



Material tailoring and analysis of functionally graded isotropic and incompressible linear elastic hollow cylinders

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ABSTRACT

We use the Airy stress function to derive exact solutions for plane strain deformations of a functionally graded (FG) hollow cylinder with the inner and the outer surfaces subjected to different boundary conditions, and the cylinder composed of an isotropic and incompressible linear elastic material. For the shear modulus given by either a power law or an exponential function of the radius r , we derive explicit expressions for stresses, the hydrostatic pressure and displacements. Conversely, we find the variation with r of the shear modulus for a linear combination of the radial and the hoop stresses to have a pre-assigned variation in the cylinder; this inverse problem is usually called material tailoring. The shear modulus found while solving the inverse problem must be positive everywhere. Results for a few problems are computed and presented graphically. It seems that the Airy stress function approach is used here for the first time to analyze two-dimensional problems for incompressible materials. When studying axisymmetric deformations of an FG cylinder, it is found that for the hoop stress to be uniform through the cylinder thickness the shear modulus must be proportional to the radial coordinate r as found earlier by Batra [Batra RC. Optimal design of functionally graded incompressible linear elastic cylinders and spheres. *AIAA* 2008;46(8):2005–7.] and for the maximum in-plane shear stress to be constant the shear modulus must vary as r^2 . The expression for the maximum in-plane shear stress in terms of pressures and the radii of the inner and the outer surfaces of the cylinder is a universal result valid for all materials for which the shear modulus is proportional to r^2 . For a hollow cylinder fixed on the inner surface and subjected to tangential tractions on the outer surface (or vice versa) the through-the-thickness in-plane shear stress distribution is also universal and is determined by surface tractions and the outer radius of the cylinder; it is independent of the spatial variation of the shear modulus.

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1. Introduction

With the increase in the use of rubberlike materials in various engineering and biological applications, and the fact that the response of an incompressible material may be different from that of a compressible material, research on the analysis and design of components composed of isotropic and incompressible linear elastic materials is gaining importance. Whereas only isochoric (volume preserving) deformations are admissible in incompressible materials, a compressible material can undergo both isochoric and non-isochoric deformations. Rubberlike materials are generally assumed to be incompressible. Because of the incompressibility constraint, the constitutive equation involves a hydrostatic pressure that cannot be determined from the deformation field but is found by solving the boundary-value problem. The equation

corresponding to the incompressibility constraint and three equations expressing the balance of linear momentum are solved for the three components of displacements and the hydrostatic pressure at a point. However, the pressure field can be determined uniquely only if normal tractions are prescribed on a part of the boundary. In general, the solution of a boundary-value problem for an incompressible material cannot be obtained from that of the corresponding problem for a compressible material by setting Poisson's ratio equal to 0.5. For isotropic, unconstrained, homogeneous and linear elastic materials, the solution for a plane stress problem can be obtained from that for a plane strain problem by modifying Young's modulus and Poisson's ratio. However, such is not the case for incompressible materials.

Inhomogeneities may be introduced in rubberlike materials either during vulcanization or uneven interaction with thermal, radiative and oxidative environments [1]. Here we consider only those inhomogeneous rubberlike materials for which material properties vary continuously in one or more directions, and call

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them Functionally Graded Incompressible Materials (FGIMs). An advantage of FGIMs is that the material property can be tailored to optimize the performance of a structure.

For a given spatial variation of material properties, one can analyze initial-boundary-value problems and delineate points where maximum stresses and deflections occur, and also find frequencies of structures. Alternatively, one can find the spatial variation of material properties so as to optimize a suitable combination of stresses (or another design variable). We study these two classes of problems for plane strain deformations of a hollow cylinder. Below we briefly review the literature on FG cylinders.

Horgan and Chan [2] analyzed two-dimensional (2D; plane stress/strain) deformations by assuming the material to be isotropic, compressible and linear elastic with Young's modulus varying only in the radial direction by a power law relation but keeping Poisson's ratio constant; Li and Peng [3] have recently extended this work to include spatial variation of Poisson's ratio. Jabbari et al. [4] used the method of separation of variables and the Fourier series to analyze 2D steady-state thermoelastic deformations of a hollow thick cylinder with material properties, except Poisson's ratio, depending on the radius by a power law function. Shao and Ma [5] scrutinized thermo-mechanical deformations of FG hollow circular cylinders subjected to mechanical loads and linearly increasing temperature on the boundary by employing the Laplace transform technique, assuming the solution of the resulting ordinary differential equations in the form of a series, and taking the thermo-mechanical properties to be temperature independent and varying continuously in the radial direction only.

We note that Lechnitskii's book [6] has solutions for several problems involving inhomogeneous linear elastic materials. One could also divide the thickness of an FG cylinder into several layers, regard material properties in each layer as uniform, and use the approach outlined in Timoshenko and Goodier's book [7] for composite cylinders. With an increase in the number of layers, the solution for the layered cylinder will approach that for the FG cylinder; Pan and Roy [8], and Liew et al. [9] followed this approach to analyze deformations of a cylinder composed of an FG linear elastic compressible material. By assuming that all elastic constants are power law functions of the radius with the same exponent, Tarn [10], and Tarn and Chang [11] have provided exact solutions for FG anisotropic cylinders subjected to thermal and mechanical loads. Oral and Anlas [12] expressed governing equations for an inhomogeneous cylindrical anisotropic body in terms of stress potentials, and provided closed-form expressions for the potentials and stress distribution when Young's modulus is a power law function of the radius and Poisson's ratio is constant. Obata and Noda [13] found steady-state thermal stresses in FG hollow cylinders and spheres, and Kim and Noda [14] used the Green function to solve the corresponding transient problem. Most of these works have considered a known power law variation of the elastic modulus.

The literature on FGIMs is limited. Batra [15] studied numerically, with the finite element method, axisymmetric deformations of a cylinder made of a Mooney–Rivlin material with two material parameters varying smoothly in the radial direction, and compared his results with the analytical solution of the problem; he did not call the inhomogeneous material an FGIM. Bilgili [16] investigated axial shearing deformations of a homogeneous and isotropic hollow rubber tube under isothermal and non-isothermal conditions. Bilgili [17] presented closed-form analytical solutions to rectilinear shearing of rubber slabs, and found that the spatial variation of the shear modulus can induce highly localized stresses in them. Batra [18] used the principle of virtual work to derive a higher-order shear and normal deformable theory for a plate comprised of an FGIM. Batra [19] derived closed-form solutions for axisymmetric plane strain deformations of a FG hollow circular cylinder and a hollow sphere loaded on inner and outer surfaces by uniform hydrostatic

pressures with the shear modulus an arbitrary function of the radius. He found that the optimal hoop or the circumferential stress in a cylinder and a sphere is a constant and occurs for the linear variation in the radial direction of the shear modulus. Batra and Iaccarino [20] obtained exact solutions for axisymmetric plane strain deformations of a FG circular cylinder composed of an isotropic and incompressible second-order elastic material with the two moduli varying only in the radial direction. Batra and Bahrami [21] studied axisymmetric deformations of a circular cylinder composed of an inhomogeneous Mooney–Rivlin material with the two material parameters varying continuously through the cylinder thickness either by a power law or an affine relation. They showed that when the two material parameters are linear functions of the radius the hoop stress in an internally pressurized cylinder is uniform.

Whereas the afore-mentioned investigations study boundary-value problems, the other challenging problem is that of finding the spatial variation of material properties to achieve a given objective, i.e., tailoring material properties for producing the desired stress distribution in a given body and under prescribed boundary conditions. For plane strain axisymmetric deformations of an FG cylinder composed of an orthotropic compressible material, Leissa and Vagins [22] assumed that all material moduli are proportional to each other and found their spatial variation to make either the hoop stress or the maximum in-plane shear stress uniform in the cylinder. We analytically study here, using the Airy stress function, the two classes of problems for a cylinder made of an FGIM. We also analyze plane strain/stress axisymmetric and non-axisymmetric deformations of an FG hollow cylinder with the shear modulus varying only in the radial direction. It seems that the Airy stress method has not been used earlier to analyze 2D problems for incompressible linear elastic materials.

