

LOCATIONS OF OPTIMAL STRESS POINTS IN HIGHER-ORDER ELEMENTS

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SUMMARY

The locations of optimal stress points in Lagrangian and serendipity elements are determined by using the symbolic mathematical tool MATHEMATICA[®]. It is found that, for the Lagrange family of elements of order more than two, the co-ordinates of optimal stress points slightly differ from those of the reduced Gauss integration points. Some of the serendipity family of elements have either none or only one optimal stress point at the element centre. Thus, when using higher-order elements in the p - or hp -version, it is more desirable to employ the Lagrange family of elements. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS optimal stress points; superconvergent patch recovery; hierarchical finite element

1 INTRODUCTION

The finite element method has been extensively used in the analysis and design of structural members where the maximum shear stress or the maximum principal stress plays a critical role. In the displacement-based finite element formulation, the stresses are not as accurately determined as the nodal displacements. Barlow¹ showed the existence of optimal points in an element where stresses are most accurate. He computed the locations of these points for first- and second-order bar, beam and plane elements and found these points to be coincident with the locations of reduced Gauss quadrature points. These optimal stress points have drawn more attention recently because of the superconvergent patch recovery^{2,3} (SPR) wherein nodal values of stresses are obtained by using the least squares fit to the stresses evaluated at the sampling points. Thus the SPR technique depends upon the precise location of sampling points. Based on Barlow's result for lower-order elements, the reduced Gauss integration points have been taken as the optimal stress points for higher-order elements.^{2,3} However, for higher-order elements, the locations of these two sets of points are slightly different, and are given herein for the Lagrange and serendipity families of elements.

2 BAR ELEMENTS

Our technique of finding the locations of optimal stress points is exactly the same as that employed by Barlow,¹ and is briefly sketched for the sake of completeness. Consider an n th-order bar element for which the displacement field in the local co-ordinate ξ is given by⁴

$$u_a = [1, \xi, \xi^2, \dots, \xi^n] \mathbf{a} = \mathbf{P}_n \mathbf{a} \quad (1)$$

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where $\mathbf{a}$ is the unknown coefficient vector. Substituting into (1) the local co-ordinates of nodes, we obtain the following relationship between the nodal displacements δ_a and the coefficient vector $\mathbf{a}$:

$$\delta_a = \mathbf{A}\mathbf{a} \quad (2)$$

Here $\mathbf{A}$ is the $(n+1) \times (n+1)$ matrix of nodal co-ordinates. Now, consider the displacement field

$$u_b = [1, \xi, \xi^2, \dots, \xi^{n+1}] \mathbf{b} = \mathbf{P}_{n+1} \mathbf{b} \quad (3)$$

containing a complete polynomial of degree $(n+1)$ in the same n th-order element with $n+1$ nodes. Substituting the local co-ordinates of $n+1$ nodes into (3), we conclude that

$$\delta_b = \mathbf{B}\mathbf{b} \quad (4)$$

where $\mathbf{B}$ is $(n+1) \times (n+2)$ matrix and δ_b is the nodal displacement vector for the n th-order element. Setting $\delta_a = \delta_b$ gives

$$\mathbf{a} = \mathbf{A}^{-1} \mathbf{B} \mathbf{b} \quad (5)$$

The first-order derivatives (or strains) of the displacement fields (1) and (3) may be written as

$$\frac{du_a}{dx} = \frac{d\xi}{dx} \frac{d\mathbf{P}_n}{d\xi} \mathbf{a}, \quad \frac{du_b}{dx} = \frac{d\xi}{dx} \frac{d\mathbf{P}_{n+1}}{d\xi} \mathbf{b} \quad (6)$$

where x is the global co-ordinate along the bar. The locations of optimal stress points are given by equating du_a/dx and du_b/dx . Since the Jacobian $d\xi/dx$ depends only upon the element geometry, it is the same in the two cases. Thus the locations of optimal stress points are given by

$$\left[\frac{d\mathbf{P}_n}{d\xi} \mathbf{A}^{-1} \mathbf{B} - \frac{d\mathbf{P}_{n+1}}{d\xi} \right] \mathbf{b} \equiv \mathbf{C} \mathbf{b} = 0 \quad (7)$$

