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Energy-Momentum Tensors in Nonsimple Elastic Dielectrics

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Abstract. We use Noether's theorem to derive energy-momentum tensors for a simple elastic material, a nonsimple elastic material of grade two, a simple elastic dielectric and a nonsimple elastic dielectric of grade two. The technique is easily extendable to a nonsimple elastic dielectric of any grade.

Introduction

The concept of the force acting on a defect, e.g., an impurity, vacant lattice site, dislocation, inclusion, void or a crack, is related, in a broad sense, to the notion of an inhomogeneity. Eshelby [1] showed that the force on a defect in an elastic body equals the integral of an energy-momentum tensor over a closed surface enclosing only this defect; subsequently he [2] derived the energy-momentum tensor for a second-grade elastic material which is nonsimple according to Noll [3]. Recently Maugin and Trimarco [4] used a general variational principle and the concept of pseudomomentum to derive the energy-momentum tensor for a second-grade elastic material. By invoking the ideas of a *basic reference* configuration, Maugin and Epstein [5], Epstein and Maugin [6], Maugin and Trimarco [4] and Maugin et al. [7] have derived energy-momentum tensors for simple electromagnetic elastic materials.

Here we use Noether's theorem [8] to derive the energy-momentum tensor for simple and nonsimple elastic dielectrics. This approach requires considerably less work as compared to the techniques employed previously and is easily extendable to electromagnetic materials of grade N. We note that Noether's theorem has been used by Knowles and Sternberg [9] to derive conservation laws in linearized and finite elastostatics, by Golebiewska-Herrmann [10] to obtain a unified formulation leading to all conservation laws of continuum mechanics, by Pak and Herrmann [11] to obtain conservation laws and the material momentum tensor for an elastic dielectric, and by Maugin [12] to obtain pseudo-momentum and Eshelby's material tensor in electromagneto-mechanical framework. Maugin [12] noted that the work can be extended to nonsimple hyperelastic solids but did not provide any results. Maugin and Trimarco [13] have applied Noether's theorem to study Eshelby's tensor for nematic liquid crystals.

Noether's Theorem

We state a version of Noether's theorem [8] appropriate for our work; according to Soper [14] earlier versions of the theorem were given by Hamel [15]. For fields $\phi_J(\mathbf{X}), J = 1,2,\dots,N$, depending upon coordinates $X^\alpha, \alpha = 1,2,\dots,M$, the Lagrangian $\mathcal{L}$ in general will be a function of $\mathbf{X}$, ϕ and derivatives of ϕ up to some finite order. That is

$$\mathcal{L} = \mathcal{L}(\phi_J, \partial_\alpha\phi_J, \partial_\alpha\partial_\beta\phi_J, \dots; X^\alpha), \tag{1}$$

where

$$\partial_\alpha\phi_J = \partial\phi_J/\partial X^\alpha, \qquad \partial_\alpha\partial_\beta\phi_J = \partial^2\phi_J/\partial X^\alpha\partial X^\beta. \tag{2}$$

The variation of the Hamiltonian action $\mathcal{A}$ is given by

$$\delta\mathcal{A} = \int \frac{\delta\mathcal{L}}{\delta\phi_J}\delta\phi_J\,\mathrm{d}X, \tag{3}$$

where

$$\frac{\delta\mathcal{L}}{\delta\phi_J} = \frac{\partial\mathcal{L}}{\partial\phi_J} - \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial_\alpha\phi_J}\right) + \partial_\alpha\partial_\beta\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\phi_J)}\right) + \tag{4}$$

Throughout this paper a repeated index implies summation over the range of the index.

Consider invertible and properly smooth transformations

$$\begin{aligned} X^\alpha &= X^\alpha(\overline{\mathbf{X}}; \epsilon), \\ \phi(\mathbf{X}) &= \Phi(\overline{\phi}(\overline{\mathbf{X}}), \mathbf{X}; \epsilon) \end{aligned} \tag{5}$$

such that for $\epsilon = 0, \overline{\mathbf{X}} = \mathbf{X}$ and $\overline{\phi} = \phi$. Let

$$\overline{\mathcal{A}} = \int \mathcal{L}\left(\overline{\phi}_J(\overline{\mathbf{X}}), \frac{\partial\overline{\phi}_J(\overline{\mathbf{X}})}{\partial\overline{X}^\alpha}, \frac{\partial^2\overline{\phi}_J(\overline{\mathbf{X}})}{\partial\overline{X}^\beta\partial\overline{X}^\alpha}, \dots; \overline{\mathbf{X}}\right) \mathrm{d}\overline{\mathbf{X}}, \tag{6}$$

$$\mathcal{A}(\epsilon) = \int \mathcal{L}\left(\phi_J(\mathbf{X}), \frac{\partial\phi_J(\mathbf{X})}{\partial X^\alpha}, \frac{\partial^2\phi_J(\mathbf{X})}{\partial X^\beta\partial X^\alpha}, \quad ; \mathbf{X}\right) \mathrm{d}\mathbf{X}. \tag{7}$$

