



Technical Note

Relations between buckling loads of functionally graded Timoshenko and homogeneous Euler–Bernoulli beams

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ABSTRACT

Analytical relations between the critical buckling load of a functionally graded material (FGM) Timoshenko beam and that of the corresponding homogeneous Euler–Bernoulli beam subjected to axial compressive load have been derived for clamped–clamped (C–C), simply supported–simply supported (S–S) and clamped–free (C–F) edges. However, no such relation is found for clamped–simply supported (C–S) beams. For C–S beams, the transcendental equation has been derived to find the critical buckling load for the FGM Timoshenko beam which is similar to that for a homogeneous Euler–Bernoulli beam. For the FGM beams Young's modulus, E , and Poisson's ratio, ν , are assumed to vary through the thickness. The significance of this work is that for the C–C, S–S and C–F FGM Timoshenko beams, the critical buckling load can be easily found from that of the corresponding homogeneous Euler–Bernoulli beam and two constants whose values depend upon the through-the-thickness variations of E and ν . For the C–S FGM Timoshenko beam the transcendental equation for the determination of the critical buckling load is similar to that for the corresponding homogeneous Euler–Bernoulli beam.

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1. Introduction

Functionally graded materials (FGMs) are composites in which continuous spatial variation of constituents can be designed to either alleviate stress concentrations near voids, defects and interfaces or material properties optimized for achieving a desired goal. Consequently, FGMs have enormous potential for technological and engineering applications especially in extreme thermal environment where stress concentration due to high temperature gradients can be either minimized or significantly reduced. For beams, plates and shells the gradient in the composition is usually taken to be in the thickness direction. Analytical solutions for structures made of FGMs are difficult to find because governing equations involve coefficients that depend upon spatial coordinates. An ideal situation will be to express the load bearing capacity of a FGM structure in terms of that of a homogeneous structure under the constraints of both structures having the same geometry and boundary conditions. We study such a problem in this paper.

Because of the enormous literature on FGMs, we briefly review papers closely related to the current work. Benatta et al. [1] and Sallai et al. [2] analytically solved static bending deformations of simply supported FGM hybrid beams subjected to uniformly distributed transverse loads by using a higher-order shear deforma-

tion theory and gave numerical results for the deflection, and the transverse normal and the transverse shear stresses. Kadoli et al. [3] used the finite element method and the third-order shear deformation theory (TSDT) to analyze static bending deformations of FGM beams with different boundary conditions (BCs) at the edges and a uniform transverse load applied on the top surface. Li [4] investigated static bending deformations and transverse vibrations of FGM Timoshenko beams (TBs) and introduced a function to uncouple governing equations for the deflection and the angle of rotation of a cross-section initially perpendicular to the neutral surface. Employing the same method, Huang and Li [5,6] used the FSDT to study bending, buckling and free vibrations of FGM circular columns with material properties continuously varying in the radial direction. Simsek [7] studied free vibrations of FGM beams using different higher-order shear deformation theories and derived governing equations by using Hamilton's principle. Ke et al. [8,9] as well as Yang and Chen [10] studied free vibrations, buckling and post-buckling of FGM TBs containing open cracks by assuming an exponential variation of material properties in the thickness direction.

Sankar [11] used the linear elasticity theory to analytically analyze deformations of simply supported FGM beams with Young's modulus varying exponentially in the thickness direction and subjected to symmetrical sinusoidal transverse loads. Zhong and Yu [12] adopted the two-dimensional linear elasticity theory to study deformations of a cantilever FGM beam with arbitrary

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through-the-thickness variation of material properties. Ding et al. [13] used the Airy stress function to study deformations of anisotropic FGM beams under various BCs at the edges.