Vectors $\mathbf{C}$ for elements of order 1–5 are listed in Table I. The locations of optimal stress points and also of reduced Gauss integration points for elements of order 1–10 are given in Table II. A

Table I. $\mathbf{C}$ vectors for a bar element of order 1–5, and locations of optimal stress points

n	$\mathbf{C}$	Optimal stress points
1	$\xi[-1, 2, -1]$	0
2	$(9\alpha/16)[-1, 3, -3, 1]$, where $\alpha = 1 - 3\xi^2$	$\pm \frac{1}{\sqrt{3}}$
3	$(8\xi\alpha/27)[1, -4, 6, -4, 1]$, where $\alpha = 5 - 9\xi^2$	$\pm \frac{\sqrt{5}}{3}, 0$
4	$(625\alpha/3072)[1, -5, 10, -10, 5, -1]$, where $\alpha = 1 - 15\xi^2 + 20\xi^4$	$\pm \frac{\sqrt{[3 \pm \sqrt{(29/5)}]}}{2\sqrt{2}}$
5	$(81\xi\alpha/25000)[-1, 6, -15, 20, -15, 6, -1]$, where $\alpha = 259 - 1750\xi^2 + 1875\xi^4$	$\pm \frac{\sqrt{(35 \pm 8\sqrt{7})}}{5\sqrt{3}}, 0$

MATHEMATICA[®] file⁵ to compute the locations of optimal stress points in an n th-order element is given in the Appendix. The results in Table II indicate that the locations of optimal stress points coincide with those of reduced Gauss points for first- and second-order elements but differ for higher-order elements.

3 PLANE ELEMENTS

We first consider the Lagrange family of elements. The displacement field $(\mathbf{u}_a, \mathbf{v}_a)$ for a 4-noded quadrilateral element can be written as

$$[\mathbf{u}_a, \mathbf{v}_a] = [1, \xi, \eta, \xi\eta][\mathbf{a}_u, \mathbf{a}_v] \quad (8)$$

Table II. Locations of optimal stress points and reduced integration points for a bar element of order 1–10

n	Optimal stress point					Reduced integration point				
1	0.000	000	000	000	000	0.000	000	000	000	000
2	± 0.577	350	269	189	626	± 0.577	350	269	189	626
3	± 0.745 0.000	355 000	992 000	499 000	923 000	± 0.774 0.000	596 000	669 000	241 000	483 000
4	± 0.822 ± 0.271	216 956	434 127	079 951	134 169	± 0.861 ± 0.339	136 981	311 043	594 584	953 856
5	± 0.865 ± 0.429 0.000	378 480 000	610 143 000	694 297 000	123 037 000	± 0.906 ± 0.538 0.000	179 469 000	845 310 000	938 105 000	664 683 000
6	± 0.892 ± 0.531 ± 0.176	679 927 884	195 345 889	264 786 851	306 612 544	± 0.932 ± 0.661 ± 0.238	469 209 619	514 386 186	203 466 083	152 265 197
7	± 0.911 ± 0.603 ± 0.301 0.000	349 734 098 000	929 362 673 000	075 960 974 000	623 412 045 000	± 0.949 ± 0.741 ± 0.405 0.000	107 531 845 000	912 185 151 000	342 599 377 000	759 394 397 000
8	± 0.924 ± 0.656 ± 0.393 ± 0.130	844 780 069 885	999 435 322 552	189 752 232 834	023 793 124 301	± 0.960 ± 0.796 ± 0.525 ± 0.183	289 666 532 434	856 477 409 642	497 413 916 495	536 627 329 650
9	± 0.935 ± 0.697 ± 0.463 ± 0.231 0.000	010 524 873 660 000	096 300 825 687 000	431 046 561 225 000	202 188 129 311 000	± 0.968 ± 0.836 ± 0.613 ± 0.324 0.000	160 031 371 253 000	239 107 432 423 000	507 326 700 403 000	626 636 590 809 000
10	± 0.942 ± 0.729 ± 0.520 ± 0.311 ± 0.103	915 774 041 636 821	732 318 425 635 422	607 838 499 682 787	524 830 254 542 923	± 0.973 ± 0.865 ± 0.679 ± 0.433 ± 0.148	906 063 409 395 874	528 366 568 394 338	517 688 299 129 981	172 985 024 247 631