The action is said to be invariant under transformations (5) if

$$\mathcal{A}(\epsilon) = \overline{\mathcal{A}} \equiv \mathcal{A}(0) \quad \forall\epsilon \text{ and } \forall\phi_J(\mathbf{X}), \quad J = 1,2,\dots,N. \tag{8}$$

Noether's theorem states that if the action $\mathcal{A}$ is invariant under a set of transformations of the coordinates and the fields, then there exist conserved currents $\mathcal{J}^\alpha$ such that

$$\partial_\alpha \mathcal{J}^\alpha \delta\epsilon = \frac{\delta\mathcal{L}}{\delta\phi_J}\delta\phi_J, \tag{9}$$

where

$$\begin{aligned}
-\mathcal{J}^\alpha \delta\epsilon &= \mathcal{L}\frac{\partial X^\alpha}{\partial\epsilon}\delta\epsilon + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi_J)}\delta\phi_J + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\phi_J)}\partial_\beta\phi_J \\
&\quad -\partial_\beta\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\phi_J)}\right)\delta\phi_J + \cdot
\end{aligned}$$

It can be shown that

$$\frac{\delta\phi_J}{\delta\epsilon} = \frac{\partial\Phi_J}{\partial\epsilon} - (\partial_\alpha\phi_J)\frac{\partial X^\alpha}{\partial\epsilon}.$$

It is evident from equation (9) that when fields $\phi_J(\mathbf{X})$ satisfy the Euler-Lagrange equations of motion, viz.,

$$\frac{\delta\mathcal{L}}{\delta\phi_J} = 0,$$

the current $\mathcal{J}^\alpha$ is conserved.

Substitution from (10) and (11) into (9) yields

$$\begin{aligned}
-\mathcal{J}^\alpha &= \mathcal{L}\frac{\partial X^\alpha}{\partial\epsilon} + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi_J)}\left[\frac{\partial\Phi_J}{\partial\epsilon} - (\partial_\beta\phi_J)\frac{\partial X^\beta}{\partial\epsilon}\right] \\
&\quad + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\phi_J)}\partial_\beta\left[\frac{\partial\Phi_J}{\partial\epsilon} - (\partial_\gamma\phi_J)\frac{\partial X^\gamma}{\partial\epsilon}\right] \\
&\quad -\partial_\beta\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\beta\phi_J)}\right)\left[\frac{\partial\Phi_J}{\partial\epsilon} - (\partial_\gamma\phi_J)\frac{\partial X^\gamma}{\partial\epsilon}\right] +
\end{aligned}$$

Note that X^α can include time, and $\phi_J, J = 1, 2, \ldots, N$, are arbitrary fields involved in the Lagrangian. Suppose now that the Lagrangian does not depend upon one of the coordinates, say X^λ. Then the action is invariant under the transformations

$$\begin{aligned}
X^\alpha(\overline{X}, \epsilon) &= \overline{X}^\alpha + \epsilon\delta^\alpha_\lambda, \\
\Phi_J(\overline{\phi}, \mathbf{X}, \epsilon) &= \overline{\phi}_J(\overline{\mathbf{X}}).
\end{aligned} \tag{14a}$$

and

$$\frac{\partial X^\alpha}{\partial \epsilon} = \delta^\alpha_\lambda, \qquad \frac{\partial \Phi_J}{\partial \epsilon} = 0.$$

Here δ^λ_μ is the Kronecker delta which equals one when $\lambda = \mu$ and zero otherwise. Equations (13) and (14b) result in the following:

$$\mathcal{J}^\mu_\lambda \equiv T^\mu_\lambda = -\mathcal{L}\delta^\mu_\lambda + (\partial_\lambda\phi_J)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_J)} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\alpha\phi_J)}(\partial_\alpha\partial_\lambda\phi_J) - \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\partial_\mu\phi_J)}\right)\partial_\lambda\phi_J +$$

where T^μ_λ is the canonical energy-momentum tensor whose components have the physical dimensions of an energy density. Equation (15) will be used below repeatedly. In terms of the energy-momentum tensor, the force F_λ on a defect or an inhomogeneity in the reference configuration is given by

$$F_\lambda = \int_S T^\mu_\lambda N_\mu \, \mathrm{d}S,$$

where the closed surface S encloses the defect in the reference configuration, and $\mathbf{N}$ is an outward unit normal to S.

