Journal of Elasticity **39**: 265–271, 1995.

Saint-Venant's Principle for Linear Elastic Porous Materials

R. C. BATRA
Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0219, USA

and

J. S. YANG
Department of Mechanical Engineering, Rensselaer Polytechnic Institute, Troy, NY 12180, USA

Received 15 February 1995

Abstract. Toupin's version of the Saint-Venant's principle in linear elasticity is generalized to the case of linear elastic porous materials. That is, it is shown that, for a straight prismatic bar made of a linear elastic material with voids and loaded by a self-equilibrated system of forces at one end only, the internal energy stored in the portion of the bar which is beyond a distance s from the loaded end decreases exponentially with the distance s.

Introduction

Mathematical versions of Saint-Venant's principle in linear elasticity due to Sternberg, Knowles, Zanaboni, Robinson and Toupin have been discussed by Gurtin [1] in his monograph. Later developments of the principle for Laplace's equation, isotropic, anisotropic, and composite plane elasticity, three-dimensional problems, nonlinear problems, and time-dependent problems are summarized in the review articles by Horgan and Knowles [2] and by Horgan [3]. For a linear elastic homogeneous prismatic body of arbitrary length and cross-section loaded on one end only by an arbitrary system of self-equilibrated forces, Toupin [4] showed that the elastic energy $U(s)$ stored in the part of the body which is beyond a distance s from the loaded end satisfies the inequality

$$U(s) \leqslant U(0)\exp[-(s-l)/s_c(l)]\,. \tag{1}$$

The characteristic decay length $s_c(l)$ depends upon the maximum and the minimum elastic moduli of the material and the smallest nonzero characteristic frequency of the free vibration of a slice of the cylinder of length l. By using Ericksen's [5] estimate for the norm of the stress tensor in terms of the strain energy density, one can show that $s_c(l)$ depends on the maximum elastic modulus and not on the minimum elastic modulus.

Inequalities similar to (1) have been obtained by Berglund [6] for linear elastic micropolar prismatic bodies, by Batra [7–9] for non-polar and micropolar linear elastic helical bodies and prismatic bodies of linear elastic materials with microstructure, and by Batra and Yang [10] for linear piezoelectric materials. Herein we prove a similar result for a straight prismatic body made of a linear elastic material with voids.

We assume that the cross-sections are materially uniform in the sense that one cross-section can be obtained from the other by a rigid body motion. Thus the material properties are independent of the axial coordinate of the point. Ericksen [5] has discussed material uniformity in more general terms.

Governing Equations for Linear Elastic Porous Materials

Let the finite spatial region occupied by the linear elastic porous body with voids be V, the boundary surface of V be S, the unit outward normal of S be n_i, and S be partitioned as

$$S_u \cup S_T = S\ ,\quad S_u \cap S_T = \emptyset. \tag{2}$$

The governing equations without body sources and boundary conditions for quasistatic deformations of the body in rectangular Cartesian coördinates are [11, 12]

$$\begin{aligned}
T_{ij,i} &= 0\,, \quad h_{i,i} + g = 0 \quad \text{in} \quad V,\\
T_{ij} &= \frac{\partial W}{\partial S_{ij}} = C_{ijkl}S_{kl} + B_{ij}\phi + D_{ijk}\phi_{,k} \quad \text{in} \quad V,\\
g &= -\frac{\partial W}{\partial \phi} = -B_{ij}S_{ij} - \xi\phi - d_i\phi_{,i} \quad \text{in} \quad V,\\
h_i &= \frac{\partial W}{\partial \phi_{,i}} = D_{kli}S_{kl} + d_i\phi + A_{ij}\phi_{,j} \quad \text{in} \quad V,\\
S_{ij} &= \tfrac{1}{2}(u_{j,i} + u_{i,j}) \quad \text{in} \quad V,\\
u_i &= \bar{u}_i \quad \text{on} \quad S_u, \quad n_iT_{ij} = \bar{t}_j \quad \text{on} \quad S_T, \quad n_ih_i = \bar{h} \quad \text{on} \quad S_h,
\end{aligned} \tag{3}$$

