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Saint-Venant's Problem for Porous Linear Elastic Materials

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Abstract. We use a semi-inverse method to study deformations of a straight, prismatic, homogeneous body made of a porous, linear elastic, and isotropic material and loaded only at its end faces by self equilibrated forces. As in the classical theory, the problem is reduced to solving plane elliptical problems. It is shown that the Clebsch/Saint-Venant and Voigt hypotheses are not valid for this problem.

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1. Introduction

Since Saint-Venant [1, 2] solved the problem of extension, bending, torsion, and flexure of a prismatic body made of a homogeneous and isotropic linear elastic material and loaded at its end faces only, there has been considerable interest in generalizing it [3–12]. Clebsch [3] observed that in Saint-Venant's solutions the surface tractions on a plane passing through the axis of the prismatic body are parallel to the axis. Voigt [4, 5] hypothesized that the stress tensor is either constant along the axis or depends linearly upon the axial coordinate. Other investigators [6–12] have analysed the problem for inhomogeneous and anisotropic linear elastic bodies, elastic dielectrics, microstretch elastic solids, and piezoelectric materials. Here we study the Saint-Venant problem for a linear elastic porous material. The theory for such materials has been developed by Nunziato and Cowin [13] who have also studied the bending of a beam [14] made of this material. They applied surface tractions equipollent to a bending moment only at the end faces of the beam and found that surface tractions were also required on its lateral walls. Batra and Yang [15] have proved Toupin's version of the Saint-Venant principle for linear elastic porous materials.

The solution for the Saint-Venant problem has been reduced to that of solving two plane elliptic problems; their solutions will give the warping of the cross-section and in-plane displacements as a function of the axial and in-plane coordinates. It is also shown that the Saint-Venant/Clebsch and Voigt hypotheses are not valid for this problem. An appropriate criterion is that the second derivative with respect to the axial coordinate of the in-plane components of the stress tensor must vanish. Moreover, the vanishing of the first derivative with respect to the axial coordinate of the in-plane components of the stress tensor is not equivalent to the vanishing of these components as is the case in classical linear elasticity [4, 5].

2. Formulation of the Problem

Equations governing quasistatic deformations of a linear elastic, porous, isotropic, and homogeneous material in the absence of body forces and extrinsic equilibrated forces are

$$\mathrm{Div}\,\mathbf{T} = \mathbf{0}, \qquad \mathrm{Div}\,\mathbf{h} + g = 0, \tag{1}$$

where

$$\begin{aligned} \mathbf{T} &= \lambda(\mathrm{tr}\,\mathbf{E})\mathbf{1} + 2\mu\mathbf{E} + \beta\phi\mathbf{1}, \\ \mathbf{h} &= \alpha\,\mathrm{Grad}\,\phi, \qquad g = -\zeta\phi - \beta(\mathrm{tr}\,\mathbf{E}). \end{aligned} \tag{2}$$

Here $\mathbf{T}$ is a stress tensor, $\mathbf{h}$ the flux of porosity ϕ, g the density of self-equilibrated body forces for the porous material, Div the three-dimensional divergence operator, λ and μ Lamé's constants, α, β and ζ are material constants that characterize the effect of porosity, tr is the trace operator, $\mathbf{E}$ the infinitesimal strain tensor, $\mathbf{1}$ the three-dimensional identity tensor, and Grad the three-dimensional gradient operator.

Let $\mathbf{e}$ be a unit vector along the axis of the prismatic body. We set

$$\begin{aligned} \mathbf{u} &= w\mathbf{e} + \mathbf{v}, \qquad \mathbf{h} = \alpha(\phi'\mathbf{e} + \mathrm{grad}\,\phi), \\ \mathbf{T} &= \sigma\mathbf{e}\otimes\mathbf{e} + \mathbf{t}\otimes\mathbf{e} + \mathbf{e}\otimes\mathbf{t} + \hat{\mathbf{T}}, \\ \mathbf{E} &= \mathrm{Sym\ Grad}\,\mathbf{u} = \varepsilon\mathbf{e}\otimes\mathbf{e} + \boldsymbol{\gamma}\otimes\mathbf{e} + \mathbf{e}\otimes\boldsymbol{\gamma} + \hat{\mathbf{E}}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} \varepsilon &= w' \equiv \tfrac{\partial w}{\partial z}, \qquad \boldsymbol{\gamma} = \tfrac{1}{2}(\mathbf{v}' + \mathrm{grad}\,w), \\ \hat{\mathbf{E}} &= \mathrm{Sym\ grad}\,\mathbf{v} = \tfrac{1}{2}(\mathrm{grad}\,\mathbf{v} + (\mathrm{grad}\,\mathbf{v})^T), \end{aligned} \tag{4}$$