2. Problem formulation

A schematic sketch of the problem studied is depicted in Fig. 1. We consider an infinitely long hollow cylinder of inner radius r_{in} , outer radius r_{ou} , assume that a plane strain state of deformation prevails in the cylinder, and describe its deformations by using cylindrical coordinates (r, θ, z) with the origin at the center of the cross-section and the z -axis along cylinder's centroidal axis.

In the absence of body forces equilibrium equations are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad (1a, b)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2}{r} \sigma_{r\theta} = 0,$$

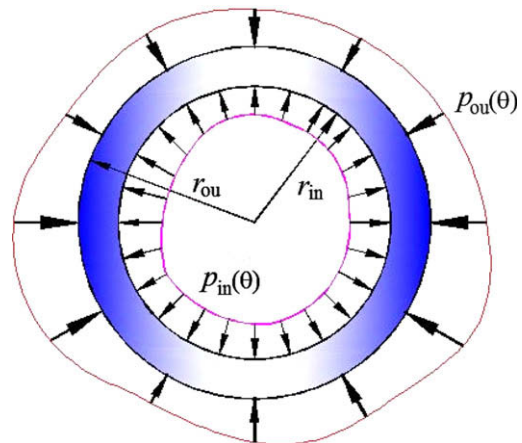


Fig. 1. Schematic sketch of the problem studied.

where σ_{rr} , $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ are the stress components. The pertinent boundary conditions on the inner and the outer surfaces are

$$\text{at } r = r_{in}, \quad \sigma_{rr}(r_{in}, \theta) = -p_{in}(\theta) \quad \sigma_{r\theta}(r_{in}, \theta) = q_{in}(\theta), \quad (2a, b)$$

$$\text{at } r = r_{ou}, \quad \sigma_{rr}(r_{ou}, \theta) = -p_{ou}(\theta), \quad \sigma_{r\theta}(r_{ou}, \theta) = q_{ou}(\theta). \quad (2c, d)$$

That is, pressures $p_{in}(\theta)$ and $p_{ou}(\theta)$ and tangential tractions $q_{in}(\theta)$ and $q_{ou}(\theta)$ act on cylinder's inner and outer surfaces, respectively. The prescribed surface tractions (2) must have null resultant force and moment in order for the problem to have a solution. For the traction-value problem displacements can only be determined within a rigid body motion.

The cylinder is assumed to be composed of an FGIM. Thus only isochoric (volume preserving) deformations satisfying

$$\varepsilon_{rr} + \varepsilon_{\theta\theta} = 0, \quad (3)$$

are admissible. Here ε_{rr} , $\varepsilon_{\theta\theta}$ and $\varepsilon_{r\theta}$ are components of the infinitesimal strain tensor. Assuming that the shear modulus, $G(r)$, varies only in the radial direction, constitutive equations are

$$\begin{aligned} \sigma_{rr} &= -p(r, \theta) + 2G(r)\varepsilon_{rr}, \\ \sigma_{\theta\theta} &= -p(r, \theta) + 2G(r)\varepsilon_{\theta\theta}, \\ \sigma_{r\theta} &= G(r)\varepsilon_{r\theta}, \end{aligned} \quad (4)$$

where the hydrostatic pressure $p(r, \theta)$ is not determined from the deformation field, but from Eqs. (1)–(4).

Stresses satisfying Eqs. (1)–(4) must also satisfy the following compatibility condition written in terms of strain components.

$$\frac{\partial^2 \varepsilon_{\theta\theta}}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2} + \frac{2}{r} \frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \frac{1}{r} \frac{\partial \varepsilon_{rr}}{\partial \theta} = \frac{1}{r} \frac{\partial^2 \varepsilon_{r\theta}}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial \varepsilon_{r\theta}}{\partial \theta}, \quad (5)$$

which can be simplified by using Eq. (3).

3. A general solution

We introduce the Airy stress function, $\varphi(r, \theta)$, and observe that stresses computed from

$$\sigma_{rr} = \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \varphi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right), \quad (6a, b, c)$$

identically satisfy equilibrium Eq. (1). Substituting for stresses from Eq. (6) into Eq. (4) and using Eq. (3) we get the following expression for the hydrostatic pressure.

$$p(r, \theta) = -\frac{1}{2r^2} \frac{\partial^2 \varphi}{\partial \theta^2} - \frac{1}{2r} \frac{\partial \varphi}{\partial r} - \frac{1}{2} \frac{\partial^2 \varphi}{\partial r^2}. \quad (6d)$$

Substitution for the hydrostatic pressure from Eq. (6d) and for stresses from Eq. (6a,b,c) into Eq. (4), solving the resulting equations for strains, and then using the compatibility Eq. (5) we obtain the following partial differential equation for the Airy stress function $\varphi(r, \theta)$:

$$\begin{aligned} \frac{\partial^4 \varphi}{\partial r^4} + \frac{1}{r^4} \frac{\partial^4 \varphi}{\partial \theta^4} + \frac{2}{r^2} \frac{\partial^4 \varphi}{\partial r^2 \partial \theta^2} + y_1(r) \frac{\partial^3 \varphi}{\partial r^3} - y_2(r) \frac{\partial^3 \varphi}{\partial r \partial \theta^2} - y_3(r) \frac{\partial^2 \varphi}{\partial r^2} \\ + y_4(r) \frac{\partial^2 \varphi}{\partial \theta^2} + y_5(r) \frac{\partial \varphi}{\partial r} = 0, \end{aligned} \quad (7)$$

where

$$y_1(r) = \frac{2}{r} - \frac{2}{G(r)} \frac{dG(r)}{dr},$$

$$y_2(r) = \frac{2}{r^3} + \frac{2}{r^2 G(r)} \frac{dG(r)}{dr},$$

$$y_3(r) = \frac{1}{rG(r)} \frac{dG(r)}{dr} - \frac{2}{G^2(r)} \left(\frac{dG(r)}{dr} \right)^2 + \frac{1}{G(r)} \frac{d^2 G(r)}{dr^2} + \frac{1}{r^2},$$

$$y_4(r) = \frac{3}{r^3 G(r)} \frac{dG(r)}{dr} - \frac{2}{r^2 G^2(r)} \left(\frac{dG(r)}{dr} \right)^2 + \frac{1}{r^2 G(r)} \frac{d^2 G(r)}{dr^2} + \frac{4}{r^4},$$

$$y_5(r) = \frac{1}{r^3} + \frac{1}{r^2 G(r)} \frac{dG(r)}{dr} - \frac{2}{rG^2(r)} \left(\frac{dG(r)}{dr} \right)^2 + \frac{1}{rG(r)} \frac{d^2 G(r)}{dr^2}.$$