Different from the conventional analytical and numerical approaches of analyzing static and dynamic responses of FGM structures, a few investigations have focused on finding relations between deflections, buckling loads and natural frequencies of FGM structures and those of the corresponding homogeneous ones. By examining numerical results for deflections, buckling loads and natural frequencies of FGM plates available in the literature Abrate [14,15] concluded that these quantities for a FGM plate are nearly proportional to those of its homogeneous counterpart with the proportionality factor depending upon through-the-thickness variation of the elastic moduli. By searching for similarities between differential equations for eigenvalues of a simply supported FGM polygonal plate and those of the frequency of a clamped membrane geometrically identical to the midsurface of the plate, Cheng and Batra [16] presented a relation between the eigenvalues (critical load or vibration frequency) of the FGM plate and those of the membrane. Zhang and Zhou [17] defined a physical neutral surface that is different from the geometric midsurface of a plate, omitted the stretching-bending coupling terms in the governing equations of FGM plates and derived a similarity relation between the buckling load and the natural frequency of thin rectangular simply supported FGM plates and those of homogeneous plates. Ma and Wang [18] used the TSDT to express axisymmetric bending deflection and buckling loads of FGM circular plates in terms of those of thin circular plates. Li and Liu [19] derived analytical formulations for the proportional coefficients between deflections, buckling loads and natural frequencies of FGM Euler–Bernoulli beams (EBBs) with arbitrary variation of Young's modulus in the thickness direction and those of the corresponding homogeneous beams; they proved that the proportionality factors are independent of BCs.

Here we find relationships between critical buckling loads of FGM TBs and those of the corresponding homogeneous and FGM EBBs. It is shown that critical buckling loads of FGM TBs can be found from those of the corresponding EBBs and two constants whose values depend upon the through-the-thickness variation of Young's modulus in the FGM TB. Whereas closed-form relations exist for C–C, S–S and C–F beams, the critical loads for C–S beams are solutions of similar transcendental equations with no closed-form relations among buckling loads for the FGM TB and the corresponding EBB.

2. Problem formulation

We consider a beam of uniform rectangular cross-section A , width b , depth h , length l , and made of an isotropic and linear elastic (Hookean) FGM, and use rectangular Cartesian coordinate axes with the x -axis along the geometric centroidal axis and the z -axis in the thickness direction to describe its deformations; Fig. 1. Furthermore, we assume that Young's modulus E and Poisson's ratio ν continuously vary in the thickness direction z according to the relation:

$$P(z) = P_b \psi_p(z) \quad (1)$$



Fig. 1. Schematic sketch of a FGM beam, and coordinate axes.

where $P_b = P(-h/2)$ and $P_t = P(h/2)$, respectively, denote values of P at the bottom and the top surfaces of the beam, and $\psi_p(z)$ is a smooth continuous function of z satisfying $\psi_p(-h/2)=1$ and $\psi_p(h/2)=P_t/P_b$.

The displacement field for the TB can be written as:

$$u(x, z) = (z - z_0)\varphi(x), \quad w(x, z) = w_0(x) \quad (2)$$

where $u(x, z)$ and $w(x, z)$ are, respectively, the axial and the transverse displacements of a point of the beam, $w_0(x)$ is the deflection, $\varphi(x)$ the angle of rotation of a cross-section about the y -axis, and z_0 the z -coordinate of the neutral surface, i.e.,

$$z_0 = \frac{B_1}{A_1}, \quad A_1 = \int_A E dA, \quad B_1 = \int_A zE dA \quad (3)$$

For infinitesimal deformations, the strain–displacement and the constitutive relations are

$$\varepsilon_x(x, z) = (z - z_0) \frac{d\varphi}{dx}, \quad \gamma_{xz}(x, z) = \left(\frac{dw_0}{dx} + \varphi \right) \quad (4)$$

$$\sigma_x = E(z - z_0) \frac{d\varphi}{dx}, \quad \tau_{xz} = \frac{E}{2(1 + \nu)} \left(\frac{dw_0}{dx} + \varphi \right) \quad (5)$$

Thus the axial and the transverse forces, F_A and F_s , and the moment, M , on a cross-section are given by:

$$\begin{aligned} F_A &= \int_A \sigma_x dA = (B_1 - z_0 A_1) \frac{d\varphi}{dx} = 0, \quad M = \int_A \sigma_x z dA \\ &= (D_1 - B_1^2/A_1) \frac{d\varphi}{dx}, \quad F_s = \int_A \tau_{xz} dA = C_1 \left(\frac{dw_0}{dx} + \varphi \right), \end{aligned} \quad (6)$$

where

$$\begin{aligned} A_1 &= AE_b \phi_1, \quad B_1 = AhE_b \phi_2, \quad C_1 = \kappa \int_A \frac{E}{2(1 + \nu)} dA \\ &= \kappa AE_b \phi_4, \quad D_1 = \int_A z^2 E dA = IE_b \phi_3, \end{aligned} \quad (7)$$

$$\phi_1 = \frac{1}{A} \int_A \psi_E(z) dA, \quad \phi_2 = \frac{1}{Ah} \int_A \psi_E(z) z dA, \quad (8a)$$

$$\phi_3 = \frac{1}{I} \int_A \psi_E(z) z^2 dA, \quad \phi_4 = \frac{1}{A} \int_A \frac{\psi_E(z)}{2(1 + \nu)} dA \quad (8b)$$

The shear correction factor, κ , is taken to equal 5/6 for a rectangular cross-section, $I = bh^3/12$, and coefficients ϕ_i are non-dimensional.

