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ROTATIONAL DEPENDENCE OF THE SUPERCONVERGENT PATCH RECOVERY AND ITS REMEDY FOR 4-NODE ISOPARAMETRIC QUADRILATERAL ELEMENTS

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SUMMARY

The superconvergent patch recovery (SPR) with bilinear interpolation functions usually gives good values of recovered stresses in an element patch. However, when 4-node quadrilateral elements meeting at a node are rigidly rotated with the essential and natural boundary conditions unchanged, the recovered stresses obtained by the SPR change and depend upon the local rotation of the patch. This can be remedied either by including higher-order terms in the polynomials for the assumed stress distribution in an element patch, or by using linear interpolation functions, which gives inferior accuracy of the recovered stresses near the boundaries of the domain. Additional sampling points are suggested to compute the higher-order terms.
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KEY WORDS Zienkiewicz-Zhu's error; quadrilateral elements; nodal stresses; plane elastic problems

1 INTRODUCTION

The superconvergent patch recovery[1–3] (SPR) technique, proposed by Zienkiewicz and Zhu, improves the accuracy of nodal stresses in linear and higher-order finite elements. These recovered stresses when used in Zienkiewicz–Zhu's error estimate[2,4,5] also improve the accuracy of the error estimate.[2] In the SPR, an element patch consists of all elements sharing a node. A polynomial is assumed to represent the distribution of a stress component in the element patch, and the unknown parameters in the polynomial are calculated by the least-squares fit to the values of the stress component at sampling points in the patch. Thus, the accuracy of recovered stresses usually depends on the degree of the assumed polynomial. For example, for a 'regular mesh' discretized with bilinear isoparametric quadrilateral elements, an element patch usually consists of four elements, the assumed polynomial is a bilinear form and each element has one sampling point coincident with the optimal stress point.

However, as the local mesh is rotated without changing the boundary conditions and the applied loading, global stresses obtained by the SPR change significantly and their calculation becomes impossible for an angle of rotation of 45 degrees. This is because the number of terms in the assumed polynomial for nodal stresses is not enough to fully represent the local rotation of the mesh. This occurs both in a regular and an irregular mesh.

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This phenomenon can be avoided either by approximating the stress distribution in an element patch with a polynomial of order high enough to represent the rotation, or low enough to exclude the rotation. Additional sampling points are suggested to determine unknown parameters associated with the higher-order polynomial.

2 THE SPR INDEPENDENT OF MESH ROTATION

Figure 1 shows a typical element patch and the associated sampling points in a regular mesh discretized with bilinear quadrilateral elements. Each component σ_p^* of the stress field in the patch is expressed as

$$\sigma_p^* = \mathrm{Pa} \tag{1}$$

where

$$\mathrm{P} = [1 \quad x \quad y \quad xy], \ \mathrm{a} = [a_1 \quad a_2 \quad a_3 \quad a_4]^{\mathrm{T}} \tag{2}$$

For simplicity, we write (1) as

$$\sigma_p^* = f(1, x, y, xy) \tag{3}$$

Then, as the finite element mesh is locally rotated through an angle θ without any change of the loading and boundary conditions, the stress component in the rotated co-ordinates becomes

$$\sigma_p^* = \tilde{f}(1, x', y', x'y', x'^2, y'^2) \tag{4}$$

because $x' = x \cos\theta + y \sin\theta$ and $y' = -x \sin\theta + y \cos\theta$. If one uses the locally rotated mesh and computes recovered stresses by using (1), then these would differ significantly from those