where $\mathbf{a}_u$ and $\mathbf{a}_v$ are unknown coefficients and (ξ, η) are the local co-ordinates of a point. The displacement fields $(\mathbf{u}_b, \mathbf{v}_b)$ containing polynomial terms up to a complete quadratic are

$$[\mathbf{u}_b, \mathbf{v}_b] = [1, \xi, \eta, \xi\eta, \xi^2, \eta^2][\mathbf{b}_u, \mathbf{b}_v] \quad (9)$$

where $\mathbf{b}_u$ and $\mathbf{b}_v$ are the unknown coefficients. The spatial derivatives for these two displacement fields are given by

$$\begin{bmatrix} \frac{\partial \mathbf{u}_a}{\partial \xi} & \frac{\partial \mathbf{v}_a}{\partial \xi} \\ \frac{\partial \mathbf{u}_a}{\partial \eta} & \frac{\partial \mathbf{v}_a}{\partial \eta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \eta \\ 0 & 0 & 1 & \xi \end{bmatrix} [\mathbf{a}_u, \mathbf{a}_v] \quad (10)$$

$$\begin{bmatrix} \frac{\partial \mathbf{u}_b}{\partial \xi} & \frac{\partial \mathbf{v}_b}{\partial \xi} \\ \frac{\partial \mathbf{u}_b}{\partial \eta} & \frac{\partial \mathbf{v}_b}{\partial \eta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \eta & 2\xi & 0 \\ 0 & 0 & 1 & \xi & 0 & 2\eta \end{bmatrix} [\mathbf{b}_u, \mathbf{b}_v] \quad (11)$$

By substituting for nodal co-ordinates into (8) and (9) we compute nodal displacements. Setting these nodal displacements equal to each other, we obtain a relation between $(\mathbf{a}_u, \mathbf{a}_v)$ and $(\mathbf{b}_u, \mathbf{b}_v)$. The locations of optimal stress points are obtained by equating the spatial derivatives of $\mathbf{u}$ and $\mathbf{v}$ in (10) and (11), and substituting for $(\mathbf{a}_u, \mathbf{a}_v)$ in terms of $(\mathbf{b}_u, \mathbf{b}_v)$. The result is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -2\xi & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\eta \end{bmatrix} [\mathbf{b}_u, \mathbf{b}_v] = \mathbf{0} \quad (12)$$

Equation (12) holds for every choice of $\mathbf{b}_u$ and $\mathbf{b}_v$ if and only if $\xi = \eta = 0$. Thus the location of the optimal stress point coincides with that of the reduced Gauss integration point. The optimal stress points for quadratic and cubic Lagrange elements are similarly found and the pertinent equations are given in Table III. It is clear that their local co-ordinates coincide with those of the one-dimensional element given in Table II.

For a quadratic serendipity element, Barlow¹ found the optimal stress points to be $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ in local co-ordinates, and are coincident with those of the Lagrange quadratic element. The locations of optimal stress points for the cubic, quartic and fifth-order serendipity elements are given in Table IV. It should be noted that both cubic and 5th-order elements have only one optimal stress point at the centre of the element, and the quartic element has no optimal stress point within the element. Furthermore, even though the cubic and fifth-order elements have an optimal stress point at the centre, it is difficult to accurately evaluate the stress components elsewhere since only one sampling point is available for the superconvergent patch recovery.