$\mathbf{u}$ is the displacement field, and grad (div) is the two-dimensional gradient (divergence) operator with respect to coordinates in the cross-section $\mathcal{A}$. Thus w gives the displacement and z the coordinate of a point along the axis of the prismatic body, $\mathbf{v}$ the components of displacement in the plane of the body, ε the axial strain, σ the axial stress, $\hat{\mathbf{T}}$ the in-plane stress tensor, $\hat{\mathbf{E}}$ the in-plane infinitesimal strain tensor, $\mathbf{t}$ the shear stress on the cross-section $\mathcal{A}$, $\boldsymbol{\gamma}$ the shear strain corresponding to the shear stress $\mathbf{t}$, and the tensor product $\otimes$ between two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$ for every vector $\mathbf{c}$. The constitutive relations (2) and the decompositions (3) yield

$$\begin{aligned} \hat{\mathbf{T}} &= [\lambda(\operatorname{tr}\hat{\mathbf{E}} + \varepsilon) + \beta\phi]\hat{\mathbf{1}} + 2\mu\hat{\mathbf{E}}, \\ \sigma &= \lambda(\operatorname{tr}\hat{\mathbf{E}} + \varepsilon) + 2\mu\varepsilon + \beta\phi, \qquad \mathbf{t} = \mu(\mathbf{v}' + \operatorname{grad} w), \end{aligned} \tag{5}$$

where $\hat{\mathbf{1}}$ is the two-dimensional identity tensor. Substitution from (3) and (4) into (1) yields

$$\begin{aligned} &(\lambda + \mu)\operatorname{div}\mathbf{v}' + (\lambda + 2\mu)w'' + \beta\phi' + \mu\Delta_R w = 0, \\ &\mu\mathbf{v}'' + (\lambda + \mu)\operatorname{grad} w' + \beta\operatorname{grad}\phi + 2\mu\Delta_R\mathbf{v} + (\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{v} = 0, \\ &\alpha\phi'' + \alpha\Delta_R\phi - \zeta\phi - \beta(\operatorname{div}\mathbf{v} + w') = 0, \end{aligned} \tag{6}$$

where Δ_R is the 2-dimensional Laplace operator in $\mathcal{A}$. Equations (6) are the field equations for the determination of $\mathbf{v}$, w, and ϕ, and correspond to Navier's equations in elastostatics; the latter are obtained by setting $\beta = 0$ in $(6)_{1,2}$. This form of the equations exploits the geometry of the prismatic body (e.g. see DiCarlo [16]). We assume that the strain energy density is positive definite; thus [13]

$$\mu \geqslant 0, \quad \lambda + \tfrac{2}{3}\mu \geqslant 0, \quad \zeta \geqslant 0, \quad \alpha \geqslant 0, \quad (\lambda + \mu)\zeta - 4\beta^2 \geqslant 0. \tag{7}$$

For the prismatic body $\mathcal{A} \times [0, l]$ of axial length l, we assume that its mantle $\partial\mathcal{A} \times [0, l]$ is traction free and it is loaded at the ends. Thus pertinent boundary conditions are

$$\begin{aligned} &\hat{\mathbf{T}}\mathbf{n} = \mathbf{0}, \quad \mathbf{t} \cdot \mathbf{n} = 0, \quad \operatorname{grad}\phi \cdot \mathbf{n} = 0, \quad \text{on } \partial\mathcal{A} \times [0, l], \\ &\int_{\mathcal{A}} \mathbf{T}\mathbf{n}\,\mathrm{d}A = \mathbf{F}, \qquad \int_{\mathcal{A}} \mathbf{r} \times \mathbf{T}\mathbf{n}\,\mathrm{d}A = \mathbf{M}, \int_{\mathcal{A}} \mathbf{h} \cdot \mathbf{n}\,\mathrm{d}A = H. \end{aligned} \tag{8}$$

Here $\mathbf{F}$ and $\mathbf{M}$ are the resultant force and resultant moment applied at the end faces $\mathcal{A} \times \{0\}$ and $\mathcal{A} \times \{l\}$, H is the resultant flux of porosity, and $\mathbf{n}$ is the outward directed unit normal to the surface.

