled as a neo-Hookean material

Figure 3 depicts the pressure distribution $f(\Theta)$ at the contact surface for the four problems with $u_0 = 0{\cdot}05$ in. As expected, the pressure distribution is symmetric about the centreline in the elastic case but is asymmetric for viscoelastic problems. Also, the peak pressure in the elastic case is roughly twice that for problem 1. The total load P per unit length of the roll cover required for the indentation and given by

$$P = R_0 \int_{-\Theta_0-\beta}^{\Theta_0-\beta} f(\Theta)\, d\Theta,$$

along with Θ_0 and β for different problems are listed in Table II.

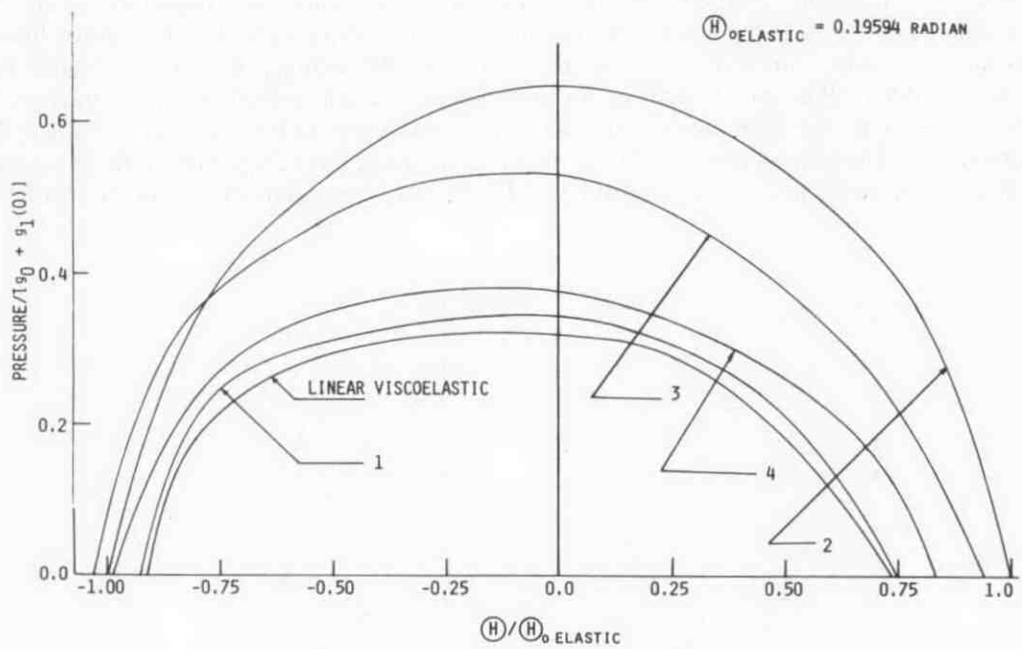

Figure 3. Pressure distribution at the contact surface

Table II. Load and contact width

Problem no.	Load P (lb/in.)	$2\Theta_0$(rad)	β(rad)
1	11·73	0·3266	0·0136
2	23·06	0·3919	0·0
3	20·2	0·3853	0·01
4	13·07	0·356	0·0114

