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Thermodynamics of non-simple elastic materials

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ABSTRACT

Elastic materials whose local state depends upon the first and second order gradients of the deformation, the temperature, its gradient and the time rate of change of the temperature are studied according to an inequality proposed by Green and Laws. It is shown that in such materials *either* thermal disturbances can propagate with finite speed in the linear theory, and the constitutive quantities do not depend upon the second order gradients of the deformation *or* the constitutive quantities may depend upon the second order gradients of the deformation and in the linear theory thermal disturbances do not propagate with finite speed. In the latter case the entropy inequality reduces to the Clausius-Duhem inequality.

Introduction

In [1] Green and Laws proposed an entropy inequality for the entire body and showed that for homogeneous bodies it reduces to the following inequality

$$\rho_0\dot{\eta} + \left(\frac{q_A}{\phi}\right)_{,A} - \rho_0\frac{r}{\phi} \geqq 0. \tag{1.1}$$

Here η is the specific entropy, ρ_0 is the mass density in the reference configuration, r is the supply per unit mass of the internal energy, $\boldsymbol{q}$ is the heat flux per unit surface area in the reference configuration, $\phi > 0$ is a constitutive quantity, a superposed dot indicates material time differentiation and $,A$ stands for differentiation with respect to coordinates X_A in the reference configuration. The entropy inequality (1.1) is more general than the Clausius-Duhem inequality in which ϕ is taken equal to the absolute temperature T. For simple heat conductors [1] and for simple elastic materials [2] studied according to the inequality (1.1) Green et al recovered results obtained earlier by Müller [3, 4] using his own inequality for supply free bodies. For these materials Green et al showed that ϕ is a function of T and $\dot{T}$ and assuming that

$$\frac{\partial \phi}{\partial \dot{T}} \neq 0 \tag{1.2}$$

they proved [1] that in the linear theory thermal disturbances can propagate with finite speed.

In [5] inequality (1.1) is used to study non-simple heat conductors whose local state depends upon the empirical temperature θ and all of its first and second order derivatives with respect to time and the space variable. For such isotropic heat conductors, it is shown that ϕ is a function of θ and all of its first order derivatives. Propagation of weak thermal disturbances is also studied and a uniqueness theorem for the linear theory is proved.

In this paper we study non-simple thermoelastic materials whose local state depends upon the empirical temperature θ, its gradient, the time rate of change of temperature, the deformation gradient $\boldsymbol{F}$ and its gradient and show that ϕ depends upon θ and $\dot{\theta}$. Furthermore, we show that *either* $\partial\phi/\partial\dot{\theta} = 0$, *or* the specific internal energy ε, η, $\boldsymbol{q}$ and the stress tensor $\boldsymbol{S}$ do not depend upon the second order gradients of the deformation. When $\partial\phi/\partial\dot{\theta} = 0$, thermal disturbances do not propagate with finite speed in the linear theory. Also the inequality (1.1) reduces to the Clausius-Duhem inequality. However, in this case, ε and η can depend upon the gradient of the deformation gradient. This differs from Gurtin's result [6] that the presence of higher order gradients of the deformation than the first in the constitutive equations for ε and η is ruled out by the Clausius-Duhem inequality. This is due to the fact that whereas in here $\dot{\theta}$ is included as a constitutive variable, Gurtin considers fields of deformation, temperature and temperature gradient defined over the entire body as the constitutive variable.

Non-simple elastic materials whose local state is characterized by the temperature, the deformation gradient and the first and second order gradients of the temperature and the deformation gradient were studied by Chen et al [7] according to an entropy inequality proposed by Gurtin and Williams [8].

2. Preliminaries

We refer the deformation of the continuum to a fixed set of rectangular Cartesian axes and study materials for which the following balance laws hold.

$$\begin{aligned} &\rho J = \rho_0, \quad J = \det \boldsymbol{F}, \qquad F_{iA} = x_{i,A}, \\ &\rho_0 \dot{v}_i = S_{iA,A} + \rho_0 b_i, \\ &\rho_0 \dot{\varepsilon} = -q_{A,A} + S_{iA}\dot{x}_{i,A} + \rho_0 r. \end{aligned} \tag{2.1}$$