3. A Saint-Venant/Almansi Solution

Following Saint-Venant [1, 2] and Almansi [17], we assume

$$w = w_0 + zw_1 + \frac{z^2}{2}w_2 + \frac{z^3}{6}w_3, \tag{9}$$

and similar expressions for $\mathbf{v}$ and ϕ, where $w_0, w_1, \ldots, \phi_3$ are functions defined on the plane $\mathcal{A}$. For the mantle to be free of surface tractions and moments per unit length, we must have $w_3 = \phi_3 = 0$. Henceforth, we assume that w_3 and ϕ_3 vanish identically. From (9), (4) and $(3)_3$ we obtain

$$\hat{\mathbf{T}} = \hat{\mathbf{T}}_0 + z\hat{\mathbf{T}}_1 + \frac{z^2}{2}\hat{\mathbf{T}}_2 + \frac{z^3}{3}\hat{\mathbf{T}}_3, \tag{10}$$

and, by equating like powers of z on both sides of (6) and $(8)_1$, the following partial differential equations:

$$F\mathbf{v}_3 = \mathbf{0}, \tag{11}$$

$$F\mathbf{v}_2 + \beta \operatorname{grad} \phi_2 = \mathbf{0}, \tag{12}$$

$$F\mathbf{v}_1 + \beta \operatorname{grad} \phi_1 + (\lambda + \mu) \operatorname{grad} w_2 + \mu\mathbf{v}_3 = \mathbf{0}, \tag{13}$$

$$F\mathbf{v}_0 + \beta \operatorname{grad} \phi_0 + (\lambda + \mu) \operatorname{grad} w_1 + \mu\mathbf{v}_2 = \mathbf{0}, \tag{14}$$

$$\mu\Delta_R w_2 + (\lambda + \mu) \operatorname{div} \mathbf{v}_3 = 0, \tag{15}$$

$$\mu\Delta_R w_1 + (\lambda + \mu) \operatorname{div} \mathbf{v}_2 + \beta\phi_2 = 0, \tag{16}$$

$$\mu\Delta_R w_0 + (\lambda + \mu) \operatorname{div} \mathbf{v}_1 + \beta\phi_1 + (\lambda + 2\mu)w_2 = 0, \tag{17}$$

$$\operatorname{div} \mathbf{v}_3 = 0, \tag{18}$$

$$\alpha\Delta_R\phi_2 - \zeta\phi_2 - \beta \operatorname{div} \mathbf{v}_2 = 0, \tag{19}$$

$$\alpha\Delta_R\phi_1 - \zeta\phi_1 - \beta (\operatorname{div} \mathbf{v}_1 + w_2) = 0, \tag{20}$$

$$\alpha\Delta_R\phi_0 - \zeta\phi_0 + \alpha\phi_2 - \beta(\operatorname{div} \mathbf{v}_0 + w_1) = 0, \tag{21}$$

in $\mathcal{A}$, and the following boundary conditions on $\partial\mathcal{A}$:

$$(\mathbf{v}_{i+1} + \operatorname{grad} w_i) \cdot \mathbf{n} = \mathbf{0}, \quad i = 0, 1, 2, \tag{22}$$

$$(2\mu \operatorname{Sym} \operatorname{grad} \mathbf{v}_i + \lambda(\operatorname{div} \mathbf{v}_i)\hat{\mathbf{1}} + \beta\phi_i\hat{\mathbf{1}})\mathbf{n} = \mathbf{0}, \quad i = 0, 1, 2, \tag{23}$$

$$(2\mu \operatorname{Sym} \operatorname{grad} \mathbf{v}_3 + \lambda(\operatorname{div} \mathbf{v}_3)\hat{\mathbf{1}})\mathbf{n} = \mathbf{0}, \tag{24}$$

$$\operatorname{grad} \phi_i \cdot \mathbf{n} = 0, \quad i = 0, 1, 2, \tag{25}$$

where $F = [2\mu\Delta_R + (\lambda + 2\mu) \operatorname{grad} \operatorname{div}]$ is the differential operator appearing in Navier's equations.