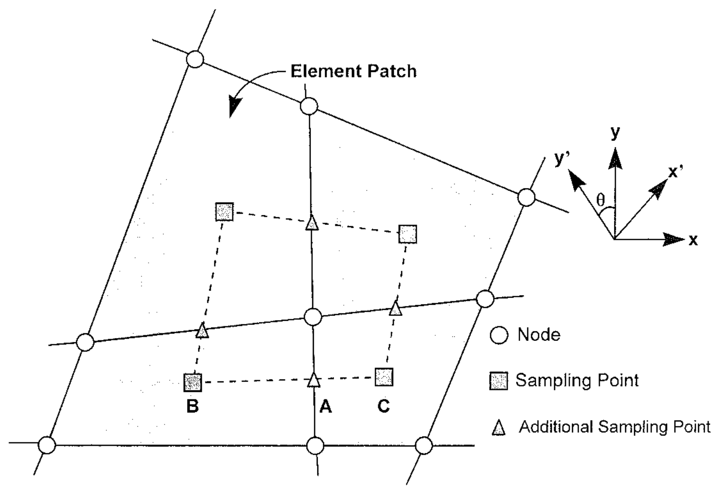

Figure 1. Element patch and sampling points

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obtained by using a regular mesh because of the absence of terms involving x^2 and y^2. In general, then, the polynomial P in (1) should be a complete polynomial of degree 2; namely,

$$\mathrm{P} = [1 \quad x \quad y \quad xy \quad x^2 \quad y^2] \tag{5}$$

Better accuracy of recovered stresses is expected in an arbitrary mesh because of added higher-order terms. However, the polynomial (5) has six terms. Thus six unknowns are to be determined, and we need at least two additional sampling points. However, four additional sampling points, shown in Figure 1, are employed for symmetry in this paper. Each additional sampling point is located on the common side between two adjoining elements where the line joining two interior sampling points intersects it. Stress values at these additional sampling points can be easily obtained by applying the SPR in one dimension. For example, in order to obtain stress values at the additional sampling point A in Figure 1, the SPR technique is applied to points B and C; it is equivalent to interpolating stresses at A from those at B and C.

Alternatively, one could omit the term xy in the polynomial P; however, the recovered stresses lose some accuracy near the boundaries of the body.

3. NUMERICAL EXAMPLES

We illustrate for two example problems the effect of employing the conventional and additional sampling points in the SPR, and the lower-order polynomial in P. In each case, Young's modulus and Poisson's ratio are set to 10 and 0·3, respectively. Figure 2 shows a finite element mesh for a cantilever beam with height/length equal to 0·2. In Figure 3, we have plotted contours of the

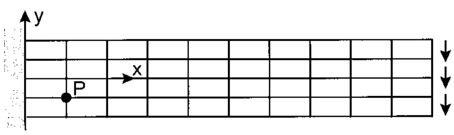

Figure 2. Cantilever beam subjected to uniform pressure at the end face

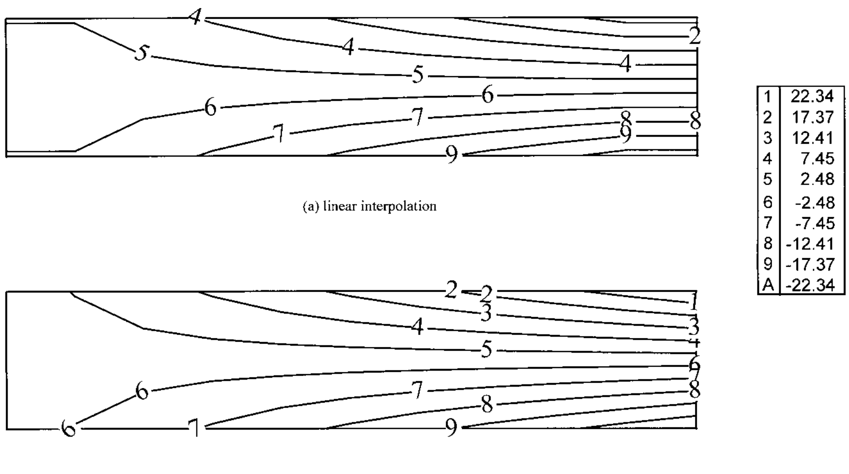

Figure 3. Contours of the bending stress obtained by the SPR technique with linear and bilinear interpolation functions