Energy-Momentum Tensor for Nonsimple Elastic Dielectrics

We consider quasistatic deformations of a nonsimple elastic dielectric of grade 2; the results are easily extendable to an elastic dielectric of grade N. Also, in principle, one can consider the dynamic case by modifying the Lagrangian to include the kinetic energy density. Before applying equation (15) to an elastic dielectric of grade 2, we apply it to the other three cases listed in the abstract. In each case, because of the invariance of the strain energy density under uniform translations, the action is invariant under the transformations (14a).

a. Simple elastic materials

For these materials,

$$\mathcal{L} = -\hat{W}(\mathbf{F};\mathbf{X}), \tag{17}$$

where $\hat{W}$ is the strain-energy density per unit volume in the reference configuration, $\mathbf{F} = \partial x/\partial\mathbf{X}$ is the deformation gradient, a two-point tensor. In order to use equation (15) we rewrite the Lagrangian as

$$\mathcal{L} = -W(x^i, \partial_\alpha x^i; X^\alpha)$$

and set $\phi_J = x^i\delta_{iJ}$. With the notation $F^i_\lambda = \partial x^i/\partial X^\lambda$, equation (15) yields

$$T^\mu_\lambda = W\delta^\mu_\lambda - F^i_\lambda \frac{\partial W}{\partial F^i_\mu},$$

which is the energy-momentum tensor derived by Eshelby [1].

b. Nonsimple elastic materials of grade 2

For these materials

$$\begin{aligned}\mathcal{L} &= -\hat{W}(\mathbf{F}, \nabla\mathbf{F}; \mathbf{X}), \\ &= -W(x^i, \partial_\lambda x^i, \partial_\lambda\partial_\mu x^i; X^\mu)\end{aligned}$$

and with $G^i_{\mu\nu} = \partial_\mu\partial_\nu x^i = \partial F^i_\nu/\partial X^\mu$, equation (15) gives

$$T^\mu_\lambda = W\delta^\mu_\lambda - F^i_\lambda \frac{\partial W}{\partial F^i_\mu} - \frac{\partial W}{\partial G^i_{\mu\nu}} G^i_{\nu\lambda} + \frac{\partial}{\partial X^\nu}\left(\frac{\partial W}{\partial G^i_{\mu\nu}}\right) F^i_\lambda, \tag{21}$$

which agrees with the energy-momentum tensors derived by Eshelby [2] and Maugin and Trimarco [4]. We note that they used different reasoning to arrive at this result.

c. Simple elastic dielectrics

In terms of the electrostatic potential ϕ, the electroelastic field $\mathbf{E}$ can be expressed as

$$E_i = -\frac{\partial\phi}{\partial x^i} = -\frac{\partial\phi}{\partial X^\lambda}\frac{\partial X^\lambda}{\partial x^i} = \hat{E}_\lambda \frac{\partial X^\lambda}{\partial x^i},$$

where $\hat{E}_\lambda \equiv -\partial\phi/\partial X^\lambda$. The Lagrangian can be written as

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}Je_0\mathbf{E}\cdot\mathbf{E} - \hat{W}(\mathbf{F}, \mathbf{E}; \mathbf{X}) \\ &= \tfrac{1}{2}Je_0\mathbf{E}\cdot\mathbf{E} - W(x^i, \partial_\lambda x^i, \phi, \partial_\lambda\phi; X^\mu),\end{aligned}$$

where $J = \det(\mathbf{F})$ and e_0 is the dielectric constant for the vacuum. In order to use Noether's theorem we note that $\hat{\mathbf{E}}$ is independent of $\mathbf{F}$. With $\phi_J = \{x^i\delta_{iJ}, \phi\}$, equation (15) gives