On recalling that the solution of Navier's equations subjected to null tractions is a rigid body motion, Equations (11) and (24) have the solution $\mathbf{v}_3 = \mathbf{v}_3^0 + \omega_3\mathbf{e} \times \mathbf{r}$, where $\mathbf{v}_3^0$ and ω_3 are constants. Since the mantle is traction free, the torque on every cross-section is the same. It requires that $\omega_3 = 0$. Thus

$$\mathbf{v}_3 = \mathbf{v}_3^0, \quad \hat{\mathbf{T}}_3 = \mathbf{0}, \tag{26}$$

and Equation (18) is identically satisfied. Equation (15) now implies that w_2 is a harmonic function, and (22) with $i = 2$ gives $(\operatorname{grad} w_2) \cdot \mathbf{n} = -\mathbf{v}_3^0 \cdot \mathbf{n}$. Thus

$$w_2 = w_2^0 - \mathbf{v}_3^0 \cdot \mathbf{r}. \tag{27}$$

In (27) and below, quantities with superscript zero denote constants.

Scarpetta [19] and Iesan [18] have shown that the boundary-value problem defined by Equations (12), (19), (23), and (25) (for $i = 2$) has a unique solution for $\mathbf{v}_2$ and ϕ_2 to within a rigid body motion. Since null fields satisfy these equations, therefore,

$$\phi_2 = 0, \qquad \mathbf{v}_2(\mathbf{r}) = \mathbf{v}_2^0 + \omega_2 \mathbf{e} \times \mathbf{r} \tag{28}$$

give every solution of the problem. The reasoning given above to conclude $\omega_3 = 0$ also gives $\omega_2 = 0$. From (16) and (28) we conclude that w_1 is a harmonic function, and the boundary condition (22) with $i = 1$ requires that

$$w_1 = w_1^0 - \mathbf{v}_2^0 \cdot \mathbf{r}. \tag{29}$$

Equations (13), (20), (23), and (25) (for $i = 1$) can be simplified to

$$\begin{aligned} &F\mathbf{v}_1 + \beta \operatorname{grad} \phi_1 = \lambda \mathbf{v}_3^0, \\ &\alpha \Delta_R \phi_1 - \zeta \phi_1 - \beta \operatorname{div} \mathbf{v}_1 = \beta(w_2^0 - \mathbf{v}_3^0 \cdot \mathbf{r}), \quad \text{in } \mathcal{A} \\ &\hat{\mathbf{T}}_1 \mathbf{n} = \mathbf{0}, \quad (\operatorname{grad} \phi_1) \cdot \mathbf{n} = 0, \quad \text{on } \partial\mathcal{A}. \end{aligned} \tag{30}$$

Scarpetta's [19] and Iesan's [18] theorems imply that the boundary-value problem (30) has a unique solution which must depend linearly upon $\mathbf{v}_3^0$ and w_2^0. From Equations $(1)_1$, $(3)_3$, $(8)_1$ we conclude that

$$\int_{\mathcal{A}} \sigma' \, \mathrm{d}A = -\int_{\mathcal{A}} (\operatorname{div} \mathbf{t}) \, \mathrm{d}A = -\int_{\partial\mathcal{A}} \mathbf{t} \cdot \mathbf{n} \, \mathrm{d}s = 0, \tag{31}$$

and, therefore,

$$(\lambda + 2\mu) w_2^0 \mathcal{A} = -\beta \int_{\mathcal{A}} \phi_1 \, \mathrm{d}A - \lambda \int_{\mathcal{A}} (\operatorname{div} \mathbf{v}_1) \, \mathrm{d}A + \mathbf{v}_0^3 \cdot \bar{\mathbf{r}} \mathcal{A}(\lambda + 2\mu) \tag{32}$$

$$\equiv \mathbf{v}_0^3 \cdot \bar{\mathbf{r}}_B \mathcal{A}(\lambda + 2\mu), \tag{33}$$

where $\bar{\mathbf{r}}$ is the position vector of the centroid of the cross-section of area $\mathcal{A}$, and $\bar{\mathbf{r}}_B$ is defined by Equations (32) and (33). We also note that $\hat{\mathbf{T}}_1$ may not vanish, but $\mathbf{t}_1 = \mathbf{t}_2 = \mathbf{0}$.

Once w_0 and ϕ_0 are found by solving the elliptic problem defined by (17), (21), (23) and (25) (for $i = 0$), the complete solution to the Saint-Venant problem is determined. Because of the presence of ϕ_0 and ϕ_1 in these equations, the porosity will influence the warping of the cross-section.