4 A NUMERICAL EXAMPLE

Consider a square cantilever plate subjected to uniform pressure on the top surface, and let the plate be discretized into four elements as shown in Figure 1. Our intent is to compare errors in

Table III. **C** matrices and the optimal stress points for Lagrangian quadratic and cubic elements

	Quadratic element	Cubic element
$\mathbf{P}_n$	$[1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^2\eta, \xi\eta^2, \xi^2\eta^2]$	$[1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \xi^3\eta, \xi^2\eta^2, \xi\eta^3, \xi^3\eta^2, \xi^2\eta^3, \xi^3\eta^3]$
$\mathbf{P}_{n+1}$	$[1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \xi^2\eta^2]$	$[1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \xi^4, \xi^3\eta, \xi^2\eta^2, \xi\eta^3, \eta^4, \xi^3\eta^2, \xi^2\eta^3, \xi^3\eta^3]$
C	$\begin{bmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 1 - 3\xi^2, 0 \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 - 3\eta^2 \end{bmatrix}$	$\begin{bmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{20}{9}\xi - 4\xi^3, 0 \\ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{20}{9}\eta - 4\eta^3 \end{bmatrix}$
Solutions of $\mathbf{Cb} = \mathbf{0}$	$\left(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}} \right)$	$\left(\pm \frac{\sqrt{5}}{3}, \pm \frac{\sqrt{5}}{3} \right)$ or $(0, 0)$

Table IV. **C** matrices and the optimal stress points for serendipity cubic, quartic and fifth order elements

	Cubic element	Quartic element
$\mathbf{P}_n$	$[1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \xi^3\eta, \xi\eta^3]$	$[1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \xi^4, \xi^3\eta, \xi^2\eta^2, \xi\eta^3, \eta^4, \xi^4\eta, \xi\eta^4]$
$\mathbf{P}_{n+1}$	$[1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \xi^4, \xi^3\eta, \xi^2\eta^2, \xi\eta^3, \eta^4]$	$[1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \xi^4, \xi^3\eta, \xi^2\eta^2, \xi\eta^3, \eta^4, \xi^5, \xi^4\eta, \xi^3\eta^2, \xi^2\eta^3, \xi\eta^4, \eta^5]$
C	$\begin{bmatrix} \{\mathbf{0}\}, \frac{20}{9}\xi - \xi^3, \{\mathbf{0}\}, 0, 0, 2\xi(1 - \eta^2), 0, 0, 0 \\ \{\mathbf{0}\}, 0, \{\mathbf{0}\}, 0, 0, 2(1 - \xi^2)\eta, 0, 0, \frac{20}{9}\eta - \eta^3 \end{bmatrix}$	$\begin{bmatrix} \{\mathbf{0}\}, (-1 + 15\xi^2 - 20\xi^4)/4, \{\mathbf{0}\}, 0, 0, 0, (1 - 3\xi^2)(-1 + \eta^2), 0, 0, 2\xi\eta(1 - \eta^2), 0, 0, 0 \\ \{\mathbf{0}\}, 0, \{\mathbf{0}\}, 0, 0, 0, 2\xi\eta(1 - \xi^2), 0, 0, (-1 + \xi^2)(1 - 3\eta^2), 0, 0, (-1 + 15\eta^2 - 20\eta^4)/4 \end{bmatrix}$
Solutions of $\mathbf{Cb} = \mathbf{0}$	$(0, 0)$	—
5th-order element		
$\mathbf{P}_n$	$[1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \xi^4, \xi^3\eta, \xi^2\eta^2, \xi\eta^3, \eta^4, \xi^5, \xi^4\eta, \xi^3\eta^2, \xi^2\eta^3, \xi\eta^4, \eta^5, \xi^5\eta, \xi^4\eta^2, \xi^3\eta^3, \xi\eta^5]$	
$\mathbf{P}_{n+1}$	$[1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3, \xi^4, \xi^3\eta, \xi^2\eta^2, \xi\eta^3, \eta^4, \xi^5, \xi^4\eta, \xi^3\eta^2, \xi^2\eta^3, \xi\eta^4, \eta^5, \xi^6, \xi^5\eta, \xi^4\eta^2, \xi^3\eta^3, \xi^2\eta^4, \xi\eta^5, \eta^6]$	
C	$\begin{bmatrix} \{\mathbf{0}\}, 2\xi(-259 + 1750\xi^2 - 1875\xi^4)/625, \{\mathbf{0}\}, 0, 0, 0, 0, 4\xi(17 - 25\xi^2)(\eta^2 - 1)/25, \{\mathbf{0}\}, 2\xi(9 - 25\eta^2)(\eta^2 - 1)/25, 0, 0, 0 \\ \{\mathbf{0}\}, \{\mathbf{0}\}, 0, 0, 0, 0, 0, 2\eta(9 - 25\xi^2)(\xi^2 - 1)/25, \{\mathbf{0}\}, 4\eta(-17 + 25\eta^2)(\xi^2 - 1)/25, 0, 0, 2\eta(-259 + 1750\eta^2 - 1875\eta^4)/625 \end{bmatrix}$	
Solutions of $\mathbf{Cb} = \mathbf{0}$	$(0, 0)$	