where u_i is the displacement, T_{ij} the stress tensor, S_{ij} the strain tensor, ϕ the change in volume fraction, h_i the equilibrated stress vector, and g the intrinsic equilibrated body force. S_u and S_T are parts of the boundary S on which mechanical displacement and traction are prescribed as $\bar{u}_i$ and $\bar{t}_i$, respectively; and the normal component of the equilibrated stress vector n_ih_i is prescribed as $\bar{h}$ on S. Throughout this paper, a repeated index implies summation over the range of the index, and a comma followed by an index j stands for partial differentiation with respect to x_j. $W(S_{ij}, \phi, \phi_{,i})$ is the internal energy density function given by

$$\begin{aligned}
W &= \tfrac{1}{2}C_{ijkl}S_{ij}S_{kl} + \tfrac{1}{2}\xi\phi^2 + \tfrac{1}{2}A_{ij}\phi_{,i}\phi_{,j}\\
&\quad + B_{ij}S_{ij}\phi + D_{ijk}S_{ij}\phi_{,k} + d_i\phi\phi_{,i},
\end{aligned} \tag{4}$$

which is assumed to be a positive definite, homogeneous quadratic function of the ten variables S_{ij}, ϕ, $\phi_{,i}$ [10, 11]. To save some writing we denote the ordered triplet $(S_{ij}, \phi, \phi_{,i})$ by $\mathbf{\Gamma}$ and write W as

$$W = \tfrac{1}{2}\mathbf{\Gamma} \cdot \mathbf{E}\mathbf{\Gamma}\,. \tag{5}$$

Thus $\mathbf{E}$ is a linear transformation from a 10-dimensional linear space into a 10-dimensional linear space. Because of the positive definiteness of W

$$\frac{\partial W}{\partial \mathbf{\Gamma}} \cdot \frac{\partial W}{\partial \mathbf{\Gamma}} = \mathbf{E}\mathbf{\Gamma} \cdot \mathbf{E}\mathbf{\Gamma} = \mathbf{\Gamma} \cdot \mathbf{E}^2\mathbf{\Gamma} \leqslant a_M \mathbf{\Gamma} \cdot \mathbf{E}\mathbf{\Gamma} = 2a_M W, \tag{6}$$

where a_M is the supremum of the eigenvalues of $\mathbf{E}$.

Formulation of the Problem

Consider an unstressed prismatic bar with materially uniform cross-sections and made of a linear elastic porous material. Introduce a fixed rectangular Cartesian coordinate system so that in the unstressed reference configuration the x_3-axis coincides with the axis of the bar, one end is contained in the plane $x_3 = 0$ and for points in the bar $x_3 \geqslant 0$. Since the cross-sections of the bar are assumed to be materially uniform, $\mathbf{E}$ depends only on x_1 and x_2. Hence

$$W = W(S_{ij}, \phi, \phi_{,i}, x_A), \quad A = 1, 2 \tag{7}$$

in which W is a homogeneous quadratic function of the indicated variables except x_A.

An infinitesimal rigid body displacement is described by a uniform translation c_i and a rotation $b_{ji} = -b_{ij}$. The displacements associated with a rigid body displacement are

$$w_i = c_i + b_{ji}x_j. \tag{8}$$

Thus if

$$v_i = u_i + w_i, \tag{9}$$

then

$$S_{ij}(\mathbf{v}) = S_{ij}(\mathbf{u}), \tag{10}$$

and $W(S_{ij}, \phi, \phi_{,i}, x_A)$ is unchanged.