We assume that the stress function can be written as

$$\varphi(r, \theta) = \varphi_r(r) \varphi_\theta(\theta). \quad (8)$$

Substitution from Eq. (8) into Eq. (7) gives the following differential equation for the unknown functions φ_r and φ_θ :

$$\frac{d^4 \varphi_\theta}{d\theta^4} + f_1(\varphi_r) \frac{d^2 \varphi_\theta}{d\theta^2} + f_2(\varphi_r) \varphi_\theta = 0, \quad (9)$$

where

$$\begin{aligned} f_1(\varphi_r) &= 4 + \frac{2r^2}{\varphi_r} \left(\frac{d^2 \varphi_r}{dr^2} - \frac{1}{r} \frac{d\varphi_r}{dr} \right) \\ &\quad + \frac{1}{G(r)} \frac{dG(r)}{dr} \left(3r - \frac{2r^2}{\varphi_r} \frac{d\varphi_r}{dr} - \frac{2r^2}{G(r)} \frac{dG(r)}{dr} \right) + \frac{r^2}{G(r)} \frac{d^2 G(r)}{dr^2}, \end{aligned} \quad (10a)$$

$$\begin{aligned} f_2(\varphi_r) &= \frac{r^4}{\varphi_r} \frac{d^4 \varphi_r}{dr^4} + \frac{2r^4}{\varphi_r} \left(\frac{1}{r} - \frac{1}{G(r)} \frac{dG(r)}{dr} \right) \frac{d^3 \varphi_r}{dr^3} \\ &\quad + \frac{r^4}{\varphi_r G(r)} \left(\frac{2}{G(r)} \left(\frac{dG(r)}{dr} \right)^2 - \frac{d^2 G(r)}{dr^2} - \frac{1}{r} \frac{dG(r)}{dr} - \frac{G(r)}{r^2} \right) \frac{d^2 \varphi_r}{dr^2} \\ &\quad + \frac{r^3}{\varphi_r G(r)} \left(\frac{G(r)}{r^2} + \frac{1}{r} \frac{dG(r)}{dr} - \frac{2}{G(r)} \left(\frac{dG(r)}{dr} \right)^2 + \frac{d^2 G(r)}{dr^2} \right) \frac{d\varphi_r}{dr} \end{aligned} \quad (10b)$$

and we have tacitly assumed that $\varphi_r \neq 0$ and $G(r) \neq 0$. It is realistic to assume that the shear modulus everywhere is positive. Assuming that $\frac{df_1(\varphi_r)}{d\varphi_r} \neq 0$ and $\varphi_\theta \neq 0$, the result of differentiation of both sides of Eq. (9) with respect to r can be written as

$$\frac{1}{\varphi_\theta} \frac{d^2 \varphi_\theta}{d\theta^2} = -\frac{df_2(\varphi_r)}{dr} \bigg/ \frac{df_1(\varphi_r)}{dr} = -\lambda^2, \quad (11a)$$

where λ is a constant. Thus for $\lambda \neq 0$,

$$\varphi_\theta = C_1 \cos(\lambda\theta) + C_2 \sin(\lambda\theta), \quad (11b)$$

where C_1 and C_2 are constants. For $\lambda = 0$

$$\varphi_\theta = C_1 \theta + C_2. \quad (11c)$$

The constant λ can be associated with the circumferential wave number. Substitution for φ_θ from Eq. (11) into Eq. (9) gives the following fourth-order ordinary differential equation with variable coefficients for determining the function φ_r :

$$\lambda^4 - \lambda^2 f_1(\varphi_r) + f_2(\varphi_r) = 0. \quad (12)$$

Having found φ_r and φ_θ for different values of λ we get the following expression for the stress function $\varphi(r, \theta)$ from Eq. (8):

$$\varphi(r, \theta) = \sum_i \varphi_r(\lambda_i, r) \varphi_\theta(\lambda_i, \theta). \quad (13)$$

The corresponding stresses and the hydrostatic pressure are computed from Eqs. (6a,b,c) and (6d). Strains are then found from Eq. (4) and displacements by integrating the following strain-displacement relations:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \quad \varepsilon_{r\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \quad (14)$$

where u_r and u_θ are displacements in the radial and the circumferential directions respectively.

A boundary-value problem for a linear elastic material has a unique solution, within a superimposed rigid body motion, provided that the shear modulus is everywhere positive. For some boundary-value problems it may suffice to consider only one value of λ in (11a,b) while for other problems, one may need to express the Airy stress function in terms of a series with terms corresponding to different values of λ in Eq. (11b). When the boundary conditions vary as $\sin(m\theta)$ or $\cos(m\theta)$ then it suffices to consider only one value of $\lambda = m$ in Eq. (11b).

4. Solutions for specified variations of the shear modulus

4.1. Power law variation

We first consider the case when the shear modulus is given by $G(r) = G_0(r/r_{ou})^n$ (15)

where G_0 equals the shear modulus at a point of the outer surface, and the index n is a real number. For $n = 0$, the shear modulus is constant, and the cylinder material is homogeneous. Substitution from Eq. (15) into Eq. (10) and the result into Eq. (12) yields

$$\frac{d^4\varphi_r}{dr^4} + \frac{2(1-n)}{r} \frac{d^3\varphi_r}{dr^3} + \frac{n^2 - 2\lambda^2 - 1}{r^2} \frac{d^2\varphi_r}{dr^2} + \frac{(n+1)(2\lambda^2 - n + 1)}{r^3} \frac{d\varphi_r}{dr} + \frac{\lambda^2(n^2 - 2n - 4 + \lambda^2)}{r^4} \varphi_r = 0. \quad (16)$$

For $n = 1$ and $n = -1$, some terms in Eq. (16) vanish. Solutions of Eq. (16) for vanishing and non-vanishing λ are given below.

Case 1: $\lambda = 0$

For $n = 0$, Eq. (16) has the solution

$$\varphi_r = C_1 \ln r + C_2 r^2 + C_3 r^2 \ln r + C_4, \quad (17a)$$

where constants C_1, C_2, C_3, C_4 are to be determined from the boundary conditions. Note that constants C_1, C_2, C_3, C_4 appearing in different expressions below and also in Eq. (11) need not have the same values. The stress function in Eq. (17a) is the same as that for a compressible material.

For $n = -2$, and $n = 2$, solutions of Eq. (16), respectively, are

$$\varphi_r = C_1 r^{-2} + C_2 \ln r + C_3 r^2 + C_4, \quad (17b)$$

$$\varphi_r = C_1 r^2 + C_2 r^2 \ln r + C_3 r^4 + C_4, \quad (17c)$$

For $n \neq -2, 2$ and 0 , we get

$$\varphi_r = C_1 r^2 + C_2 \frac{r^{n+2}}{n+2} + C_3 \frac{r^n}{n} + C_4. \quad (17d)$$

Case 2: $\lambda \neq 0$.

For $n = 0$, and $\lambda = 1$ and $\lambda \neq 1$, solutions of Eq. (16), respectively, are

$$\varphi_r = C_1 r^{-1} + C_2 r + C_3 r \ln r + C_4 r^3, \quad (18a)$$

$$\varphi_r = C_1 r^{-\lambda} + C_2 r^{2-\lambda} + C_3 r^\lambda + C_4 r^{2+\lambda}. \quad (18b)$$

For $n \neq 0$, Eq. (16) has the solution

$$\varphi_r = \sum_{i=1}^4 C_i r^{n_i}, \quad (18c)$$

where

$$n_1 = \frac{1}{2}(2 + n - n_6), \quad n_2 = \frac{1}{2}(2 + n + n_6), \quad n_3 = \frac{1}{2}(2 + n - n_7),$$

$$n_4 = \frac{1}{2}(2 + n + n_7), \quad n_5 = \sqrt{n^2 + 4\lambda^2 - n^2\lambda^2},$$

$$n_6 = \sqrt{4 + n^2 + 4\lambda^2 - 4n_5}, \quad n_7 = \sqrt{4 + n^2 + 4\lambda^2 + 4n_5}.$$

4.1.1. Axisymmetric deformations of a homogeneous hollow cylinder

As noted above, for a cylinder composed of a homogeneous material, $n = 0$. For axisymmetric deformations it suffices to set $\lambda = 0$ in Eq. (12). Thus the Airy stress function is given by Eq. (17a), and substitution from Eq. (17a) into Eq. (6) gives

$$\begin{aligned} \sigma_{rr} &= C_1 r^{-2} + 2C_2 + 2C_3 \ln r + C_3, \\ \sigma_{\theta\theta} &= -C_1 r^{-2} + 2C_2 + 2C_3 \ln r + 3C_3, \\ \sigma_{r\theta} &= 0, \quad p(r, \theta) = -2(C_2 + C_3 \ln r + C_3). \end{aligned} \quad (19a, b, c, d)$$