For a beam made of a homogeneous material of Young's modulus, E_b , hereafter called the reference homogeneous beam, we get $\phi_1 = \phi_3 = 1$, $\phi_2 = 0$ and $\phi_4 = 1/[2(1 + \nu)]$. Assuming that ν is independent of z and

$$\psi_E = 1 + (r_E - 1)(\eta + 1/2)^n \quad (9)$$

where $r_E = E_t/E_b$, $\eta = z/h$ and the constant n is such that

$$\begin{aligned} \phi_1 &= 1 + \frac{r_E - 1}{n + 1}, \quad \phi_2 = \frac{n(r_E - 1)}{(n + 1)(n + 2)}, \\ \phi_3 &= 1 + \frac{3(r_E - 1)(n^2 + n + 2)}{(n + 1)(n + 2)(n + 3)} \end{aligned}$$

are well defined. For through-the-thickness exponential variation of E , the function $\psi_E(z)$ is assumed to be given by

$$\psi_E(z) = e^{\beta(\eta+1/2)} \quad (11)$$

with $\beta = \ln r_E$. Functions ϕ_1 , ϕ_2 , and ϕ_3 have the following values:

$$\phi_1 = \frac{1}{\beta} (r_E - 1), \quad \phi_2 = \frac{1}{\beta} \left[\frac{r_E + 1}{2} - \phi_1 \right], \quad \phi_3 = 3\phi_1 - \frac{24}{\beta} \phi_2 \quad (12)$$

We introduce following non-dimensional variables

$$(\xi, W, \delta) = (x, w_0, h)/l, \quad P = \frac{pl^2}{E_b I}, \quad c = \frac{1}{(\phi_3 - 12\phi_2^2/\phi_1)},$$

$$c_s = \frac{\delta^2}{12\kappa\phi_4} \quad (13)$$

where p is the axial compressive force applied at the ends. In Eq. (13) the dimensionless constant, c , represents the inhomogeneity parameter, and equals 1 for a homogeneous beam. The constant, c_s , related to the shear deformation, goes to zero with the decrease in δ or the height of the beam. The constant $c_s = 0$ for an FGM EBB.

Constitutive relations relating M and F_s to W and ϕ , and equilibrium equations are

$$M = \frac{E_b I}{c l} \frac{d\phi}{d\xi}, \quad F_s = \frac{E_b I}{c_s l^2} \left(\frac{dW}{d\xi} + \phi \right) \quad (14a, b)$$

$$\frac{dM}{dx} = F_s, \quad \frac{dF_s}{dx} = p \frac{d^2 w_0}{dx^2} \quad (15a, b)$$

Combining Eqs. (14) and (15) gives

$$\frac{c_s}{c} \frac{d^2 \phi}{d\xi^2} = \frac{dW}{d\xi} + \phi, \quad \frac{d^2 W}{d\xi^2} + \frac{d\phi}{d\xi} = c_s P \frac{d^2 W}{d\xi^2} \quad (16a, b)$$

Eliminating ϕ from Eq. (16a,b) yields

$$\frac{d^4 W}{d\xi^4} + \Lambda^2 \frac{d^2 W}{d\xi^2} = 0, \quad (17a)$$

where

$$\Lambda = \sqrt{cP/(1 - c_s P)}. \quad (17b)$$

For $c_s = 0$, Eq. (17a) reduces to

$$\frac{d^4 W_E}{d\xi^4} + cP_E \frac{d^2 W_E}{d\xi^2} = 0 \quad (18)$$

that governs deformations of an FGM EBB quantities for which are indicated by subscript 'E'. Setting $c = 1$ in Eq. (18) we obtain Eq. (19) for the reference homogeneous EBB.

$$\frac{d^4 W_E^*}{d\xi^4} + P_E^* \frac{d^2 W_E^*}{d\xi^2} = 0 \quad (19)$$

Here and below quantities with superscript star are for EBB made of the reference homogeneous material.