The equations and boundary conditions for quasistatic deformations of the prismatic bar are

$$\left(\frac{\partial W}{\partial S_{ij}}\right)_{,i} = 0, \quad \left(\frac{\partial W}{\partial \phi_{,i}}\right)_{,i} - \frac{\partial W}{\partial \phi} = 0 \quad \text{in} \quad V,$$
$$n_i T_{ij} = \bar{t}_j, \quad n_i h_i = 0 \quad \text{on} \quad S, \tag{11}$$

where we have assumed that $S_T = S$. We are interested in the case when the part $x_3 = 0$ of the boundary S is loaded and the remainder of the boundary is traction free; hence $\bar{t}_j$ is nonzero only at $x_3 = 0$. In order that there exists a solution to (11), the applied loads must be self-equilibrated and must satisfy

$$\int_{C_0} \bar{t}_i \, \mathrm{d}S = 0, \quad \int_{C_0} \epsilon_{lkj} x_j \bar{t}_k \, \mathrm{d}S = 0. \tag{12}$$

Here moments are taken with respect to the origin, ϵ_{ijk} is the alternating tensor, and C_s is the cross-section of the body contained in the plane $x_3 = s$. With the definition

$$U(s) = \int_{x_3 \geqslant s} W \, \mathrm{d}V, \tag{13}$$

we state and prove below the

THEOREM. *If a prismatic body made of a linear elastic porous material and with materially uniform cross-sections is loaded on C_0 by a self-equilibrated force system, then*

$$U(s) \leqslant U(0) \exp[-(s-l)/s_c(l)], \tag{14}$$

where

$$s_c(l) = 2(a_M/\lambda_0(l))^{1/2}, \tag{15}$$

$\lambda_0(l)$ is the smallest nonzero eigenvalue of the following eigenvalue problem

$$-\left(\frac{\partial W}{\partial S_{ij}}\right)_{,i} = \lambda u_j, \quad -\left(\frac{\partial W}{\partial \phi_{,i}}\right)_{,i} + \frac{\partial W}{\partial \phi} = \lambda \phi \quad \text{in} \quad V,$$
$$n_i T_{ij} = 0, \quad n_i h_i = 0 \quad \text{on} \quad S, \tag{16}$$

for a slice of the prismatic body of axial length l. In (16) V is the region of the slice between $x_3 = s$ and $x_3 = s + l$, S is the total boundary surface of V.

Proof of the Theorem. Recalling (13) and W is a homogeneous quadratic function of the indicated variables except x_A, we have by Euler's theorem

$$\begin{aligned} U(s) &= \frac{1}{2}\int_{x_3 \geqslant s} \left(\frac{\partial W}{\partial S_{ij}} S_{ij} + \frac{\partial W}{\partial \phi}\phi + \frac{\partial W}{\partial \phi_{,i}}\phi_{,i}\right) \mathrm{d}V \\ &= \frac{1}{2}\int_{x_3 \geqslant s} \left(\frac{\partial W}{\partial S_{ij}} u_{j,i} + \frac{\partial W}{\partial \phi}\phi + \frac{\partial W}{\partial \phi_{,i}}\phi_{,i}\right) \mathrm{d}V \\ &= \frac{1}{2}\int_{C_s} \left(n_i \frac{\partial W}{\partial S_{ij}} u_j + n_i \frac{\partial W}{\partial \phi_{,i}}\phi\right) \mathrm{d}S \\ &= -\frac{1}{2}\int_{C_s} \left(\frac{\partial W}{\partial S_{3j}} u_j + \frac{\partial W}{\partial \phi_{,3}}\phi\right) \mathrm{d}S, \end{aligned} \tag{17}$$

where we have used the strain-displacement relation $(3)_6$, the divergence theorem and $n_k = -\delta_{3k}$ on C_s.

Using the inequality

$$2\int_V fh\,\mathrm{d}V \leqslant \alpha\int_V f^2\,\mathrm{d}V + \frac{1}{\alpha}\int_V h^2\,\mathrm{d}V, \tag{18}$$

which holds for $\alpha > 0$ and is a consequence of the Schwarz and the geometric-arithmetic mean inequalities (e.g. see Toupin [4]), we obtain

$$-\frac{1}{2}\int_{C_s}\frac{\partial W}{\partial S_{3j}}u_j\,\mathrm{d}S \leqslant \frac{1}{4}\left(\alpha_1\int_{C_s}\frac{\partial W}{\partial S_{3j}}\frac{\partial W}{\partial S_{3j}}\,\mathrm{d}S + \frac{1}{\alpha_1}\int_{C_s}u_ju_j\,\mathrm{d}S\right). \tag{19}$$