Here $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$ gives the present position of the material particle that occupied place $\boldsymbol{X}$ in the reference configuration, ρ is the mass density at time t, $\boldsymbol{b}$ is the specific body force and S_{iA} is the Piola-Kirchoff stress tensor. Introducing the definition

$$\psi = \varepsilon - \eta\phi \tag{2.2}$$

and eliminating r from $(2.1)_5$ and (1.1) we obtain the following inequality

$$\rho_0(\dot{\psi} + \eta\dot{\phi}) - S_{iA}\dot{x}_{i,A} + \frac{q_A \phi_{,A}}{\phi} \leq 0. \tag{2.3}$$

Consider a material characterized by the five response functions $\hat{\psi}$, $\hat{\eta}$, $\hat{\phi}$, $\hat{\boldsymbol{S}}$ and $\hat{\boldsymbol{q}}$ which are assumed to be functions of the deformation gradient $\boldsymbol{F}$, its gradient $G_{iAB} = x_{i,AB}$, the empirical temperature θ, its gradient $g_A = \theta_{,A}$ and the time rate of change of temperature $\dot{\theta}$. Thus

$$\begin{aligned} \psi &= \hat{\psi}(\boldsymbol{F}, \boldsymbol{G}, \theta, \dot{\theta}, \boldsymbol{g}), & \eta &= \hat{\eta}(\boldsymbol{F}, \boldsymbol{G}, \theta, \dot{\theta}, \boldsymbol{g}), \\ \boldsymbol{S} &= \hat{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}, \theta, \dot{\theta}, \boldsymbol{g}), & \phi &= \hat{\phi}(\boldsymbol{F}, \boldsymbol{G}, \theta, \dot{\theta}, \boldsymbol{g}), \\ \boldsymbol{q} &= \hat{\boldsymbol{q}}(\boldsymbol{F}, \boldsymbol{G}, \theta, \dot{\theta}, \boldsymbol{g}). \end{aligned} \tag{2.4}$$

Assume that the response functions (2.4) are twice continuously differentiable functions of their arguments and that all quantities are referred to a reference configuration in which the body is homogeneous. Substitution of (2.4) into $(2.1)_{3,4}$ gives field equations for $\boldsymbol{x}$ and θ. Since the constitutive functions (2.4) are assumed to be functions of $\boldsymbol{G}$, the material is not simple in the sense of Noll [9].

3. Restrictions from the entropy inequality

Referring the reader to [1, 2, 5, 10] for details, we use an argument due to Coleman and Noll [10] and conclude that the following are necessary and sufficient conditions in order that every solution of the field equations satisfies (2.3).

$$\hat{\boldsymbol{S}} = \rho_0 \left(\frac{\partial \hat{\psi}}{\partial \boldsymbol{F}} + \hat{\eta} \frac{\partial \hat{\phi}}{\partial \boldsymbol{F}} \right) \tag{3.1}$$

$$\frac{\partial \hat{\psi}}{\partial \boldsymbol{G}} + \hat{\eta} \frac{\partial \hat{\phi}}{\partial \boldsymbol{G}} = \boldsymbol{0}, \tag{3.2}$$

$$\frac{\partial \hat{\psi}}{\partial \dot{\theta}} + \hat{\eta} \frac{\partial \hat{\phi}}{\partial \dot{\theta}} = 0,$$

$$\rho_0 \left(\frac{\partial \hat{\psi}}{\partial \boldsymbol{g}} + \hat{\eta} \frac{\partial \hat{\phi}}{\partial \boldsymbol{g}} \right) + \frac{\partial \hat{\phi}}{\partial \dot{\theta}} \frac{\hat{\boldsymbol{q}}}{\hat{\phi}} = \boldsymbol{0},$$

$$\frac{\hat{q}_{(A}}{\hat{\phi}} \frac{\partial \hat{\phi}}{\partial g_{B)}} = 0,$$

$$\frac{\hat{q}_{A)}}{\hat{\phi}} \frac{\partial \hat{\phi}}{\partial G_{i(BC}} = 0,$$

$$\rho_0 \left(\frac{\partial \hat{\psi}}{\partial \theta} + \hat{\eta} \frac{\partial \hat{\phi}}{\partial \theta} \right) \dot{\theta} + \frac{\hat{q}_{A)}}{\hat{\phi}} \frac{\partial \hat{\phi}}{\partial F_{i(B}} G_{iAB} + \frac{\partial \hat{\phi}}{\partial \theta} \frac{\hat{q}_A g_A}{\hat{\phi}} \leq 0.$$