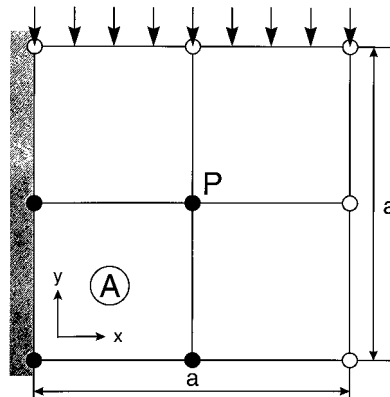


Figure 1. A cantilever plate

stresses at the optimal stress points and the reduced Gauss integration points. The error e in the stresses is defined as the difference between their values for the n th and $(n + 1)$ th-order elements and normalized by stresses for the $(n + 1)$ th-order element. Figures 2 and 3 compare the errors in stresses at the optimal stress points and the reduced integration points in element A when the plate is modelled with Lagrangian cubic and quartic hierarchical elements. It is evident that the errors at the optimal stress points are lower than those at the reduced Gauss integration points. Thus, it is expected that stresses recovered by the SPR from their values at the optimal stress points will have better accuracy than those from their values at the reduced integration points.

Figure 4 exhibits the convergence of the shear stress at point P obtained by using the superconvergent patch recovery. Whereas the optimal stress points were employed as sampling points for Lagrangian hierarchical elements, reduced Gauss integration points were used for serendipity elements. The shear stress is normalized with that obtained by using a mesh with 100×100 4-noded quadrilateral elements. It is clear from the results plotted in Figure 4 that the shear stress obtained from the Lagrange family of elements converges as the order of the element is increased but that derived from the serendipity family of elements appears to diverge. This is because a serendipity element of order higher than three either has only one or no sampling point within the element.

5 CONCLUSIONS

We have computed locations of optimal stress points for bar and quadrilateral elements of various orders. It is found that these locations may differ from those of the reduced integration points. The serendipity elements of order 3 and higher have either one or no optimal stress point within the element. Thus the Lagrange family of hierarchical elements should be used in p - or hp -versions to get better accuracy of the stresses.

The optimal stress points for the hierarchical and isoparametric Lagrange elements are the same since both elements have the same polynomial terms even though they have different shape functions.