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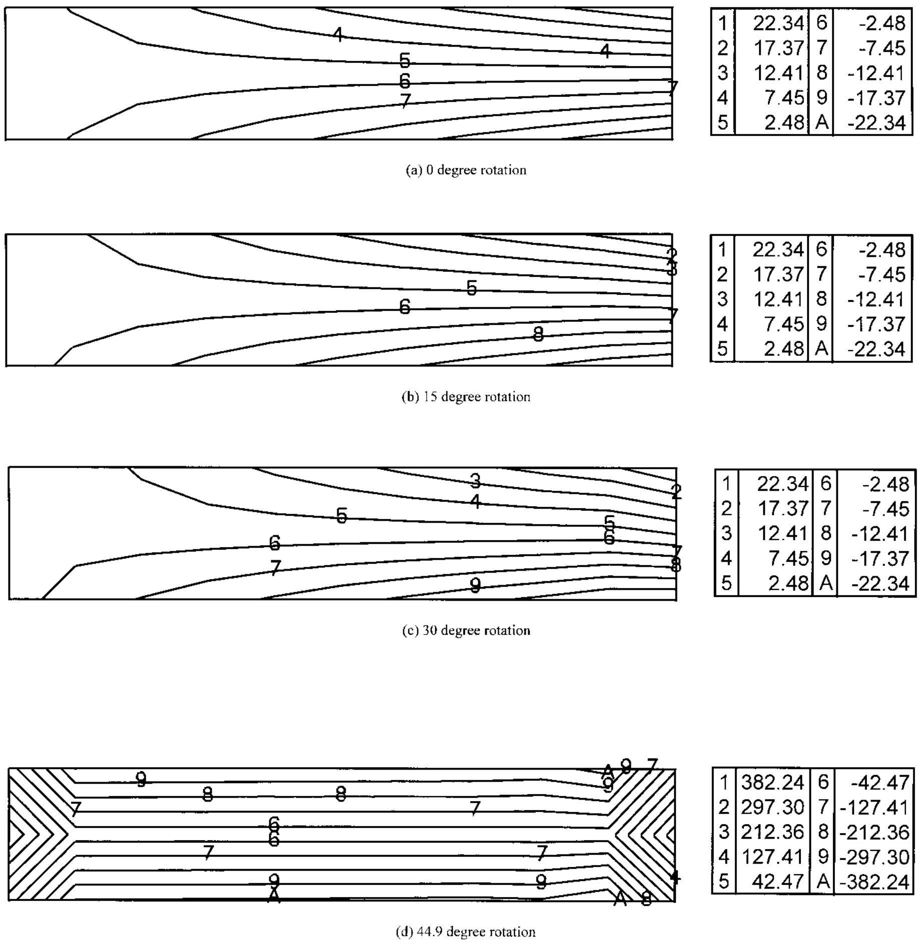

Figure 4. Contours of the bending stress obtained by the SPR with only the conventional sampling points as the mesh is locally rotated

bending stress recovered by using the linear and the bilinear interpolation functions in the SPR technique. Even though there is no rotational dependence exhibited by the SPR technique with a linear interpolation function, the accuracy of the recovered stresses near the two edges of the beam is less than that obtained with the bilinear interpolation function. Figure 4 depicts contours of the bending stress obtained by the SPR with only the conventional sampling points for different angles of rotation of the mesh. For the regular mesh of Figure 2, as expected, the magnitude of the bending stress is symmetrical about the horizontal centroidal axis, but this symmetry disappears when the mesh is rotated through 15, 30 and 44·9 degrees. However, when additional sampling points are also considered the contours of the bending stress are independent of the rotation of the mesh. Figure 5 shows the bending stresses at the point P obtained by the SPR with conventional sampling points only and with conventional and additional sampling points; these are normalized by the bending stress at point P for no rotation of the mesh. It is clear that the bending stress obtained with the conventional sampling points strongly depends on