Heretofore and hereafter the round parantheses around the indices indicate symmetrization about the indices A, B, C *etc.* Assuming that $\hat{\boldsymbol{q}} \neq \boldsymbol{0}$, we conclude from (3.5) and (3.6) that

$$\frac{\partial \hat{\phi}}{\partial \boldsymbol{g}} = \boldsymbol{0}, \qquad \frac{\partial \hat{\phi}}{\partial \boldsymbol{G}} = \boldsymbol{0},$$

and now from (3.2) that

$$\frac{\partial \hat{\psi}}{\partial \boldsymbol{G}} = \boldsymbol{0}. \tag{3.9}$$

Differentiation of (3.3) and (3.4) with respect to $\boldsymbol{G}$ and the use of (3.8) and (3.9) gives

$$\frac{\partial \hat{\eta}}{\partial \boldsymbol{G}} \frac{\partial \hat{\phi}}{\partial \dot{\theta}} = \boldsymbol{0}, \qquad \frac{1}{\hat{\phi}} \frac{\partial \hat{\phi}}{\partial \dot{\theta}} \frac{\partial \hat{\boldsymbol{q}}}{\partial \boldsymbol{G}} = \boldsymbol{0}. \tag{3.10}$$

Thus *either*

$$\frac{\partial \hat{\phi}}{\partial \dot{\theta}} = 0,$$

or

$$\frac{\partial \hat{\eta}}{\partial \boldsymbol{G}} = \boldsymbol{0}, \qquad \frac{\partial \hat{\boldsymbol{q}}}{\partial \boldsymbol{G}} = \boldsymbol{0}. \tag{3.12}$$

If (3.12) holds, then from (3.1), (3.9) and (2.2) we obtain

$$\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{G}} = \boldsymbol{0}, \qquad \frac{\partial \hat{\varepsilon}}{\partial \boldsymbol{G}} = \boldsymbol{0}. \tag{3.13}$$

(3.9), $(3.8)_2$, (3.12) and (3.13) imply that the left-hand side of inequality (3.7) is linear in $\boldsymbol{G}$ and since the inequality has to hold for all values of $\boldsymbol{G}$ for which the constitutive functions (2.4) are defined, therefore,

$$\frac{\partial \hat{\phi}}{\partial \boldsymbol{F}} = \boldsymbol{0}. \tag{3.14}$$

We now attempt to establish (3.14) even when (3.11) holds. However, in the remainder of this section, we do not commit ourselves to either of the two alternatives (3.11) and (3.12). We assume that the heat flux vanishes whenever the temperature gradient does i.e.

$$\hat{\boldsymbol{q}}(\boldsymbol{F}, \boldsymbol{G}, \theta, \dot{\theta}, \boldsymbol{0}) = \boldsymbol{0}. \tag{3.15}$$

In view of (3.15) we write $\hat{\boldsymbol{q}}$ as

$$\hat{q}_A = -K_{AB}(\boldsymbol{F}, \boldsymbol{G}, \theta, \dot{\theta}, \boldsymbol{g}) g_B. \tag{3.16}$$

We note that the left-hand side of (3.7) has its maximum value namely zero in a process in which $\dot{\theta} = 0$, $\boldsymbol{g} = \boldsymbol{0}$, usually called equilibrium. This definition of an equilibrium process is slightly more general than that given in [7] according to which $\boldsymbol{G}$ would also have to be a zero tensor. The necessary conditions for the left-hand side of (3.7) to be maximum are that

$$\left.\frac{\partial \hat{\psi}}{\partial \dot{\theta}}\right|_E + \hat{\eta}|_E \left.\frac{\partial \hat{\phi}}{\partial \dot{\theta}}\right|_E = 0, \tag{3.17}$$

$$K_{A)C}|_E \left.\frac{\partial \hat{\phi}}{\partial F_{i(B}}\right|_E G_{iAB} = 0, \tag{3.18}$$

$$\left.\frac{\partial^2\hat{\psi}}{\partial\theta\partial\dot{\theta}}\right|_E + \left.\frac{\partial\hat{\eta}}{\partial\dot{\theta}}\right|_E \left.\frac{\partial\hat{\phi}}{\partial\theta}\right|_E + \hat{\eta}|_E \left.\frac{\partial^2\hat{\phi}}{\partial\theta\partial\dot{\theta}}\right|_E \leqq 0,$$

$$\left[\left.\frac{\partial K_{A)C}}{\partial g_D}\right|_E \left.\frac{\partial\hat{\phi}}{\partial F_{i(B}}\right|_E G_{iAB} + \left.\frac{\partial\hat{\phi}}{\partial\theta}\right|_E K_{CD}|_E\right]\xi_C\xi_D \geqq 0, \tag{3.20}$$