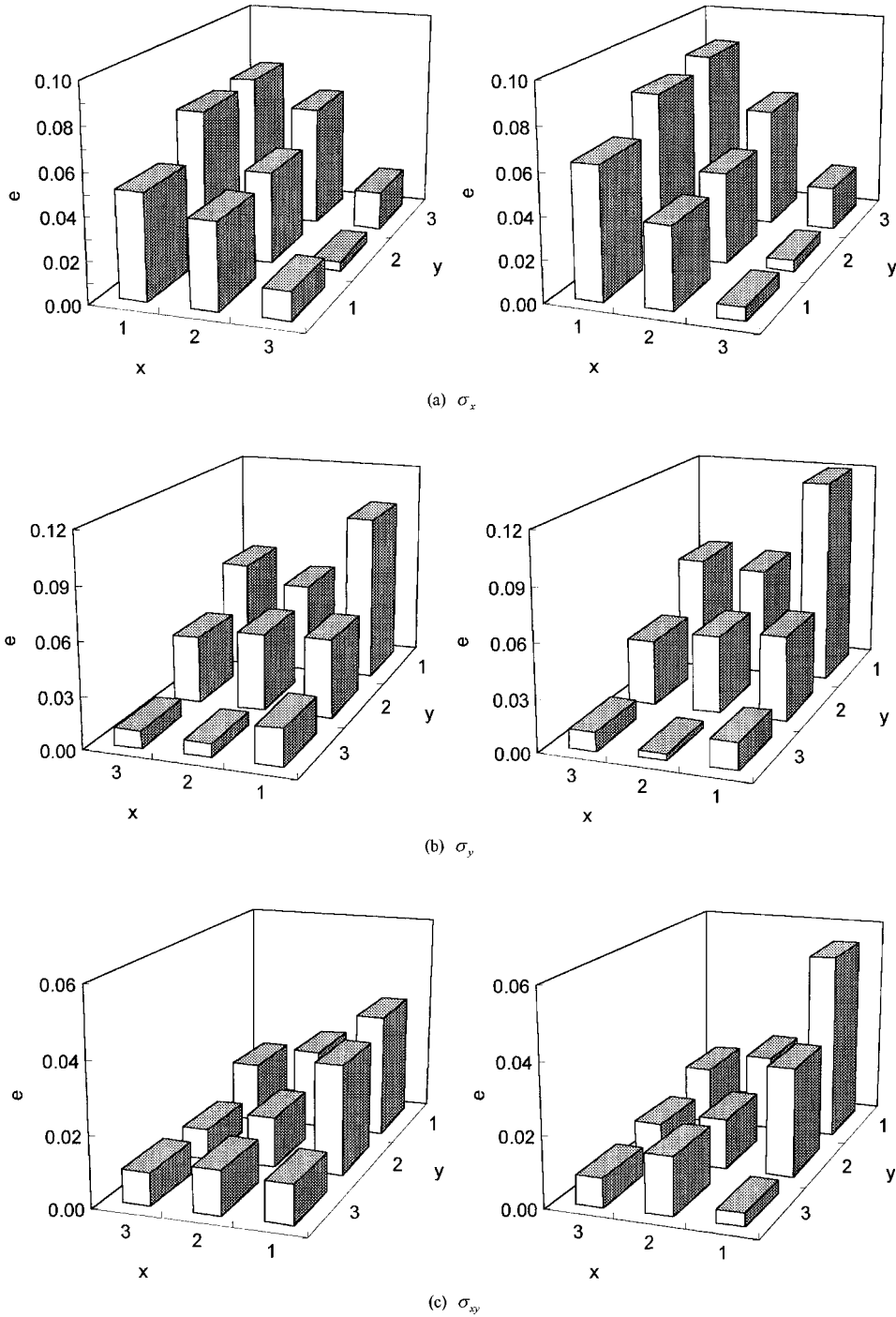


Figure 2. Errors in stresses in a Lagrangian cubic element (left figures for the optimal stress points and right figures for the reduced Gauss integration points)