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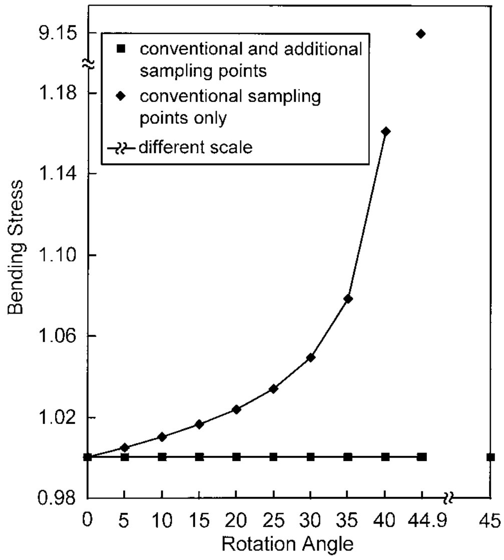

Figure 5. Bending stress at point P vs. the angle of rotation of the mesh

the rotation of the mesh. The slope of the curve rapidly increases when the angle of rotation is close to 45°, and we cannot calculate the bending stress for an angle of rotation of 45° because then the matrix P^TP is singular. However, the stresses obtained with the conventional and additional sampling points do not depend on the angle of rotation of the mesh.

Figure 6 exhibits a finite element model of a circular plate with a circular hole at its centre and subjected to a uniform pressure on the inner surface of the hole. For this problem, the solution should be independent of the angular position of a point. The distribution of the maximum principal stress recovered by the SPR technique with linear and bilinear interpolation functions is plotted in Figure 7. As for the beam problem (cf. Figure 3), the SPR technique with the bilinear interpolation functions gives more accurate values of the principal stresses near the boundaries. Figure 8 evinces, for different angles of rotation of the mesh, the distribution of the maximum principal stress obtained by the SPR technique with conventional and additional sampling points. The stress obtained with the conventional sampling points shows non-uniform

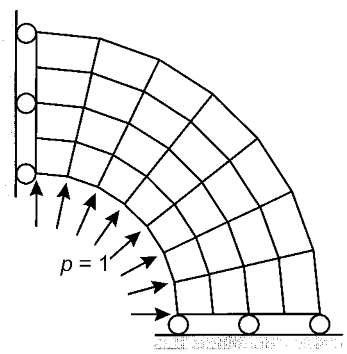

Figure 6. Finite element model and boundary condition for an annular circular plate

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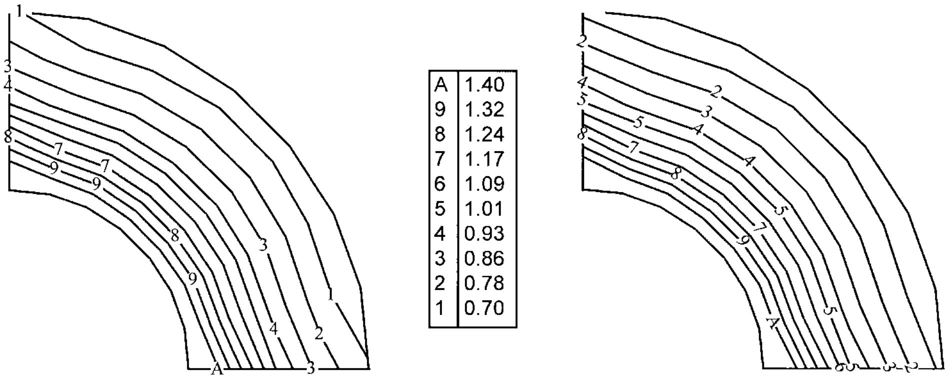
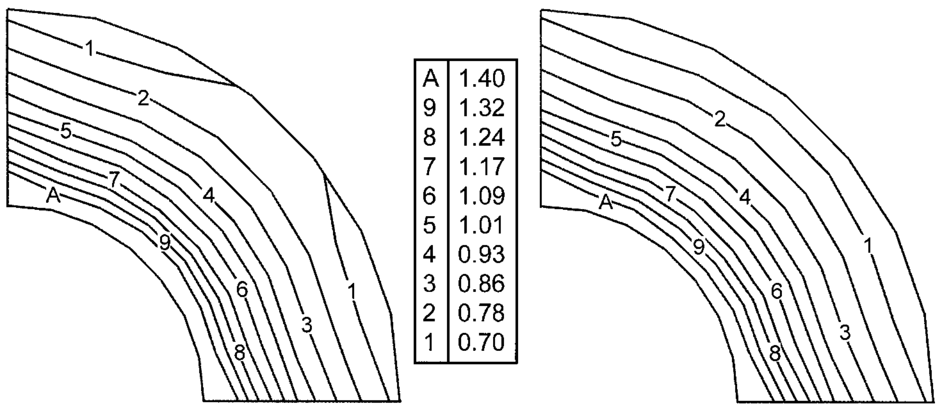