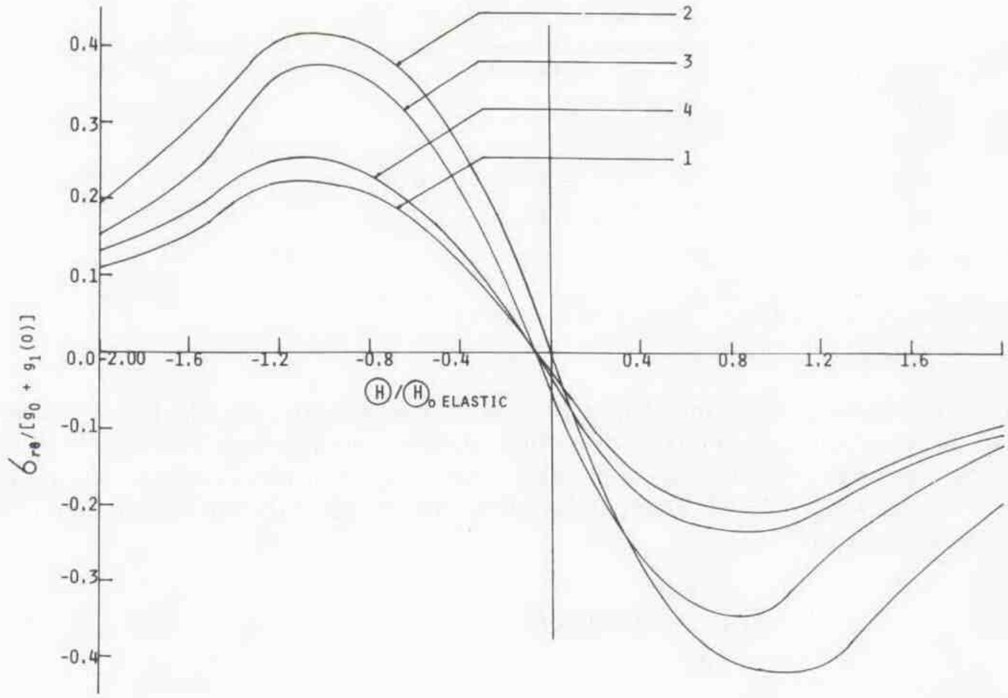

Figure 4. Shear stress distribution near the bond surface

Among the problems studied, the maximum principal strain that occurred at any Gauss (integration) point was 15·5 per cent.Even at these strains the pressure distribution for linear and nonlinear viscoelastic problems is somewhat different; the contact pressure is higher for the nonlinear problem. We remark that for linear problems the entire load on the contact surface is applied in one step. The asymmetry in the deformations of viscoelastic roll covers is rather obvious from Figure 4, which shows the shear stress distribution on the surface parallel to the bond surface. The distance between this surface and the bond surface is 7 per cent of the thickness of the roll

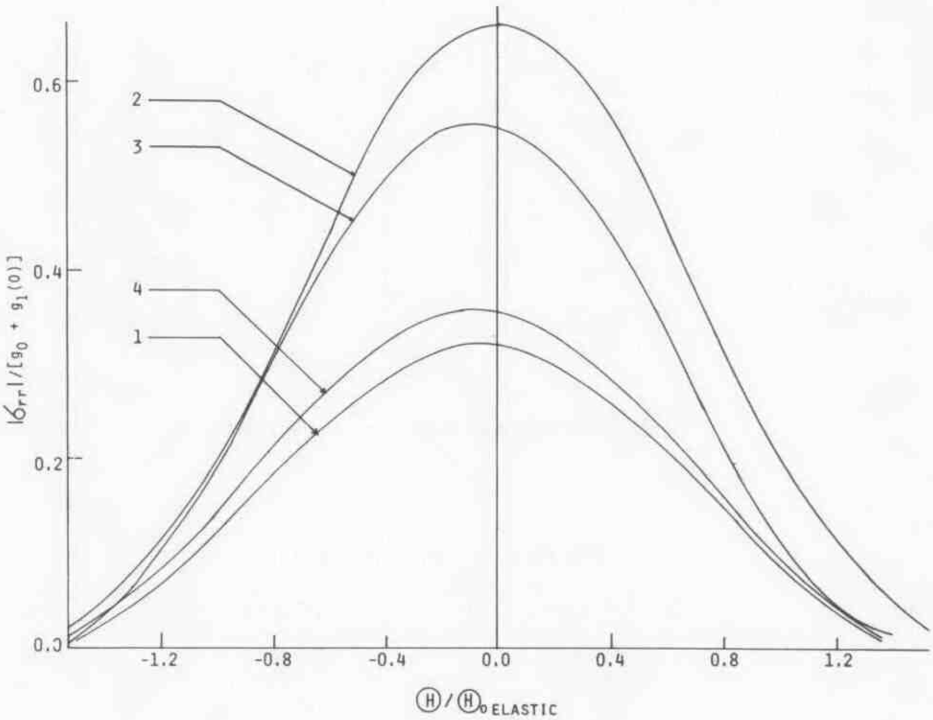

Figure 5. Radial stress distribution near the bond surface

cover. The magnitude of the maximum shear stress is approximately 70 per cent of the peak contact pressure. However, the magnitude of the maximum radial stress on this surface, as shown in Figure 5, is about the same as the peak pressure on the contact surface. Of course, the radial stress distribution on this surface is smoother and is spread over a larger width. The details of the deformation of the rubber within the vicinity of the contact width are shown in Figure 6. Lower case letters indicate the deformed positions of points whose undeformed position is marked by the corresponding upper case letters. Because of the 4-node isoparametric elements used, the element boundaries both in the undeformed and deformed configurations are shown by straight lines. There is noticeable shear strain induced in elements near the bond surface. The computed results indicate that the radial stress does not vary much through the thickness for this case. Finally, the effect of various parameters on the load required to cause the indentation is shown in Figure 7. When the thickness of the roll cover is decreased by 25 per cent, the load required to cause the same amount of indentation increases by 80 per cent.

CONCLUSIONS

The steady-state geometrically and materially nonlinear problem involving the indentation of a uniformly rotating viscoelastic roll cover indented by a rigid plane surface has been studied. The computed results indicate that the thickness of the roll cover has a predominant effect on the total load required to cause the same amount of indentation. Along a radial line near the middle of the contact surface, the radial stress does not seem to vary much. The peak radial stress at points near the bond surface is nearly the same as the peak pressure at the contact surface.

Even though it is desirable to specify when to include material nonlinearities, and/or geometric nonlinearities, we believe that the decision is problem dependent. For a lack of computer resources, we have not studied a sufficient number of problems to characterize meaningfully the effect of different non-dimensional parameters or relations among them.