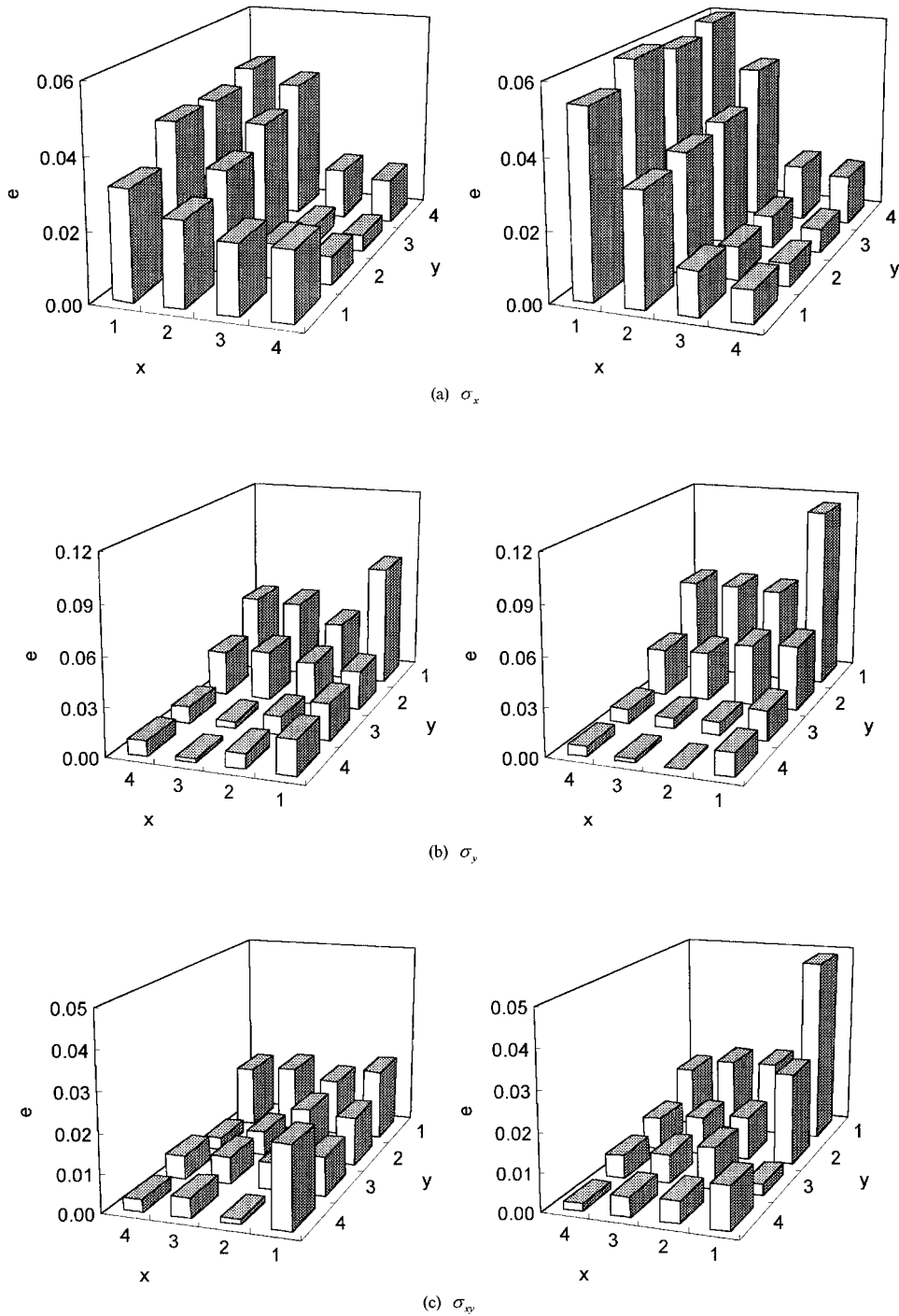


Figure 3. Errors in stresses in a Lagrangian quartic element (left figures for the optimal stress points and right figures for the reduced Gauss integration points)