(a) linear interpolation (b) bilinear interpolation

Figure 7. Contours of the maximum principal stress in the circular plate obtained by the SPR technique with linear and bilinear interpolation functions



(a) only conventional sampling points (b) conventional and additional sampling points

Figure 8. Contours of the maximum principal stress in the circular plate obtained by the SPR with conventional, and with conventional and additional sampling points

distribution along the circumference of the outer boundary, but that obtained by using the conventional and additional sampling points in the SPR is uniformly distributed in the circumferential direction. Figure 9 shows the distribution of the Zienkiewicz–Zhu error energy norm.[2,4,5] When using only the conventional sampling points in the SPR, the error distribution is unreasonable, since the error in the shaded element near the middle of the free boundary is larger than that in the shaded element near the hole. The inaccuracy of the recovered stress shown in Figure 8 yields this incorrect estimate of Zienkiewicz–Zhu's error. However, by using the conventional and additional sampling points, the computed Zienkiewicz–Zhu errors vary only in the radial direction as expected.

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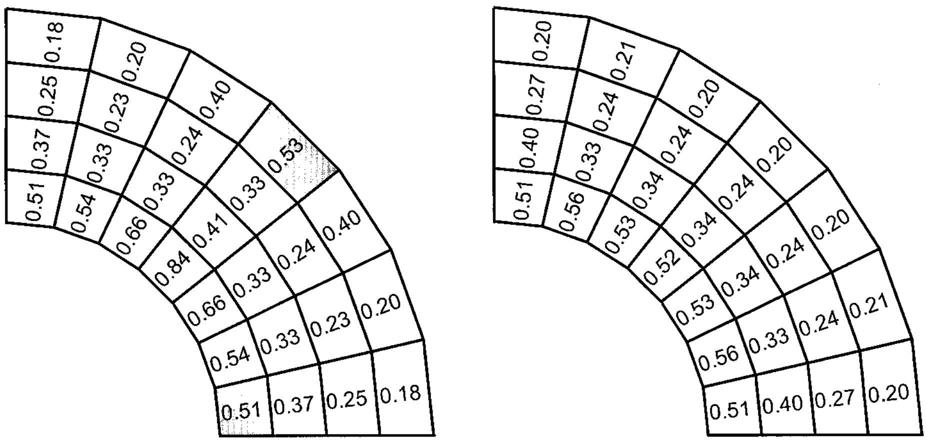

(a) conventional sampling points (b) conventional and additional sampling points

Figure 9. Distribution of Zienkiewicz–Zhu's error energy norm in the circular plate

4 CONCLUDING REMARKS

We have shown for two commonly encountered example problems that the SPR technique employing bilinear interpolation functions with conventional sampling points for bilinear quadrilateral elements gives an incorrect distribution of stresses as the mesh is locally rotated. However, when either a linear polynomial for recovered stresses or a higher-order polynomial expression for the recovered stresses and additional sampling points suggested herein are used, the recovered stresses are independent of the orientation of the mesh with respect to the global axes. The proposed modified SPR technique can be easily extended to the 8-noded brick element.

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