Boundary conditions (BCs) at simply supported, clamped and free edge, $x = 0$, of a FGM TB, respectively, are

$$W(0) = 0, W'''(0) = 0; W(0) = 0, \alpha W'''(0) + W'(0) = 0; W''(0) = 0, -\alpha W'''(0) = c_s P W'(0) \quad (20a-f)$$

where $\alpha = c_s(1 - c_s P)/c$ and $W' = dW/d\xi$. At a clamped edge of an EBB Eq. (20d) becomes $W_E'(0) = 0$, and at a free edge of an EBB Eq. (20f) is replaced by $W_E'''(0) = 0$.

3. Buckling loads of FGM TBs and EBBs

BCs for a simply supported edge of both homogeneous and FGM EBBs are the same as those for the FGM TB. However, BCs at clamped and free edges of the FGM TB need not be the same as those for the EBBs since $\phi \neq -dW/d\xi$ for the TB. Thus we first find eigenvalues of Eq. (17) for the appropriate BCs and then search for relations between buckling loads of FGM TBs and those of homogeneous EBBs. A general solution of Eq. (17) has four constants which are to be determined from two BCs at each edge of the beam. This gives an eigenvalue problem, and details for the C–S beam are given in Appendix A.

For the EBBs Eqs. (18) and (19) imply that for all BCs [19],

$$P_{Ecr} = P_{Ecr}^*/c \quad (21)$$

where P_{Ecr}^* and P_{Ecr} are, respectively, the critical buckling loads of the homogeneous and FGM EBBs of the same geometry and BCs. We note that the proportionality constant in Eq. (21) is determined by the spatial distribution of $E(z)$.

For the S–S FGM TB and the homogeneous EBB, $\Lambda_{cr} = \pi = \sqrt{P_{Ecr}^*}$. From the definition of Λ given in Eq. (17b), we get the following relation between the critical loads, P_{Tcr} and P_{Ecr}^* , of the FGM TB and the homogeneous EBB:

$$P_{Tcr} = \frac{P_{Ecr}^*}{c + c_s P_{Ecr}^*}, \quad \text{or } P_{Tcr} = \frac{P_{Ecr}}{1 + c_s P_{Ecr}} \quad (22)$$

Thus $P_{Tcr} < P_{Ecr}$. That is, the consideration of shear deformations decreases the critical buckling load. The constant c_s reflects effects of shear deformations on the critical load and is proportional to δ^2 . A relation similar to Eq. (22) for critical buckling loads of FGM circular columns with material properties varying in the radial direction was given by Huang and Li [5].

For a C–C beam the lowest eigenvalues of the FGM TB and the homogeneous EBB are given by $\Lambda_{min} = 2\pi$, and Eq. (22) again holds.

For a cantilever beam clamped at the edge $x = 0$ one can show that for both FGM TB and the homogeneous EBB, $\Lambda_{min} = \pi/2$, and Eq. (22) relates P_{Tcr} and P_{Ecr}^* .

For a beam clamped at one end, say $x = 0$, and simply supported at the other end, the condition of having nontrivial eigenvalues of the governing equation (e.g., see Appendix A) is

$$\tan \sqrt{\frac{cP}{1 - c_s P}} = \sqrt{cP(1 - c_s P)} \quad (23)$$

If we set $c = 1$ and $c_s = 0$, Eq. (23) reduces to the equation for finding the buckling load of a homogeneous EBB with C–S ends, which can be found in text books on Strength of Materials. It is obvious that Eq. (22) is not valid for the FGM TBs with C–S edges. The critical buckling load P_{Tcr} can be ascertained by numerically finding the minimum root of Eq. (23).

We have summarized in Table 1 relations between critical buckling loads for different beams. Here c_s^* is determined by setting $\phi_4 = 1/[2(1 + \nu)]$ in Eq. (13), and P_{Tcr}^* equals the critical load of the reference homogeneous TM.

It follows from Eq. (22) that for C–C, C–F and S–S FGM TBs, the critical buckling load will have an extreme value when the exponent n in Eq. (9) is a solution of

$$\frac{dc}{dn} + P^* \frac{dc_s}{dn} = 0.$$

This nonlinear algebraic equation can be numerically solved for n .