where ξ is any vector and the index E implies that the quantity is evaluated in an equilibrium process. It follows from (3.20) and (3.18) that

$$\left.\frac{\partial\hat{\phi}}{\partial\theta}\right|_E K_{AB}|_{E^*}\,\xi_A\xi_B \geqq 0, \tag{3.21}$$

$$K_{A)C}|_{E^*} \left.\frac{\partial\hat{\phi}}{\partial F_{i(B}}\right|_E = 0,$$

where the index E^* signifies that the quantity is evaluated in that subclass of equilibrium processes for which $\mathbf{G} = \mathbf{0}$. Assuming that

$$\left.\frac{\partial\hat{\phi}}{\partial\theta}\right|_E > 0,$$

we conclude from (3.21) that $K_{AB}|_{E^*}$ is positive semidefinite. Assuming that it is positive definite, we conclude from (3.22) that

$$\left.\frac{\partial\hat{\phi}}{\partial \mathbf{F}}\right|_E = \mathbf{0}. \tag{3.24}$$

Thus $\hat{\phi}|_E$ is a function of θ only. (3.24) and (3.20) imply that $K_{CD}|_E$ is positive semi-definite. One can obtain from (3.17), (3.9), (3.1), (3.3), (2.2) and (3.24) that

$$d\hat{\eta}|_E = \frac{1}{\hat{\phi}|_E}\left[d\hat{\varepsilon}|_E - \frac{\hat{S}|_E \cdot dF}{\rho_0}\right]$$

which may be interpreted as Gibb's equation for non-simple elastic materials studied here. (3.3), (3.19) and (3.23) give the following:

$$\left.\frac{\partial\hat{\eta}}{\partial\dot{\theta}}\right|_E \leqq \left.\frac{\partial\hat{\eta}}{\partial\theta}\right|_E \left.\frac{\partial\hat{\phi}}{\partial\dot{\theta}}\right|_E \Bigg/ \left.\frac{\partial\hat{\phi}}{\partial\theta}\right|_E$$

4. Remarks

Summarizing the preceeding results we note that should $\partial\hat{\phi}/\partial\dot{\theta} = 0$, then from (3.8) and (3.24), it follows that $\phi = \hat{\phi}(\theta)$ so that the entropy inequality (1.1) reduces to the

Clausius-Duhem inequality. The material is characterized by the three response functions $\hat{\psi}$, $\hat{\eta}$ and $\hat{\boldsymbol{q}}$. The response functions $\hat{\eta}$, $\hat{\varepsilon}$ and $\hat{\boldsymbol{q}}$ depend upon $\boldsymbol{G}$ so that the material is non-simple. This suggests that if one includes $\dot{\theta}$ in the list of local state variables then the Clausius-Duhem inequality does not rule out the possibility of spatial interaction. Since $\partial\hat{\phi}/\partial\dot{\theta} \neq 0$ is a necessary condition [1] for the propagation of thermal disturbances with finite speed in the linear theory, therefore, when $\partial\hat{\phi}/\partial\dot{\theta} = 0$, thermal disturbances would not propagate with finite speed in the linear theory.

When $\partial\hat{\phi}/\partial\dot{\theta} \neq 0$, it follows from (3.8), (3.9) and (3.14) that

$$\phi = \hat{\phi}(\theta, \dot{\theta}),$$

$$\psi = \hat{\psi}(\boldsymbol{F}, \theta, \dot{\theta}, \boldsymbol{g}).$$

The two functions $\hat{\phi}$ and $\hat{\psi}$ characterize the material since $\hat{\boldsymbol{S}}$, $\hat{\eta}$ and $\hat{\boldsymbol{q}}$ are determined respectively, by (3.1), (3.3) and (3.4). The theory of non-simple elastic materials reduces to that of the simple elastic materials studied in [2].

Another class of materials for which one gets results similar to that given by $(3.10)_2$ is the class of heat conductors which exhibit infinitesimal memory of the temperature history. Such materials are studied in [11] wherein it is assumed that the local state depends upon θ, $\dot{\theta}$, $\ddot{\theta}, \ldots, \overset{(N)}{\theta}$, $\theta_{,A}$ and Müller's entropy inequality is used. If one studies these materials according to the inequality (1.1) one obtains that either

$$\frac{\partial\hat{\phi}}{\partial\dot{\theta}} = 0,$$

or

$$\frac{\partial\hat{\boldsymbol{q}}}{\partial\overset{(N)}{\theta}} = 0,$$

provided $N \geqq 2$.

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