Substitution into Eq. (4) for stresses and the hydrostatic pressure from Eq. (19) and for the shear modulus from Eq. (15), we obtain expressions for the strain components which when integrated give

$$u_r = -\frac{C_1 + C_3 r^2}{2G_0 r}, \quad u_\theta = \frac{C_3 \theta r}{G_0} + f(r), \quad (20a, b)$$

where $f(r)$ is an arbitrary function of r and represents circumferential displacements due to rigid body rotation about the z -axis. Henceforth, we eliminate this rigid body motion by setting $f(r) = 0$. In order for the displacements to be single-valued, the constant C_3 in Eq. (20b) must be zero. Substitution for stresses from Eq. (19) into the boundary conditions (2a, c), constants C_1 and C_2 can be determined. We thus get the following expressions for stresses, the hydrostatic pressure and displacements:

$$\begin{aligned} \sigma_{rr} &= \frac{-p_{ou}(r_{in}^2 - r^2)r_{ou}^2 - p_{in}(r^2 - r_{ou}^2)r_{in}^2}{r^2(r_{in}^2 - r_{ou}^2)}, \\ \sigma_{\theta\theta} &= \frac{-p_{in}(r^2 + r_{ou}^2)r_{in}^2 + p_{ou}(r^2 + r_{in}^2)r_{ou}^2}{r^2(r_{in}^2 - r_{ou}^2)}, \\ \sigma_{r\theta} &= 0, \quad p(r, \theta) = \frac{p_{in}r_{in}^2 - p_{ou}r_{ou}^2}{r_{in}^2 - r_{ou}^2}, \end{aligned} \quad (21a, b, c, d)$$

and

$$u_r = \frac{(p_{ou} - p_{in})r_{in}^2 r_{ou}^2}{2G_0 r(r_{in}^2 - r_{ou}^2)}, \quad u_\theta = 0. \quad (22a, b)$$

These expressions for stresses and displacements agree with those derived in [20] where equilibrium equations are expressed in terms of displacements.

4.1.2. Axisymmetric deformations of a pressurized FG cylinder

Recall that for axisymmetric deformations $\lambda = 0$ in Eq. (12). For nonzero values of n the Airy stress function is given by Eq. (17b,c,d). Following the procedure outlined in Subsection 4.1.1, we give below the stress and the displacement fields for $n = -2, 2$ and for values other than these two.

For $n = -2$, stresses, the hydrostatic pressure and displacements are given by

$$\begin{aligned} \sigma_{rr} &= \frac{-p_{ou}(r_{in}^4 - r^4)r_{ou}^4 - p_{in}(r^4 - r_{ou}^4)r_{in}^4}{r^4(r_{in}^4 - r_{ou}^4)}, \\ \sigma_{\theta\theta} &= \frac{-p_{in}(r^4 + 3r_{ou}^4)r_{in}^4 + p_{ou}(r^4 + 3r_{in}^4)r_{ou}^4}{r^4(r_{in}^4 - r_{ou}^4)}, \\ \sigma_{r\theta} &= 0, \quad p(r, \theta) = \frac{p_{in}(r^4 + r_{ou}^4)r_{in}^4 - p_{ou}(r^4 + r_{in}^4)r_{ou}^4}{r^4(r_{in}^4 - r_{ou}^4)}, \end{aligned} \quad (23a, b, c, d)$$

$$u_r = \frac{(p_{ou} - p_{in})r_{in}^4 r_{ou}^2}{G_0 r(r_{in}^4 - r_{ou}^4)}, \quad u_\theta = 0. \quad (24a, b)$$

For $n = 2$, we get the following expressions for stresses, the hydrostatic pressure and displacements:

$$\begin{aligned}\sigma_{rr} &= \frac{(p_{ou} - p_{in}) \ln r - p_{ou} \ln r_{in} + p_{in} \ln r_{ou}}{\ln(r_{in}/r_{ou})}, \\ \sigma_{\theta\theta} &= \frac{(p_{ou} - p_{in})(1 + \ln r) + p_{in} \ln r_{ou} - p_{ou} \ln r_{in}}{\ln(r_{in}/r_{ou})}, \\ \sigma_{r\theta} &= 0, \quad p(r, \theta) = \frac{(p_{in} - p_{ou})(1 + 2 \ln r) + 2p_{ou} \ln r_{in} - 2p_{in} \ln r_{ou}}{2 \ln(r_{in}/r_{ou})},\end{aligned}\quad (25a, b, c, d)$$

$$u_r = \frac{p_{in} r_{ou}^2 - p_{ou} r_{ou}^2}{4G_0 r \ln(r_{ou}/r_{in})}, \quad u_\theta = 0, \quad (26a, b)$$

which agree with those derived in [20]. Note that

$$\sigma_{\theta\theta} - \sigma_{rr} = \frac{p_{in} - p_{ou}}{\ln r_{ou} - \ln r_{in}}, \quad (27)$$

is constant throughout the cylinder thickness. One can derive Eq. (27) simply by integrating with respect to the radius r the equilibrium Eq. (1a). Note that Eq. (27) does not involve material properties and hence is a universal relation in the sense that it is valid for all hollow cylinders irrespective of their materials and the magnitude of deformations.

For $n \neq -2$, $n \neq 2$, $n \neq 0$, stresses, the hydrostatic pressure and displacements are given by

$$\begin{aligned}\sigma_{rr} &= \frac{p_{ou}(r^2 r_{in}^n - r^n r_{in}^2) r_{ou}^2 - p_{in}(r^2 r_{ou}^n - r^n r_{ou}^2) r_{in}^2}{r^2 (r_{in}^2 r_{ou}^n - r^n r_{ou}^2)}, \\ \sigma_{\theta\theta} &= -p_{in} - \frac{(p_{ou} - p_{in})((n-1)r^{n-2} - r_{in}^{n-2})}{r_{ou}^{n-2} - r_{in}^{n-2}}, \\ p(r, \theta) &= -\frac{n(p_{in} - p_{ou})r^n r_{in}^2 r_{ou}^2 + 2r^2(p_{ou} r_{in}^n r_{ou}^2 - p_{in} r_{in}^2 r_{ou}^n)}{2r^2(r_{in}^2 r_{ou}^n - r^n r_{ou}^2)}, \quad \sigma_{r\theta} = 0,\end{aligned}\quad (28a, b, c, d)$$

$$u_r = \frac{(p_{in} - p_{ou})(n-2)r_{in}^2 r_{ou}^{2+n}}{4G_0 r (r_{in}^n r_{ou}^2 - r_{in}^2 r_{ou}^n)}, \quad u_\theta = 0. \quad (29a, b)$$

4.1.3. FG cylinder deformed by tangential tractions on the outer surface

For a FG cylinder subjected to tangential tractions q_{ou} on the outer surface with the inner surface rigidly clamped (i.e., $u_r = u_\theta = 0$ on $r = r_{in}$) we use the Airy stress function corresponding to $\lambda = 0$, and set $C_2 = 0$ in Eq. (11c). For different values of the exponent n , the Airy stress function is given below.

$$\text{For } n = 0, \quad \varphi = (C_1 \ln r + C_2 r^2 + C_3 r^2 \ln r + C_4) \theta, \quad (30a)$$

$$\text{For } n = -2, \quad \varphi = (C_1 r^{-2} + C_2 \ln r + C_3 r^2 + C_4) \theta, \quad (30b)$$

$$\text{For } n = 2, \quad \varphi = (C_1 r^2 + C_2 r^2 \ln r + C_3 r^4 + C_4) \theta, \quad (30c)$$

$$\text{For } n \neq 0, n \neq -2, n \neq 2, \quad \varphi = (C_1 r^2 + C_2 \frac{r^{n+2}}{n+2} + C_3 \frac{r^n}{n} + C_4) \theta. \quad (30d)$$