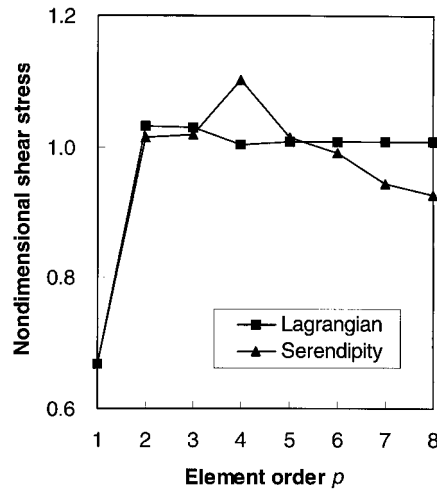


Figure 4. Convergence of the shear stress at point A obtained by the superconvergent patch recovery in Lagrangian and serendipity finite elements

APPENDIX: MATHEMATICAL DATA PROGRAM

Bar element

```
ord1 = 3;
ord2 = ord1 + 1;
sh1 = Range[1,ord1 + 1];
sh2 = Range[1,ord2 + 1];
Do[ Part[sh1,i]s^(i - 1),{i,ord1 + 1}];
Do[ Part[sh2,i]s^(i - 1),{i,ord2 + 1}];
A = IdentityMatrix[ord1 + 1];
B = Table[1, {i,ord1 + 1},{j,ord2 + 1}];
tmp = - 1;
del = 2/ord1;
Do[ tmpva = sh1/.s → tmp;
    tmpvb = sh2/.s → tmp;
    Do[ Part[A,i,j] = Part[tmpva,j],{j,ord1 + 1}];
    Do[ Part[B,i,j] = Part[tmpvb,j],{j,ord2 + 1}];
    tmp = tmp + del,{i,ord1 + 1}];
IAB = Inverse[A].B;
dsh1 = D[sh1,s];
dsh2 = D[sh2,s];
Eq = Factor[Simplify[dsh1.IAB - dsh2]]
Factor[Simplify[Solve[Eq == 0,s]]]
N[%,16]
```

Lagrangian element

```

ord1 = 3;
ord2 = ord1 + 1;
sh1 = Range[1,(ord1 + 1)*(ord1 + 1)];
sh2 = Range[1,(ord1 + 1)*(ord1 + 1) + 2];
cv = Range[1,ord1 + 1];
isum = 0;
Do[ Do[ isum = isum + 1;
        Part[sh1,isum] = s^(i - 1)*t^(j - 1);
        Part[sh2,isum] = s^(i - 1)*t^(j - 1),{i,ord1 + 1}],
{j,ord1 + 1}]
Part[sh2,(ord1 + 1)*(ord1 + 1) + 1] = s^ord2;
Part[sh2,(ord1 + 1)*(ord1 + 1) + 2] = t^ord2;
A = IdentityMatrix[(ord1 + 1)^2];
B = Table[1, {i,(ord1 + 1)^2},{j,(ord1 + 1)^2 + 2}];
del = 2/ord1;
tmp = -1;
Do[ Part[cv,i] = tmp;
    tmp = tmp + del,{i,ord1 + 1}];
jj = 0;
Do[ Do[ jj = jj + 1;
        tmpva = sh1/.{s → Part[cv,i],t → Part[cv,j]};
        tmpvb = sh2/.{s → Part[cv,i],t → Part[cv,j]};
        Do[Part[A,jj,ii] = Part[tmpva,ii],{ii,1,(ord1 + 1)^2}];
        Do[Part[B,jj,ii] = Part[tmpvb,ii],{ii,1,(ord1 + 1)^2 + 2}],{i,ord1 + 1}],
{j,ord1 + 1}];
IAB = Inverse[A].B;
dsh1 = {D[sh1,s],D[sh1,t]};
dsh2 = {D[sh2,s],D[sh2,t]};
Eq = Simplify[dsh1.IAB - dsh2]
Simplify[Solve[Eq == 0, {s,t}]]
N[%,16]

```

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