4. Clebsch/Saint-Venant and Voigt Hypotheses

The Clebsch/Saint-Venant hypothesis in classical linear elasticity is $\hat{\mathbf{T}} = \mathbf{0}$ and the Voigt hypothesis is $\mathbf{T}' = \mathbf{0}$ or $\hat{\mathbf{T}}_1 = \mathbf{0}$. Podio-Guidugli [20] has proved the equivalence of these two hypotheses. We show here that these hypotheses are not valid for a linear elastic porous material. We first recall the following result: *if* $f \in C^2(\mathcal{A})$ *and there exists a function* $\mathbf{u}$ *such that* $f\hat{\mathbf{1}} = \text{Sym grad}\,\mathbf{u}$, *then* $\Delta_R f = 0$.

We now prove that $\hat{\mathbf{T}}_2 = \mathbf{0}$ if and only if $\mathbf{v}_2 = \text{constant}$. $\hat{\mathbf{T}}_2 = \mathbf{0}$ implies that $\text{tr}\,\hat{\mathbf{T}}_2 = 0$, or

$$\text{div}\,\mathbf{v}_2 = -(\beta/(\lambda+\mu))\phi_2, \tag{34}$$

and hence

$$\text{Sym grad}\,\mathbf{v}_2 = -\beta\phi_2\hat{\mathbf{1}}/2(\lambda+\mu). \tag{35}$$

Because of the above stated result, $\Delta_R\phi_2 = 0$. Now recalling $(7)_3$, Equations (34) and (19) yield $\phi_2 = 0$. Thus Equation (35) gives

$$\mathbf{v}_2 = \mathbf{v}_2^0 + \omega_2\mathbf{e}\times\mathbf{r}, \tag{36}$$

and because the mantle is traction free, therefore $\omega_2 = 0$, which proves the result since the converse is trivial.

The second result of this section is that $\hat{\mathbf{T}}_1 = \mathbf{0}$ rules out flexure. We first note that

$$\hat{\mathbf{T}}_1 = 2\mu\,\text{Sym grad}\,\mathbf{v}_1 + \lambda\,(\text{div}\,\mathbf{v}_1)\hat{\mathbf{1}} + (\lambda w_2 + \beta\phi_2)\hat{\mathbf{1}}. \tag{37}$$

$\hat{\mathbf{T}}_1 = \mathbf{0}$ implies $\text{tr}\,\hat{\mathbf{T}}_1 = 0$ which gives $\text{div}\,\mathbf{v}_1 = -(\lambda w_2 + \beta\phi_1)/(\lambda+\mu)$, and, therefore,

$$2\,\text{Sym grad}\,\mathbf{v}_1 = -(\lambda w_2 + \beta\phi_1)\hat{\mathbf{1}}/(\lambda+\mu), \qquad \Delta_R\phi_1 = 0, \tag{38}$$

where we have used the aforestated result, and w_2 is harmonic (cf. the line following Equation (26)). From (20) we now conclude that

$$\left(\frac{\beta^2}{\lambda+\mu} - \zeta\right)\phi_1 - \frac{\beta\mu}{\lambda+\mu}w_2 = 0. \tag{39}$$

Recalling (27), (33) and (25) with $i = 1$, we conclude that $\mathbf{v}_3^0 = \mathbf{0}$ which rules out flexure.

We remark that $\hat{\mathbf{T}}_1 = f(\mathbf{r})\hat{\mathbf{1}}$ also rules out flexure. Indeed, from $(30)_1$ and $(30)_3$, one again gets $f = 0$.

Following reasoning similar to that given above, one can prove that $\hat{\mathbf{T}}_0 = \mathbf{0}$ implies $\mathbf{v}_2^0 = \mathbf{0}$ and hence no bending. It explains why Cowin and Nunziato [14] imposed nonzero tractions on the lateral surface of the porous beam deformed in bending.

Thus the Clebsch/Saint-Venant and Voigt hypotheses must be relaxed for studying the Saint-Venant problem for linear elastic porous materials.