Similarly

$$-\frac{1}{2}\int_{C_s}\frac{\partial W}{\partial \phi_{,3}}\phi\,\mathrm{d}S \leqslant \frac{1}{4}\left(\alpha_2\int_{C_s}\frac{\partial W}{\partial \phi_{,3}}\frac{\partial W}{\partial \phi_{,3}}\,\mathrm{d}S + \frac{1}{\alpha_2}\int_{C_s}\phi^2\,\mathrm{d}S\right), \tag{20}$$

and hence

$$\begin{aligned} U(s) \leqslant \frac{1}{4}&\left[\beta\int_{C_s}\left(\frac{\partial W}{\partial S_{ij}}\frac{\partial W}{\partial S_{ij}} + \frac{\partial W}{\partial \phi_{,i}}\frac{\partial W}{\partial \phi_{,i}} + \frac{\partial W}{\partial \phi}\frac{\partial W}{\partial \phi}\right)\mathrm{d}S\right. \\ &\left.+ \frac{1}{\beta}\int_{C_s}(u_ju_j + \phi^2)\,\mathrm{d}S\right], \end{aligned} \tag{21}$$

where we have set $\alpha_1 = \alpha_2 = \beta$. Substituting from (6) into (21) results in

$$U(s) \leqslant \frac{1}{4}\left[\beta\int_{C_s}2a_MW\,\mathrm{d}S + \frac{1}{\beta}\int_{C_s}(u_ju_j + \phi^2)\,\mathrm{d}S\right]. \tag{22}$$

Integration of both sides of (22) with respect to x_3 from $x_3 = s$ to $x_3 = s + l$ for some $l > 0$ and setting

$$\frac{1}{l}\int_s^{s+l} U(y)\,\mathrm{d}y = Q(s,l) \tag{23}$$

gives

$$Q(s,l) \leqslant \frac{\beta a_M}{2l}\int_{C_{s,l}} W\,\mathrm{d}V + \frac{1}{4\beta l}\int_{C_{s,l}}(u_ju_j + \phi^2)\,\mathrm{d}S, \tag{24}$$

in which

$$\begin{aligned} C_{s,l} &\equiv \{\mathbf{x} : \mathbf{x} \in V, s \leqslant x_3 \leqslant s + l\} \\ &= \text{portion of the prismatic body between the planes} \\ &\qquad x_3 = s \text{ and } x_3 = s + l. \end{aligned} \tag{25}$$

In order to bound the last integral on the right-hand side of (24) by an integral of W, we consider the eigenvalue problem (16) on $C_{s,l}$. By taking the inner product of $(16)_1$ with u_j and $(16)_2$ with ϕ, adding the respective sides of the resulting equations and integrating them over $C_{s,l}$, using the divergence theorem and the boundary conditions $(16)_{3,4}$, we obtain

$$\lambda = \frac{2\int_{C_{s,l}} W\,\mathrm{d}V}{\int_{C_{s,l}}(u_j u_j + \phi\phi)\,\mathrm{d}V}. \tag{26}$$

Since $W = 0$ for a rigid body displacement, the smallest eigenvalue is zero. In order to eliminate the rigid body displacement and thereby the possibility of zero eigenvalue we consider smooth fields v_i and ϕ that satisfy

$$\int_{C_{s,l}} (v_j v_j + \phi^2)\,\mathrm{d}V \neq 0, \quad \int_{C_{s,l}} v_j\,\mathrm{d}V = 0, \quad \int_{C_{s,l}} \epsilon_{ijk} x_j v_k\,\mathrm{d}V = 0. \tag{27}$$

As shown by Toupin [4], for a given u_i we can choose w_i in (9) such that v_i satisfies (27). Thus the lowest eigenvalue $\lambda_0(l)$ will satisfy the inequality

$$0 < \lambda_0(l) \leqslant \frac{2\int_{C_{s,l}} W\,\mathrm{d}V}{\int_{C_{s,l}}(v_j v_j + \phi\phi)\,\mathrm{d}V}. \tag{28}$$