We provide details of the solution procedure only for $n = 0$. Substitution for the stress function from Eq. (30a) into Eqs. (6a,b,c,d) gives

$$\begin{aligned}\sigma_{rr} &= \left(\frac{C_1}{r^2} + 2C_2 + C_3 + 2C_3 \ln r \right) \theta, \\ \sigma_{\theta\theta} &= \left(-\frac{C_1}{r^2} + 2C_2 + 3C_3 + 2C_3 \ln r \right) \theta, \\ \sigma_{r\theta} &= -\frac{C_1(1 - \ln r) + C_2 r^2 + C_3 r^2(1 + \ln r) - C_4}{r^2}, \\ p(r, \theta) &= -2(C_2 + C_3(1 + \ln r)) \theta.\end{aligned}\quad (31a, b, c, d)$$

Substitution for stresses from Eq. (31) and for $G(r)$ from Eq. (15) into Eq. (4), solving them for strains, substituting from strains into Eq. (14), and integrating the resulting equations we arrive at

$$u_r = -\frac{\theta(C_1 + r^2 C_3)}{2G_0 r} + f(\theta), \quad u_\theta = \frac{r \theta^2 C_3}{2G_0} - \int f(\theta) d\theta + f_1(r), \quad (32a, b)$$

where

$$\begin{aligned}f_1(r) &= -\frac{C_1 \ln r}{2G_0 r} - \frac{C_2 r \ln r}{G_0} - \frac{C_3 r \ln r(1 + \ln r)}{2G_0} - \frac{C_4}{2G_0 r} + r C_5, \\ f(\theta) &= C_6 \sin(\theta) + C_7 \cos(\theta).\end{aligned}\quad (33a, b)$$

Using boundary conditions on the inner and the outer surfaces of the hollow cylinder and requiring the displacements to be single-valued functions of θ , we get

$$\sigma_{rr} = 0, \quad \sigma_{\theta\theta} = 0, \quad \sigma_{r\theta} = \frac{q_{ou} r_{ou}^2}{r^2}, \quad (34a, b, c, d)$$

$$u_r = 0, \quad u_\theta = \frac{q_{ou}(r^2 - r_{in}^2) r_{ou}}{2G_0 r_{in}^2 r}. \quad (35a, b)$$

Following the same procedure for other values of n , we find the corresponding stress and displacement fields. Note that Eq. (34c) follows from the overall equilibrium of a hollow cylinder of inner radius r and outer radius r_{ou} , and is a universal result since it does not depend upon the shear modulus. Moreover, it is applicable for both large and small deformations. Since $u_r = 0$ for all values of n , we list below only expressions for u_θ .

$$\text{For } n = -2, \quad u_\theta = \frac{q_{ou} r (\ln r - \ln r_{in})}{G_0}. \quad (36)$$

$$\text{For } n = 2, \quad u_\theta = \frac{q_{ou}(r^4 - r_{in}^4) r_{ou}}{4G_0 r_{in}^4 r^3}. \quad (37)$$

$$\text{For } n \neq 0, n \neq -2, n \neq 2, \quad u_\theta = \frac{q_{ou}(r^2 r_{in}^{2-n} - r^{2-n}) r_{ou}^{2+n}}{G_0(2+n)r}. \quad (38)$$

Thus circumferential displacements depend upon the inhomogeneity of the material but the in-plane shear stress is independent of the shear modulus.

4.2. Exponential variation

We now assume that

$$G(r) = G_0 \exp[m(r/r_{ou})], \quad (39)$$

where G_0 and m are real numbers, and for a homogeneous material $m = 0$. Substitution from Eq. (39) into Eq. (12) yields the following 4th order ordinary differential equation for φ_r :

$$\begin{aligned}\frac{d^4 \varphi_r}{dr^4} + \left(\frac{2}{r} - \frac{2m}{r_{ou}} \right) \frac{d^3 \varphi_r}{dr^3} + \left(\frac{-2\lambda^2 - 1}{r^2} + \frac{m^2}{r_{ou}^2} - \frac{m}{r r_{ou}} \right) \frac{d^2 \varphi_r}{dr^2} \\ + \left(\frac{2\lambda^2 + 1}{r^3} + \frac{2m\lambda^2 + m}{r^2 r_{ou}} - \frac{m^2}{r r_{ou}^2} \right) \frac{d \varphi_r}{dr} \\ + \left(\frac{\lambda^4 - 4\lambda^2}{r^4} - \frac{3m\lambda^2}{r^3 r_{ou}} + \frac{m^2 \lambda^2}{r^2 r_{ou}^2} \right) \varphi_r = 0.\end{aligned}\quad (40)$$

Since $m = 0$ for a homogeneous material, it suffices to consider $m \neq 0$. For $\lambda = 0$ we get

$$\varphi_r = C_1 r^2 + C_2 f_2(r) + C_3 f_3(r) + C_4, \quad (41)$$

where

$$f_2(r) = \exp(mr/r_{ou}) (-mr_{ou}r - r_{ou}^2)/r_{ou}^2 - 2\text{Ei}(mr/r_{ou}) + m^2 r^2 \text{Ei}(mr/r_{ou})/r_{ou}^2,$$

$$f_3(r) = \exp(mr/r_{ou}) (-r_{ou}r/m + r_{ou}^2/m^2) + r^2 \text{Ei}(mr/r_{ou}),$$

$$\text{and } \text{Ei}(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt,$$

is the exponential integration function.

For $\lambda \neq 0$, Eq. (40) is solved by using the Frobenius series method, i.e., we assume a solution of the form

$$\varphi_r = \sum_{k=0}^{\infty} a_k r^{k+s}, \quad (42)$$

substitute it into Eq. (40), equate terms of like powers of r on both sides of the resulting equation, and obtain the following recursive formula for a_k :

$$a_1 = \frac{m(2s - 5s^2 + 2s^3 + 3\lambda^2 - 2s\lambda^2)a_0}{r_{ou}(s - \lambda - 1)(s - \lambda + 1)(s + \lambda - 1)(s + \lambda + 1)}, \quad (43a)$$

$$a_k = \frac{b_1 a_{k-2} + b_2 a_{k-1}}{r_{ou}^2(k + s - \lambda - 2)(k + s - \lambda)(k + s + \lambda - 2)(k + s + \lambda)}, \quad (43b)$$

$k = 2, 3,$

$$\varphi(r) = \frac{r^{1-k}(p_{ou}((k-1)r^{k+\beta+1} + (\beta+2)r_{in}^{k+\beta+1})r_{ou}^{k+1} - p_{in}r_{in}^{k+1}((k-1)r^{k+\beta+1} + (\beta+2)r_{ou}^{k+\beta+1}))}{(k-1)(\beta+2)(r_{in}^{k+\beta+1} - r_{ou}^{k+\beta+1})}, \quad \text{when } k \neq 1 \text{ or } \beta \neq -2 \text{ or } k + \beta \neq -1, \quad (49a)$$

where $b_1 = -m^2(8 + k^2 + 2k(s-3) - 6s + s^2 + \lambda^2)$, $b_2 = -mr_{ou}(9 - 2k^3 + k^2(11 - 6s) + 11s^2 - 2s^3 - 5\lambda^2 + 2s(\lambda^2 - 9) + 2k(-9 + 11s - 3s^2 + \lambda^2))$.