5. Summary

The Saint-Venant problem for a prismatic body of cross-section $\mathcal{A}$ has been reduced to finding a solution of the following two elliptic problems in the cross-section $\mathcal{A}$: the first defined by Equation (30) and the second by Equations (14), (17), (21), (22), (23) and (25) with $i = 0$ and

$$w_2 = -\mathbf{v}_3^0 \cdot (\mathbf{r} - \bar{\mathbf{r}}_B), \qquad \mathbf{v}_2(\mathbf{r}) = \mathbf{v}_2^0, \qquad w_1 = w_1^0 - \mathbf{v}_2^0 \cdot \mathbf{r}. \tag{40}$$

Let $\bar{\mathbf{v}}_1(\mathbf{r})$ and $\overline{\phi}_1(\mathbf{r})$ be a solution of the boundary-value problem defined by Equations (30) with $\mathbf{v}_3^0 = \hat{\mathbf{1}}$, then

$$\mathbf{v}_1(\mathbf{r}) = v_3^0 \bar{\mathbf{v}}_1(\mathbf{r}) + \omega_1 \mathbf{e} \times \mathbf{r}, \phi_1(\mathbf{r}) = \overline{\phi}_1(\mathbf{r}) v_3^0, \tag{41}$$

where v_3^0 is the magnitude of $\mathbf{v}_3^0$ and ω_1 is an arbitrary constant. The six constants $\mathbf{v}_3^0$, $\mathbf{v}_2^0$, ω_1, and w_1^0 characterize, respectively, the flexure, bending, torsion, and extension of the prismatic porous body as discussed below.

Extension: The only nonvanishing constant is ω_1^0 and the solution of the second plane elliptic problem to within a rigid body motion is

$$w_0 = 0, \quad \mathbf{v}_0 = -\left(\frac{\zeta\lambda - \beta^2}{2(\lambda+\mu)\zeta - \beta^2} w_1^0\right)\mathbf{r}, \quad \phi_0 = -\frac{\beta(\beta^2 + 2\mu\zeta)}{2(\lambda+\mu)\zeta - \beta^2} w_1^0. \tag{42}$$

For $\beta \neq 0$, the porosity affects noticeably the Poisson effect. Note that the denominator in (42) is positive because of inequalities (6).

Torsion: The only nonzero constant is ω_1, and there is no coupling between the displacement and porosity fields. When either nonzero tractions or nonvanishing flux of porosity is prescribed on the mantle, then the two fields will be coupled with each other.

Bending: The only nonvanishing constant is $\mathbf{v}_2^0$, and $w_0 = 0$. The functions ϕ_0 and $\mathbf{v}_0$ are solutions of Equations (14), (21), (40), (23) and (25) with $i = 0$. If one takes $\mathbf{v}_0$ and ϕ_0 to be polynomials of degree 2 and 1 in $\mathbf{r}$ respectively, (in classical linear elasticity $\mathbf{v}_0$ is a polynomial of degree 2 in $\mathbf{r}$), then Equations (14), (21), and (23) are satisfied but (25) is not. Cowin and Nunziato [14] studied bending of a porous beam and found that $\mathbf{v}_0$ and ϕ_0 are not polynomials in $\mathbf{r}$. For their solution, tractions on the lateral walls do not vanish and $\mathbf{h} \cdot \mathbf{n} = 0$ on the end faces where normal tractions equipollent to a moment only are applied.

In general, the axial stress $\sigma = \lambda \operatorname{div} \mathbf{v}_0 + (\lambda + 2\mu)(-\mathbf{v}_2^0 \cdot \mathbf{r}) + \beta\phi_0$, and the locus of points in $\mathcal{A}$ where σ vanishes may not be a straight line.

Flexure: Here the only nonzero constant is $\mathbf{v}_3^0$. We recall that the point with the position vector $\bar{\mathbf{r}}_B$ in $(40)_1$ and defined by (33) need not coincide with the centroid of the cross-section. Equations (30) determine $\mathbf{v}_1$, the part of the displacement field $\mathbf{u}$ that is linear in z. However, $\mathbf{v}_1$ need not be quadratic in $\mathbf{r}$. Consequently, Equation (17) for warping function w_0 has a source term not necessarily affine in $\mathbf{r}$. Equations (14) and (21) with boundary conditions (23) and (25) with $i = 0$ have a rigid motion solution as in classical linear elasticity. Thus warping of the cross-section and the Poisson effect are influenced by the porosity.

We note that the flux of double forces at the terminal faces of the cylinder has no effect on the Saint-Venant solutions away from these faces. An analysis of the corresponding one-dimensional problem indicated that the porosity decays exponentially away from the loaded ends. However, the deformation fields in the prismatic body and in particular the warping of the cross-section are influenced by the porosity.

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