Substitution from (28) into (24) results in the following:

$$Q(s,l) \leqslant \frac{s_c(l)}{l} \int_{C_{s,l}} W\,\mathrm{d}V, \tag{29}$$

in which

$$s_c(l) = \tfrac{1}{2}\beta a_M + \frac{2}{\lambda_0 \beta}. \tag{30}$$

We choose $\beta = 2/(a_M\lambda_0)^{1/2}$ so that $s_c(l)$ takes on the minimum value

$$s_c(l) = 2(a_M/\lambda_0)^{1/2}. \tag{31}$$

Differentiating (23) with respect to s yields

$$\frac{\mathrm{d}Q}{\mathrm{d}s} = \frac{1}{l}[U(s+l) - U(s)] = -\frac{1}{l}\int_{C_{s,l}} W\,\mathrm{d}V. \tag{32}$$

This when combined with (29) results in

$$s_c(l)\frac{\mathrm{d}Q}{\mathrm{d}s} + Q \leqslant 0. \tag{33}$$

Integrating (33) and using

$$U(s+l) \leqslant Q(s,l) \leqslant U(s), \tag{34}$$

which follows from the observation that $U(s)$ is a nonincreasing function of s, we arrive at

$$\frac{U(s_2+l)}{U(s_2)} \leqslant \exp[-(s_2-s_1)/s_c(l)]. \tag{35}$$

The choice $s_1 = 0$ and $s_2 = s - l$ gives the desired inequality (14).

REMARKS. Even though inequality (14) ensures that the energy stored in the bar beyond a distance s from the loaded end decreases exponentially with the distance s, it is difficult to find the optimum decay rate unless one considers specific cross-sections. This is an inherent weakness of Toupin's version of the Saint-Venant principle. The porosity affects the decay rate since a_M in (15) and λ_0 given by (26) and (28) depend upon it; these effects can not be delineated unless one considers a specific material and a simple cross-section of the prismatic body. Thus it is hard to quantify the effect of porosity on the decay rate of the energy.

References

1. M. E. Gurtin, *The Linear Theory of Elasticity*, Handbuch der Physik, Vol. VIa/2, ed. C. Truesdell. Berlin/Heidelberg/New York: Springer-Verlag (1972).
2. C. O. Horgan and J. K. Knowles, Recent developments concerning Saint-Venant's principle. In: J. W. Hutchinson and T. Y. Wu (eds), *Advances in Applied Mechanics*, Vol. 23. New York: Academic Press (1983) pp. 179–269.
3. C. O. Horgan, Recent developments concerning Saint-Venant's principle: an update. *Appl. Mech. Rev.* 42(11) (1989) 295–303.
4. R. A. Toupin, Saint-Venant's principle. *Arch. Rat'l Mechanics Anal.* 18 (1965) 83–96.
5. J. L. Ericksen, Uniformity in shells. *Arch. Rat'l Mechanics Anal.* 37 (1970) 73–84.
6. K. Berglund, Generalization of Saint-Venant's principle to micropolar continua, *Arch. Rat'l Mechanics Anal.* 64 (1977) 317–326.
7. R. C. Batra, Saint-Venant's principle for a helical spring. *Trans. ASME, J. Appl. Mechs.* 45 (1978) 297–311.
8. R. C. Batra, Saint-Venant's principle for a micropolar helical body. *Acta Mechanica* 42 (1982) 99–109.
9. R. C. Batra, Saint-Venant's principle in linear elasticity with microstructure. *Journal of Elasticity* 13 (1983) 165–173.
10. R. C. Batra and J. S. Yang, Saint-Venant's principle in linear piezoelectricity. *Journal of Elasticity* 38 (1995) 209–218.
11. S. C. Cowin, and J. W. Nunziato, Linear elastic materials with voids. *Journal of Elasticity* 13 (1983) 125–147.
12. D. Iesan, A theory of thermoelastic materials with voids. *Acta Mechanica* 60 (1986) 67–89.