We note that a static mechanical problem has been studied and values of material parameters have been assumed to be temperature independent. Were we to consider the temperature dependence of material parameters then it will be better to analyze

Table 1

BCs	P_{Ecr}^*	P_{Ecr}/P_{Ecr}^*	P_{Tcr}/P_{Tcr}^*	P_{Tcr}/P_{Ecr}
S–S	π^2	$1/c$	$\frac{1 + \pi^2 c_s^*}{c + \pi^2 c_s}$	$\frac{c}{c + \pi^2 c_s}$
C–C	$4\pi^2$	$1/c$	$\frac{1 + 4\pi^2 c_s^*}{c + 4\pi^2 c_s}$	$\frac{c}{c + 4\pi^2 c_s}$
C–F	$\pi^2/4$	$1/c$	$\frac{1 + \pi^2 c_s^*/4}{c + \pi^2 c_s/4}$	$\frac{c}{c + \pi^2 c_s/4}$
C–S	$2.04\pi^3$	$1/c$	no closed form relation	

thermo-elastic deformations of the beam and find the temperature rise that buckles the beam. However, if the temperature of the beam is uniformly raised then the current analysis applies provided that values of material parameters at the final temperature are considered.

4. Numerical results and discussion

In order to show that Eqs. (22) and (23) accurately predict the critical buckling load of FGM TBs, for $\delta = l/h = 5$ and 10 we have compared, respectively, in Tables 2 and 3 predictions from these equations with results found by using the shooting method to numerically solve eigenvalue problems defined by differential Eqs. (16a) and (16b) and the pertinent BCs. The FGM beam is assumed to be composed of ceramic (alumina) and metal (aluminum) with $E_t = E_c = 380$ GPa and $E_b = E_m = 70$ GPa, respectively, and $\nu_t = \nu_b = 0.23$. Thus in the computation of numerical results Poisson's ratio is assumed to be constant. Young's modulus of the FGM beam is found from Eqs. (1) and (9). Values in the first row of the tables are obtained by numerically solving the eigenvalue problem, and those in 2nd row are predictions from Eq. (22) for C–C, S–S and C–F beams and from Eq. (23) for C–S beams. In the 3rd row we have listed values from Eq. (21) which are critical buckling loads of EBBs. Values in the 1st and the 2nd rows confirm

that there is very good agreement between predictions from Eqs. (22) and (23) and those found by using the shooting method. A comparison of values in the 2nd and 3rd rows in the two tables suggests that effects of shear deformations on the critical buckling loads not only depend on the slenderness ratio but also on the end constraints. We note that for $n = 0$, the beam material is homogeneous and is ceramic. Since $E_c/E_b = 38/7$, the critical buckling load for the ceramic beam is higher than that of the FGM beam in which ceramic on a part of the cross-section is replaced by a weaker mixture of ceramic and aluminum. This holds irrespective of the consideration of shear deformation effects.

We note that the critical buckling load of an FGM beam depends upon through-the-thickness variations of E and ν only through the parameters c and c_s . Thus results for any through-the-thickness variation of E and ν can be obtained from those for the power law variation provided that the two have the same values of c and c_s .

Here we have assumed the material properties to only vary in the z -direction. Alternatively, Young's modulus and Poisson's ratio could be assumed to continuously vary both in the x - and the z -directions. Qian and Batra [20] numerically found the variation of material parameters in the x - and the z -directions to optimize the fundamental frequency of free vibrations of a cantilever rectangular plate by using a higher-order shear and normal deformable plate theory [21].

Table 2
Dimensionless critical loads of FGM beams with length to thickness ratio $l/h = 5$.

BCs	n									
	0.0	0.5	1	2	5	7	10	100	$10^{11} (\infty)$	
C–C	154.35 ^a	103.22	80.498	62.614	50.384	47.332	44.267	31.231	28.433	
	154.35 ^b	103.22	80.498	62.614	50.384	47.332	44.267	31.231	28.433	
	214.31 ^d	138.93	106.82	83.355	70.491	67.598	64.207	44.262	39.478	
C–S	97.580 ^a	64.052	49.497	38.576	32.000	30.437	28.731	19.990	17.975	
	97.580 ^c	64.052	49.497	38.576	32.000	30.437	28.731	19.990	19.975	
	109.61 ^d	71.053	54.633	43.631	36.052	34.572	32.838	22.638	20.190	
S–S	48.835 ^a	31.967	24.687	19.245	16.024	15.265	14.427	10.020	8.9959	
	48.835 ^b	31.967	24.687	19.245	16.024	15.265	14.427	10.020	8.9959	
	53.578 ^d	34.731	26.705	20.838	17.623	16.899	16.052	11.066	9.8696	
C–F	13.213 ^a	8.5782	6.6002	5.1495	4.3445	4.1620	3.9501	2.7263	2.4340	
	13.213 ^c	8.5782	6.6002	5.1495	4.3445	4.1620	3.9502	2.7262	2.4340	
	13.394 ^d	8.6829	6.6763	5.2097	4.4057	4.2249	4.0129	2.7664	2.4674	

^a Eqs. (16a and 16b) by shooting method.