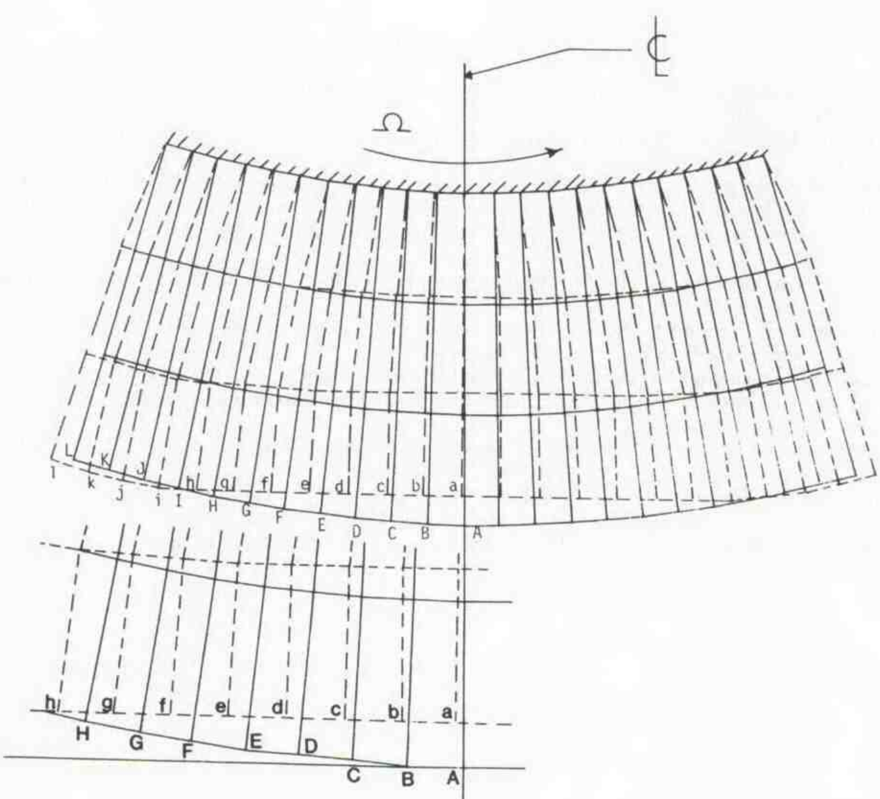

Figure 6. Details of the deformation of the rubber in the vicinity of the contact width

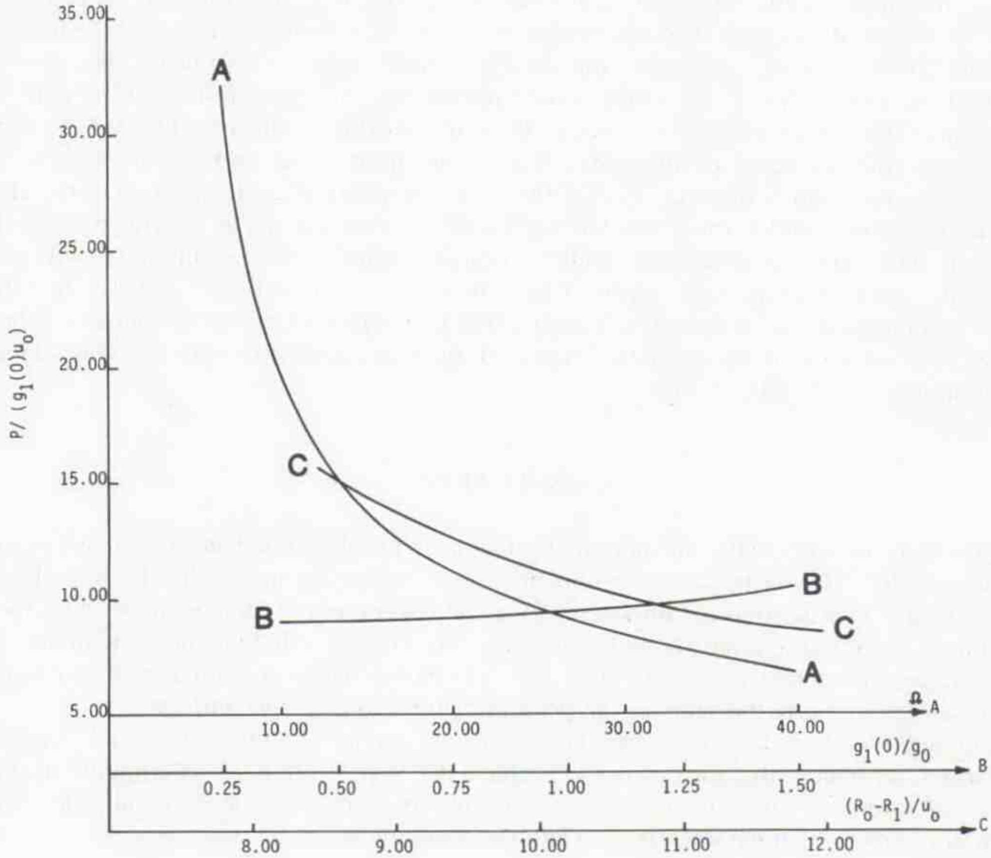

Figure 7. Effect of various parameters on the total load required

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