$$\begin{aligned}T^\mu_\lambda &= -\mathcal{L}\delta^\mu_\lambda + F^i_\lambda \frac{\partial \mathcal{L}}{\partial F^i_\mu} + \hat{E}_\lambda \frac{\partial \mathcal{L}}{\partial \hat{E}_\mu} \\ &= W\delta^\mu_\lambda - F^i_\lambda \frac{\partial W}{\partial F^i_\mu} + \hat{E}_\lambda \left[\frac{\partial W}{\partial \hat{E}_\mu} + Je_0 \frac{\partial X^\mu}{\partial x^i}\frac{\partial X^\nu}{\partial x^i}\hat{E}_\nu \right],\end{aligned}$$

which differs from the energy-momentum tensor derived by Maugin and Epstein [5] in that the last term on the right-hand side of (24) is missing in their expression, it is a contribution from the free electric field $\frac{1}{2}Je_0\mathbf{E}\cdot\mathbf{E}$ and equals its derivative with respect to $\partial_\mu\phi = -\hat{E}_\mu$. In deriving (24) we have also used the relation $\partial J/\partial\mathbf{F} = J(\mathbf{F}^{-1})^T$.

d. Nonsimple elastic dielectrics of grade 2

For these materials,

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}Je_0\mathbf{E}\cdot\mathbf{E} - \hat{W}(\mathbf{F},\mathbf{G},\mathbf{E};\mathbf{X})\\ &\tfrac{1}{2}Je_0\mathbf{E}\cdot\mathbf{E} - W(x^i,\partial_\mu x^i,\partial_\mu\partial_\nu x^i,\phi,\partial_\mu\phi;X^\mu),\end{aligned}$$

and equation (15) gives

$$\begin{aligned}T^\mu_\lambda &= -\mathcal{L}\delta^\mu_\lambda + F^i_\lambda\frac{\partial\mathcal{L}}{\partial F^i_\mu} + \hat{E}_\lambda\frac{\partial\mathcal{L}}{\partial\hat{E}_\mu} + \frac{\partial\mathcal{L}}{\partial G^i_{\mu\nu}}G^i_{\nu\lambda} - \frac{\partial}{\partial X^\nu}\left(\frac{\partial\mathcal{L}}{\partial G^i_{\nu\mu}}\right)F^i_\lambda\\ &= W\delta^\mu_\lambda - F^i_\lambda\frac{\partial W}{\partial F^i_\mu} + \hat{E}_\lambda\left[-\frac{\partial W}{\partial\hat{E}_\mu} + Je_0\frac{\partial X^\mu}{\partial x^i}\frac{\partial X^\nu}{\partial x^i}\hat{E}_\nu\right]\\ &\quad -G^i_{\nu\lambda}\frac{\partial W}{\partial G^i_{\mu\nu}} + \frac{\partial}{\partial X^\nu}\left(\frac{\partial W}{\partial G^i_{\mu\nu}}\right)F^i_\lambda.\end{aligned}$$

Equation (26) is the energy-momentum tensor for nonsimple elastic dielectrics of grade 2 and includes the previous three cases.

Conclusions

It has been shown that the use of Noether's theorem gives an expression for the energy-momentum tensor for four different classes of materials. In principle, the method is easily extendable to nonsimple elastic dielectrics of grade N. By also considering the magnetic field in the Lagrangian, the work can be extended to nonsimple electromagnetic materials of grade N; Maugin [12] has derived the Eshelby energy-momentum tensor for simple electromagnetic materials.

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Note added in Proof

The method illustrated above can also be used to derive the Ericksen tensor for nematic liquid crystals by taking

$$\begin{aligned}\mathcal{L} &= -\widehat{W}(\mathbf{n}, \mathbf{F}, \nabla\mathbf{n}) \\ &= W(x^i, n^i, \partial_\alpha x^i, \partial_\alpha n^i; X^\alpha),\end{aligned}$$

where $\widehat{W}$ is the strain-energy density per unit volume in the reference configuration, and $\mathbf{n}$ is the director field. Noting that $\mathbf{x}$ and $\mathbf{n}$ are field variables, equation (15) gives

$$\mathcal{J}^\mu_\lambda \equiv T^\mu_\lambda = W\delta^\mu_\lambda - F^i_\lambda \frac{\partial W}{\partial F^i_\mu} - \partial_\lambda n^i \frac{\partial W}{\partial(\partial_\mu n^i)},$$

which is the Ericksen tensor for nematic liquid crystals also derived recently by Maugin and Trimarco [13].