Equating the coefficient of a_0 to zero gives the indicial equation

$$(s - \lambda - 2)(s + \lambda - 2)(s - \lambda)(s + \lambda) = 0, \quad (44)$$

for determining s in terms of λ . It is evident that roots of Eq. (44) are distinct. According to the Frobenius method, the solution corresponding to the maximum root $s_{\max} = \max(s_1, s_2, s_3, s_4)$ is

$$\varphi_{r1} = \sum_{k=0}^{\infty} a_k r^{k+s_{\max}}, \quad (45a)$$

and the solution for the other roots of Eq. (44) is

$$\varphi_{ri} = \sum_{k=0}^{\infty} \{(s - s_i)a_k\}_{s=s_i} r^{k+s_i} \ln r + \sum_{k=0}^{\infty} \left\{ \frac{d}{ds} [(s - s_i)a_k] \right\}_{s=s_i} r^{k+s_i}, \quad (45b)$$

$i = 2, 3, 4,$

where we have assumed that s_1 is the maximum root of Eq. (44). Thus

$$\varphi_r(r) = \sum_{i=1}^4 c_i \varphi_{ri}, \quad (46)$$

where c_1, c_2, c_3 and c_4 are constants. Substitution from Eqs. (46) and (11) into Eq. (8) gives the stress function.

5. Material tailoring

We now study the inverse problem of finding the variation with the radius of the shear modulus for a pre-specified variation of the stress distribution.

5.1. Axisymmetric deformations of pressurized hollow cylinders

We require that the radial and the hoop stresses at a point satisfy the constraint

$$\sigma_e = k\sigma_{rr} + \sigma_{\theta\theta} = C_1 r^\beta, \quad (47)$$

where C_1, k and β are known constants, and find the corresponding variation of the shear modulus. For $k = 0$ and $\beta = 0$, Eq. (47) implies that the hoop stress is constant through the cylinder thickness, and

for $k = -1$ and $\beta = 0$ the in-plane shear stress is uniform through the cylinder thickness. Note that, except for the trivial case of null stresses everywhere, the radial stress cannot be uniform through the cylinder thickness.

For axisymmetric deformations, the stress function depends only on the radial coordinate of a point, and the compatibility equation in terms of strains is

$$\frac{d}{dr}(r\varepsilon_{\theta\theta}) - \varepsilon_{rr} = 0. \quad (48)$$

Substituting for stresses from Eq. (6) into Eq. (47) and assuming that cylinder's inner and outer surfaces are loaded by pressures only, we obtain the following expression for the stress function.

$$\varphi(r) = \frac{r^{\beta+2}(-p_{in}r_{in}^2 + p_{ou}r_{ou}^2) - (\beta+2)r_{in}^2r_{ou}^2(p_{ou}r_{in}^\beta - p_{in}r_{ou}^\beta) \ln r}{(\beta+2)(r_{in}^{\beta+2} - r_{ou}^{\beta+2})}, \quad (49b)$$

when $k = 1$ and $\beta \neq -2$,

$$\varphi(r) = \frac{r^{-k}(r_{in}^k r_{ou}^k r(p_{in}r_{in}^2 - p_{ou}r_{ou}^2) + (k-1)r_{in}r_{ou}r^k(p_{in}r_{in}^{k+1} - p_{ou}r_{ou}^{k+1}) \ln r)}{(k-1)(-r_{in}^k r_{ou} + r_{in}r_{ou}^k)}, \quad (49c)$$

when $k \neq 1$ and $\beta = -2$,

$$\varphi(r) = \frac{\ln r((-p_{in}r_{in}^2 + p_{ou}r_{ou}^2) \ln r - 2p_{ou}r_{ou}^2 \ln r_{in} + 2p_{in}r_{in}^2 \ln r_{ou})}{2 \ln(r_{in}/r_{ou})}, \quad (49d)$$

when $k = 1$ and $\beta = -2$,

$$\varphi(r) = \frac{r(p_{in}r_{in} - p_{ou}r_{ou} + (p_{ou}r_{ou} - p_{in}r_{in}) \ln r - p_{ou}r_{ou} \ln r_{in} + p_{in}r_{in} \ln r_{ou})}{\ln(r_{in}/r_{ou})}, \quad (49e)$$

when $k = 0$ and $\beta = -1$,

$$\varphi(r) = \frac{r^2((p_{in} - p_{ou})(1 - 2 \ln r) - 2p_{ou} \ln r_{in} + 2p_{in} \ln r_{ou})}{4 \ln(r_{in}/r_{ou})}, \quad (49f)$$

when $k = -1$ and $\beta = 0$.

Substitution from Eq. (49) into Eqs. (6) and (4) and then into Eq. (48) gives

$$G(r) = \frac{G_{ou}r^{\beta+1}r_{ou}^{-k-\beta-2}(-\beta r_{in}^{k+\beta+1}(p_{in}r_{in}^{k+1} - p_{ou}r_{ou}^{k+1}) + (k+1)r_{in}^{k+1}r_{ou}^{k+1}(p_{ou}r_{in}^\beta - p_{in}r_{ou}^\beta))}{-(k+\beta+1)p_{in}r_{in}^{k+1}r_{ou}^\beta + p_{ou}((k+1)r_{in}^{k+\beta+1} + \beta r_{ou}^{k+\beta+1})}, \quad (50a)$$

when $k \neq 1$ or $\beta \neq -2$ or $k + \beta \neq -1$,

$$G(r) = \frac{G_{ou}(-\beta r^{\beta+2}(p_{in}r_{in}^2 - p_{ou}r_{ou}^2) + 2r_{in}^2r_{ou}^2(p_{ou}r_{in}^\beta - p_{in}r_{ou}^\beta))}{r_{ou}^2(-(\beta+2)p_{in}r_{in}^2r_{ou}^\beta + p_{ou}(2r_{in}^{\beta+2} + \beta r_{ou}^{\beta+2}))}, \quad (50b)$$

when $k = 1$ and $\beta \neq -2$,

$$G(r) = \frac{G_{ou}r^{-k}(p_{ou}(-2r^k r_{in} + (k+1)r_{in}^k r_{ou}^{k+2} - p_{in}r_{in}^{k+2}(-2r_{ou}^k + (k+1)r_{ou}^k r_{ou}^k))}{r_{ou}(p_{in}r_{in}^{k+2}(1-k) + p_{ou}(r_{in}^k r_{ou}^2(1+k) - 2r_{in}r_{ou}^{k+1}))}, \quad (50c)$$

when $k \neq 1$ and $\beta = -2$,

$$G(r) = \frac{G_{ou}((p_{in}r_{in}^2 - p_{ou}r_{ou}^2)(1 - 2 \ln r) - 2p_{ou}r_{ou}^2 \ln r_{in} + 2p_{in}r_{in}^2 \ln r_{ou})}{p_{in}r_{in}^2 - p_{ou}r_{ou}^2(1 + 2 \ln r_{in} - 2 \ln r_{ou})}, \quad (50d)$$

when $k = 1$ and $\beta = -2$,

$$G(r) = \frac{G_{ou}r((p_{ou}r_{ou} - p_{in}r_{in})(1 - \ln r) + p_{ou}r_{ou} \ln r_{in} - p_{in}r_{in} \ln r_{ou})}{r_{ou}(p_{ou}r_{ou} - p_{in}r_{in} + p_{ou}r_{ou} \ln(r_{in}/r_{ou}))},$$

when $k = 0$ and $\beta = -1$, (50e)

$$G(r) = \frac{G_{ou}r^2}{r_{ou}^2}, \quad \text{when } k = -1 \text{ and } \beta = 0, \quad (50f)$$

where G_{ou} is the values of $G(r)$ on the outer surface.

