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A Second-Order Solution of Saint-Venant's Problem for an Elastic Pretwisted Bar Using Signorini's Perturbation Method

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Abstract. We use Signorini's expansion to analyse deformations of a straight, prismatic, isotropic, stress free, homogeneous body made of a second-order elastic material and loaded as follows. It is first twisted by an infinitesimal amount and then loaded by applying surface tractions, with nonzero resultant forces and/or moments, only at its end faces. The centroid of one end face is taken to be rigidly clamped. By using a semi-inverse method, the problem is reduced to that of solving two plane elliptic problems involving six arbitrary constants that characterize flexure, bending, extension, and torsion superimposed upon the infinitesimal twist. It is shown that the Clebsch hypothesis is not valid for this problem. A second-order Poisson's effect, not of the Saint-Venant type, and generalized Poynting effects may also occur in these problems.

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1. Introduction

Since Saint-Venant [1, 2] solved the problem of extension, bending, torsion and flexure of a prismatic body made of a linear elastic, isotropic and homogeneous material, and loaded at its end faces only, Clebsch [3] and Voigt [4, 5] have characterized these solutions. Clebsch noted that in Saint-Venant's solutions the surface tractions on a plane passing through the axis of the prismatic body are parallel to the axis, and Voigt observed that the stress tensor is at most an affine function of the axial coordinate. Iesan [6–9] has analysed the Saint-Venant problem for inhomogeneous and anisotropic linear elastic bodies, elastic dielectrics, and microstretch elastic solids. Dell'Isola and Rosa [10, 11] and Davì [12] have studied the problem for linear piezoelectric bodies and dell'Isola and Batra [13] for linear elastic porous solids.

Rivlin [14] studied the problem of extension superimposed upon an infinitesimal twist for isotropic, homogeneous prismatic bodies made of a second-order elastic material. The work has been extended to general nonlinear elastic solids by Green and Shield [15]. Green and Adkins [16] noted that when the centroid of one end-face is rigidly clamped (in the sense that its displacements and infinitesimal rotations vanish), then the compatibility conditions for the loads in Signorini's expansion method [17] are automatically satisfied. We recall that Signorini's method reduces the solution of a nonlinear elastic problem to that of a series of linear elastic problems with loads determined at each step by the solution of the previous linear problems. Truesdell and Noll [18] have discussed Signorini's expansion and summarized other works based on this method.

Here we use Signorini's expansion to analyse the Saint-Venant problem for a straight, isotropic, stress-free and homogeneous prismatic body made of a second-order elastic material with the assumption that the first term in the expansion for the displacement field corresponds to an infinitesimal twist of the body. We use a semi-inverse method to reduce the problem of the determination of the second term in the expansion to two plane elliptic problems – one for the warping function and the other for the in-plane displacements. The loads (body forces and surface tractions) in these two problems are proportional to the square of the initial twist per unit length. As pointed out by Truesdell and Noll [18] Signorini's method delivers only those solutions that are in the neighborhood of solutions of a linear elastic problem with the same loads as for the nonlinear problem. Generalized Poynting effects are shown to arise in the coupling not only of second-order extension but also of second-order torsion, bending and flexure with first-order torsion. For a cylindrical rod loaded to obtain an extension superimposed upon an initial infinitesimal twist, the two plane problems are solved and the Poynting effect [19] is delineated. When the resultant axial force vanishes, the elongation of the rod equals that found earlier by Wang and Truesdell [20].

2. Formulation of the Problem

Equations governing quasistatic deformations of a second-order elastic, isotropic and homogeneous body in the absence of body forces are

$$\mathrm{Div}\,\mathbf{T} = 0, \tag{1}$$

where

$$\begin{aligned}\mathbf{T} \;=\; \mu\Bigg[&\left(\alpha_1 I_{\mathbf{E}} + 2\mathbf{E} + \frac{\alpha_1}{2}(I_{\mathbf{HH}^T} + 2I_{\mathbf{E}}^2) + \alpha_3 I_{\mathbf{E}}^2 + \alpha_4 I\, I_{\mathbf{E}}\right)\mathbf{1}\\ &+(\alpha_5 + 2) I_{\mathbf{E}}\mathbf{E} - \alpha_1 I_{\mathbf{E}}\mathbf{H}^T - (\mathbf{H}^T)^2 + \alpha_6 \mathbf{E}^2\Bigg].\end{aligned} \tag{2}$$