^b by Eq. (22).

^c by Eq. (23).

^d by Eq. (21).

Table 3
Dimensionless critical loads of FGM beams with length to thickness ratio $l/h = 10$.

BCs	n									
	0.0	0.5	1	2	5	7	10	100	$10^{11} (\infty)$	
C–C	195.34 ^a	127.87	98.749	76.980	64.096	61.062	57.708	40.081	35.984	
	195.34 ^b	127.87	98.749	76.980	64.096	61.062	57.708	40.081	35.984	
	214.31 ^d	138.93	106.82	83.355	70.491	67.598	64.207	44.262	39.478	
C–S	106.33 ^a	69.154	52.251	41.535	34.945	33.436	31.705	21.192	19.587	
	106.33 ^c	69.154	52.251	41.535	34.945	33.436	31.705	21.192	19.587	
	109.61 ^d	71.053	54.622	42.631	36.052	34.572	32.838	22.638	20.191	
S–S	52.309 ^a	33.996	26.171	20.416	17.192	16.459	15.612	10.784	9.6357	
	52.309 ^b	33.996	26.171	20.416	17.192	16.459	15.612	10.784	9.6357	
	53.578 ^d	34.731	26.705	20.838	17.623	16.899	16.052	11.066	9.8696	
C–F	13.349 ^a	8.6566	6.6570	5.1944	4.3903	4.2091	3.9969	2.7562	2.4589	
	13.349 ^c	8.6564	6.6571	5.1945	4.3902	4.2090	3.9969	2.7562	2.4589	
	13.395 ^d	8.6828	6.6763	5.2097	4.4057	4.2248	4.0129	2.7564	2.4674	

^a Eqs. (16a and 16b) by shooting method.

^b by Eq. (22).

^c by Eq. (23).

^d by Eq. (21).

5. Conclusions

For simply supported, clamped and clamped–free beams, we have found closed-form relations between critical buckling loads of functionally graded Timoshenko and Euler–Bernoulli beams and those of a homogeneous Euler–Bernoulli beam. However, no such relation exists for the clamped–simply supported beams. For these end conditions, an algebraic eigenvalue problem is derived to determine the critical buckling load of the FGM Timoshenko beam which is similar to that for finding the critical buckling load of a homogeneous Euler–Bernoulli beam with the same end constraints. As a result, the calculation of critical buckling loads of FGM Timoshenko beams is reduced to that of finding critical buckling loads of a homogeneous Euler–Bernoulli beam with the same geometry and end constraints in conjunction with the calculation of the two constants whose values depend upon the through-the-thickness variations of Young's modulus and Poisson's ratio. Predictions from these relations are shown to agree well with the critical buckling loads found by numerically solving the eigenvalue problem with the shooting method.

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Appendix A

For a C–S FGM TB, we derive below an expression for the critical buckling load. A general solution of differential Eq. (17) is

$$W(\xi) = \beta_1 \cos \Lambda \xi + \beta_2 \sin \Lambda \xi + \beta_3 \xi + \beta_4 \quad (\text{A1})$$

where $\beta_i (i = 1, 2, 3, 4)$ are arbitrary constants. Substitution of solution (A1) into boundary conditions (20) yields the following system of linear algebraic equations:

$$\beta_1 + \beta_4 = 0 \quad (\text{A2})$$

$$(1 - \alpha \Lambda^2) \Lambda \beta_2 + \beta_3 = 0 \quad (\text{A3})$$

$$\beta_1 \cos \Lambda + \beta_2 \sin \Lambda + \beta_3 + \beta_4 = 0 \quad (\text{A4})$$

$$\beta_1 \cos \Lambda + \beta_2 \sin \Lambda = 0. \quad (\text{A5})$$

Eqs. (A2)–(A5) have a non-trivial solution if and only if Λ satisfies

$$\tan \Lambda = (1 - c_s P) \Lambda \quad (\text{A6})$$

which is the same as Eq. (23) (recall the value of Λ listed in Eq. (17b)).

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