Setting $k = 0$ and $\beta = 0$ in Eq. (50a), we get $G(r) = G_{ou}r/r_{ou}$. That is for the hoop stress to be constant in the cylinder the shear modulus must be proportional to r which agrees with the result given by Batra [19]. For $k = -1$ and $\beta = 0$, the in-plane shear stress is constant through the cylinder thickness and Eq. (50f) implies that the shear modulus must be proportional to r^2 . It follows from Eq. (27) that the value of the constant shear stress depends upon the pressures applied on the inner and the outer surfaces and their radii. For other pre-assigned variation of the linear combination of the radial and the hoop stresses, we can find the corresponding variation of the shear modulus from Eq. (50). For $n = 1$ and 2, the corresponding displacement and stress fields can be obtained from results given in Section 4.1.

5.2. FG cylinder deformed by tangential tractions on the outer surface

For the boundary conditions considered in Subsection 4.1.3 stresses are determined by the boundary conditions and the cylinder geometry. Thus they cannot be controlled by tailoring the material properties.

6. Numerical examples

6.1. Analysis of FG cylinders

Example 1: For a cylinder with $r_{in} = 0.1$ cm, $r_{ou} = 1.0$ cm, $G_0 = 1$ MPa in Eq. (15), $p_{in} = 0$, $p_{ou} = 0.1$ MPa and $0.1 \times \cos(2\theta)$ MPa, and different values of the index n in Eq. (15) we have plotted in Figs. 2 and 3 the through-the-thickness variation of the stresses.

For uniform distribution of pressure on the cylinder outer surface the hoop stress in the cylinder with $n = 1$ is a constant, but that is not the case for a non-uniform distribution of the pressure. For $n < 0$ the maximum magnitude of the hoop stress occurs at points on the inner surface of the cylinder, and large values of the hoop stress occur at points near the inner surface. It is clear that the through-the-thickness distribution of stresses can be controlled by selecting an appropriate value of n . Even though results have been plotted for integer values of n , the analysis presented above is valid for all real values of n .

Example 2: For a cylinder with $r_{in} = 0.2$ cm, $r_{ou} = 1.0$ cm, $G_0 = 1$ MPa in Eq. (39), $p_{in} = 0$, $p_{ou} = 0.1 \times \cos(6\theta)$ MPa, and different values of m in Eq. (39) we have plotted in Fig. 4 the through-the-thickness variation of stresses.

For $m = 5$ and -5 , the maximum value of the hoop stress occurs at a point on the outer surface of the cylinder, and its magnitude for $m = 5$ is nearly five times that of the hoop stress for $m = -5$.

6.2. Material tailoring

Example 3: For a cylinder with $r_{in} = 0.2$ cm, $r_{ou} = 1.0$ cm, $p_{in} = 0$, $p_{ou} = 1.0$ MPa, we find the through-the-thickness variation of the shear modulus to attain a pre-specified through-the-thickness variation of a linear combination of the radial and the hoop stresses. For $\beta = 0$, 1 and $k = -1$, 0 and 1, we have exhibited in Fig. 5 the computed variation of the shear modulus. The value of the constant C_I in Eq. (47) depends upon the pressures prescribed on cylinder's inner and outer surfaces and the cylinder geometry. Results exhibited in Fig. 5a reveal that in a homogeneous cylinder $\sigma_{rr} + \sigma_{\theta\theta}$ is a constant. For the hoop stress $\sigma_{\theta\theta}$ to be constant the shear modulus must vary linearly with the radius. The in-plane shear stress, $(\sigma_{rr} - \sigma_{\theta\theta})/2$, is constant when the shear modulus varies as r^2 . It is found from results plotted in Fig. 5b that the shear modulus needs to increase gradually from the inner surface to the outer surface in order to achieve either linearly varying hoop stress or linearly varying in-plane shear stress in the cylinder with pressure applied only on the outer surface. Furthermore, essentially the same through-the-thickness variation of the shear modulus makes the in-plane shear stress and the sum of the hoop and the radial stress vary linearly through the cylinder thickness. Note that the shear modulus on the inner surface is nearly 0.3% of that on the outer surface implying that the material on the inner surface is very soft relative to that on the outer surface. For pressure applied on the inner surface or on both surfaces of the cylinder the through-the-thickness variation of the shear modulus will be quantitatively different from that given in Fig. 5.

7. Remarks

The requirement that the shear modulus must be positive everywhere in the cylinder may rule out tailoring the shear modulus so as to attain a desired through-the-thickness variation of a linear combination of the radial and the hoop stresses.

We have not addressed how to fabricate a cylinder with the shear modulus varying by a factor of 300 over the cylinder thickness; this exercise is left for material scientists.

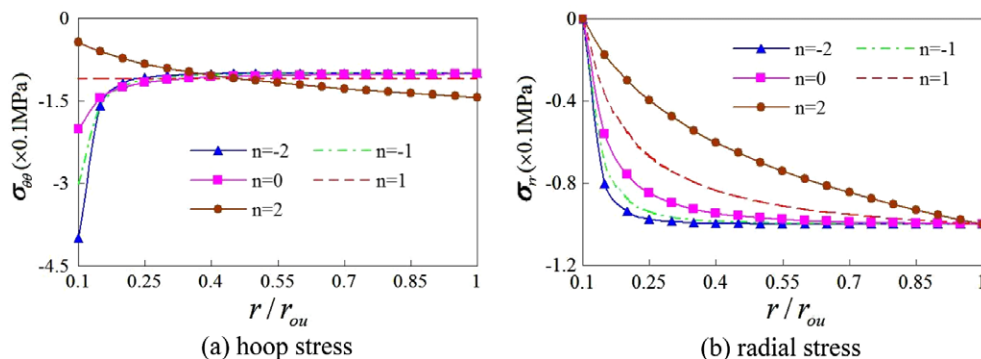


Fig. 2. Through-the-thickness variation of stresses in a cylinder for different values of n .

A material scientist can vary elastic moduli of a fiber-reinforced composite by changing the materials of the fibers and the matrix, varying continuously the fiber layout in going from the bottom

layer to the top layer (e.g., see [23]), and changing the spacing between adjacent fibers in a single-layer plate (e.g., see [24,25]). For the problem studied in [23] through-the-thickness material prop-

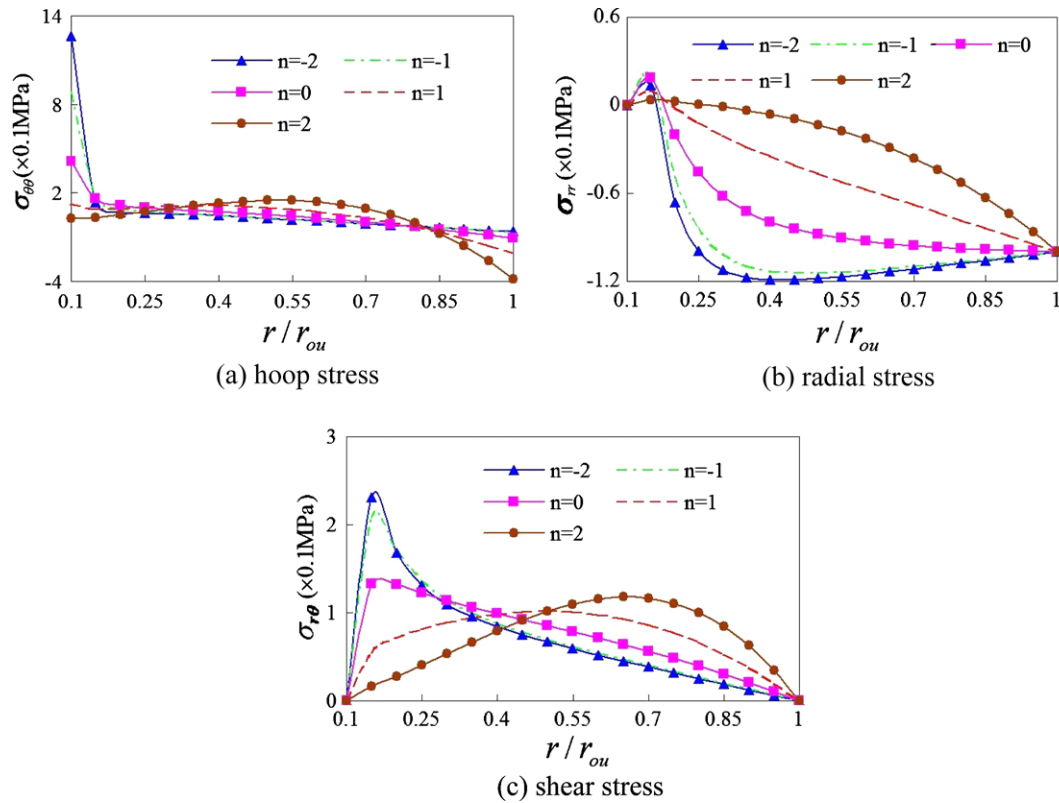


Fig. 3. Through-the-thickness variation of stresses in a cylinder for different values of n .

