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Deformation produced by a simple tensile load in an isotropic elastic body

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ABSTRACT
It is shown that a simple tensile load produces a simple extension provided the empirical inequalities (Truesdell and Noll [1], eqn. 51.27) hold.

One form of the general constitutive equation for an unconstrained isotropic homogeneous elastic material (*Cf.* § 47 of [1]) is

$$\boldsymbol{T} = f_0 \boldsymbol{1} + f_1 \boldsymbol{B} + f_{-1} \boldsymbol{B}^{-1}, \tag{1}$$

where $\boldsymbol{T}$ is the Cauchy stress tensor, $\boldsymbol{B}$ is the left Cauchy–Green tensor with respect to an undistorted configuration, and the response coefficients f_α ($\alpha = -1, 0, 1$) are functions of the principal invariants of $\boldsymbol{B}$. In order that equation (1) describe a response that is physically reasonable, Truesdell and Noll ([1], eqn. 51.27) proposed that f_α satisfy the following inequalities.

$$f_0 \leqslant 0, \quad f_1 > 0, \quad f_1 \leqslant 0. \tag{2}$$

The Cauchy–Stress tensor for a simple tensile load along the x_3-axis of a rectangular Cartesian set of axes is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T \end{bmatrix} \tag{3}$$

where T is a positive constant. The constitutive relation (1) requires that $\boldsymbol{TB} = \boldsymbol{BT}$. This together with (3) gives that

$$B_{13} = B_{23} = 0$$

Thus

$$\boldsymbol{B} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{33} \end{bmatrix} \tag{4}$$

Since $\boldsymbol{B}$ is positive definite,

$$B_{33}>0\,, \quad C\equiv B_{11}B_{22}-B_{12}^2>0\,. \tag{5}$$

Substituting from (3) and (4) into (1), we conclude that

$$\begin{aligned} 0 &= \left(f_1-\frac{1}{C}f_{-1}\right)B_{12}\,, \\ 0 &= f_0+f_1B_{11}+\frac{1}{C}f_{-1}B_{22}\,, \\ 0 &= f_0+f_1B_{22}+\frac{1}{C}f_{-1}B_{11} \\ T &= f_0+f_1B_{33}+\frac{1}{B_{33}}f_1 \end{aligned} \tag{6}$$

Substraction of $(6)_3$ from $(6)_2$ yields

$$0=\left(f_1-\frac{1}{C}f_{-1}\right)(B_{11}-B_{22}) \tag{7}$$

When inequalities (2) hold, it follows from $(6)_1$ and (7) that

$$B_{12}=0\,, \quad B_{11}=B_{22}\,. \tag{8}$$

Thus B has the form

$$\boldsymbol{B}=\operatorname{diag}(\alpha^2v^2,\,\alpha^2v^2,\,v^2)\,, \tag{9}$$

where

$$\begin{aligned} v^2 &= B_{33}\,, \\ \alpha^2 &= B_{11}/B_{33}\,. \end{aligned} \tag{10}$$

In $(10)_2$ the positive definiteness of $\boldsymbol{B}$ has been used. $\boldsymbol{B}$ given by (9) corresponds to simple tension (*Cf.* [2], eqn. 44.1). That $\alpha<1$ follows from (9), (2) and the equation obtained by subtracting $(6)_2$ from $(6)_4$.

The constitutive relation for an incompressible isotropic elastic material is

$$\boldsymbol{T}=-p\boldsymbol{1}+f_1\boldsymbol{B}+f_{-1}\boldsymbol{B}^{-1}, \tag{11}$$

where p is an arbitrary hydrostatic pressure and f_1 and f_{-1} are functions of the first and second invariant of $\boldsymbol{B}$; the third invariant of $\boldsymbol{B}$ equals 1. The E inequalities, suggested as plausible by Truesdell (Eqn. (41.14) of [3]) in 1952, are

$$f_1>0\,, \quad f_{-1}\leqslant 0\,. \tag{12}$$

Proceeding as we did before, we obtain (4), $(6)_1$ and (7) and, in view of (12), we conclude that for incompressible isotropic elastic materials $\boldsymbol{B}$ should have the form given by (9) except that α now equals $v^{-\frac{3}{2}}$ since the determinant of $\boldsymbol{B}$ must be 1.

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