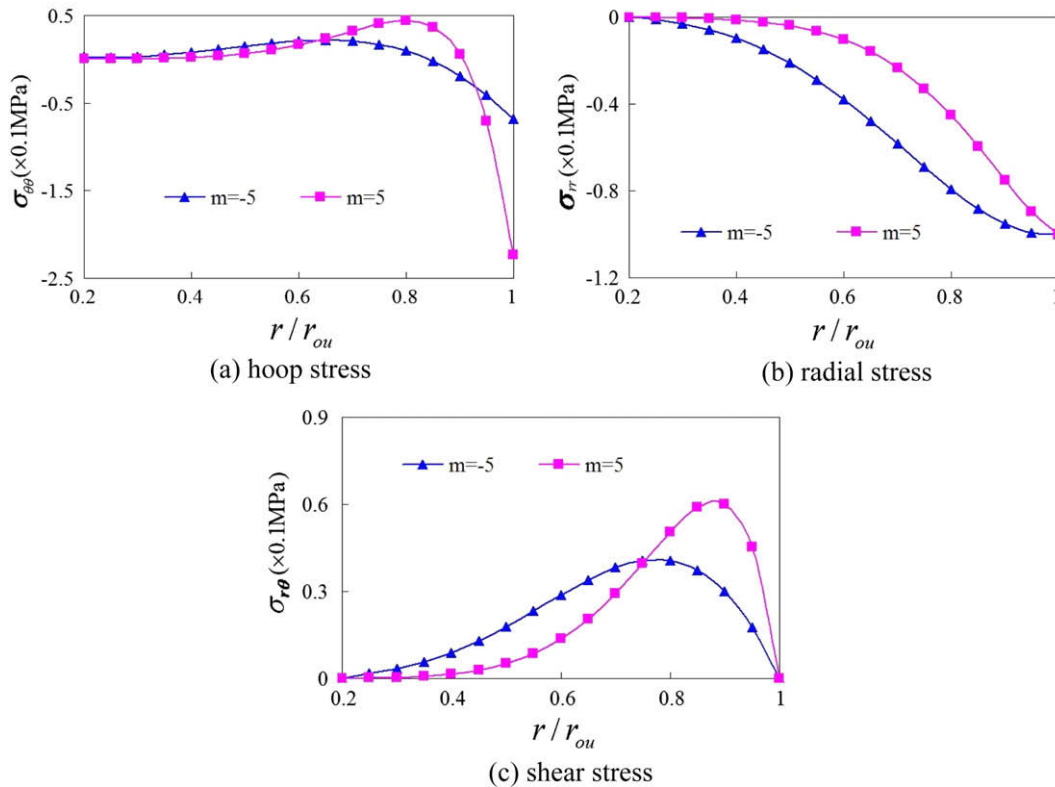


Fig. 4. Through-the-thickness variation of stresses in a cylinder for different values of m .

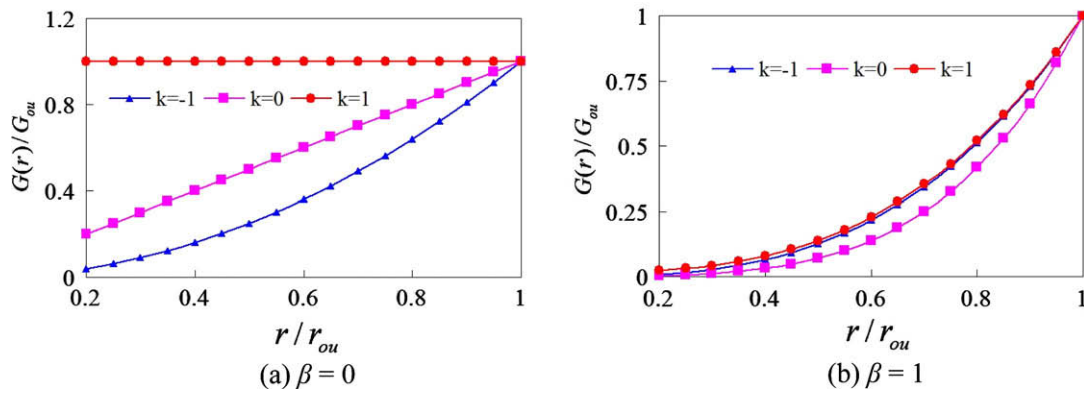


Fig. 5. Through-the-thickness variation of the shear modulus for $k = -1, 0$ and 1 ; (a) $\beta = 0$, (b) $\beta = 1$.

erties depend continuously upon the fiber orientation angle and for problems analyzed in [24,25] material properties vary continuously in one of the in-plane directions. A problem in which material properties vary continuously in both in-plane directions has been analyzed in [26]. Materials studied in [23–26] are not incompressible but present interesting possibilities that can be extended to incompressible materials especially for scaffolds used in biological applications.

For the material tailoring problem, we have found through-the-thickness variation of the shear modulus to achieve a desired variation of a linear combination of the radial and the hoop stresses. If one were to optimize more than one variable, e.g., a stress component and the first frequency of free vibrations of the structure, then one could write an objective function with appropriate weights for the stress component and the frequency to be optimized. One can then find through-the-thickness variation of the shear modulus to find an optimum value of the objective function. However, this distribution of the shear modulus will not, in general, simultaneously optimize the stress component and the frequency included in the objective function because for incompressible isotropic linear elastic materials there is only material parameter to be varied. Optimization problems involving more than one variable to be optimized have been studied, for example, in [27,28].

Batra [29] has studied the torsion of a cylinder composed of an FGIM, and shown that the axial variation of the shear modulus can be adjusted to control the angle of twist of a cross-section.

The paper [30] should have been cited earlier in the text along with others dealing with the analysis of a thermoelastic deformations of an FG cylinder. The just accepted paper [31] deals with infinitesimal deformations of a FG cylinder composed of an elastic-plastic material. Finite thermo-elasto-visco-plastic deformations of FG structures have been studied in [32–34].

8. Conclusions

We have studied the problem of material tailoring to achieve the desired stress distribution in a hollow cylinder composed of a linear elastic incompressible material with the shear modulus varying only in the radial direction. The Airy stress function approach employed herein enables one to analyze both axisymmetric and non-axisymmetric problems, and seems to be the first use of this technique for incompressible materials. It also enables one to find deformations of the cylinder with known variations of the shear modulus and prescribed displacements and tractions on the inner and the outer surfaces. Analytical solutions for displacements and stresses with the shear modulus given by either a power law or an exponential function are derived and numerical results for a few example problems are presented. One can, of course, con-

trol the through-the-thickness distribution of stresses by tailoring the through-the-thickness variation of the shear modulus; however, not all conceivable stress states can be controlled since the shear modulus must be positive everywhere in the cylinder.

Results presented here will be useful to both material scientists in designing new materials, stress analysts, and designers. Also, analytical solutions presented herein can serve as benchmarks for comparison with the approximate solutions obtained numerically.

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