Here $\mathbf{T}$ is the first Piola–Kirchhoff stress tensor, Div is the three-dimensional divergence operator with respect to coordinates in the reference configuration, μ is the shear modulus,

$$\mathbf{H} = \mathrm{Grad}\,\mathbf{u} \tag{3}$$

is the displacement gradient, $\mathbf{u} = \mathbf{x} - \mathbf{X}$ is the displacement, $\mathbf{x}$ and $\mathbf{X}$ denote respectively the position vectors of a material point with respect to a fixed rectangular Cartesian coordinate system in the present and reference configurations, and Grad is the gradient operator with respect to referential co-ordinates. Furthermore,

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^T) \tag{4}$$

is the infinitesimal strain tensor, $\lambda = \mu\alpha_1$ is a Lamé constant of linear elasticity, and $\alpha_3, \alpha_4, \alpha_5$ and α_6 are nondimensional material constants for a second-order elastic material. In (2) $I_{\mathbf{E}}$ denotes the trace of $\mathbf{E}$, $I_{\mathbf{HH}^T}$ the trace of $\mathbf{HH}^T$,

$$II_{\mathbf{E}} = \tfrac{1}{2}(I_{\mathbf{E}}^2 - I_{\mathbf{E}^2}) \tag{5}$$

is the second-invariant of $\mathbf{E}$, and $\mathbf{1}$ is the identity tensor. The reader is referred to Wang and Truesdell [20] for details of deriving the constitutive relation (2) and for references to other authors who have derived it.

In order to study the Saint-Venant problem, we consider a prismatic body $\mathcal{A} \times [0, \ell]$ with cross-section $\mathcal{A}$ and axial length ℓ, and assume that its mantle $\partial\mathcal{A} \times [0, \ell]$ is traction free. At the centroid of the end face $\mathcal{A} \times \{0\} =: \mathcal{A}_0$, we assume that

$$\mathbf{u} = \mathbf{0}, \qquad \mathbf{H} - \mathbf{H}^T = \mathbf{0}. \tag{6}$$

That is, the centroid of $\mathcal{A}_0$ is fixed and the infinitesimal rotation there vanishes. Henceforth, we place the origin of the rectangular Cartesian coordinate system at the centroid of $\mathcal{A}_0$. Green and Adkins [16] have pointed out that under conditions (6), there is no compatibility condition required by the first-order loads in the Signorini's series expansion of the solution. Further, the mantle $\partial\mathcal{A} \times [0, \ell]$ is taken to be traction free but surface tractions are applied on the end faces $\mathcal{A}_0$ and $\mathcal{A}_\ell := \mathcal{A} \times \{\ell\}$; these surface tractions are determined subsequently in the semi-inverse method used to analyse the problem. Thus

$$\mathbf{TN} = \mathbf{0} \quad \text{on } \partial\mathcal{A} \times [0, \ell], \tag{7}$$

where $\mathbf{N}$ is an outward unit normal to the boundary in the reference configuration.

In Signorini's method, we assume that the displacement field $\mathbf{u}$ has a series expansion

$$\mathbf{u} = \sum_{n=1}^{\infty} \eta^n \mathbf{u}^{(n)}, \tag{8}$$

where η is a small, yet to be determined, parameter in the problem. Thus, provided sufficient conditions of regularity,

$$\mathbf{H} = \sum_{n=1}^{\infty} \eta^n \mathbf{H}^{(n)}, \qquad \mathbf{E} = \sum_{n=1}^{\infty} \eta^n \mathbf{E}^{(n)}, \tag{9}$$

and, up to second order in η,

$$\begin{aligned} I_{\mathbf{E}} &= \eta \operatorname{tr} \mathbf{E}^{(1)} + \eta^2 \operatorname{tr} \mathbf{E}^{(2)}, \\ I_{\mathbf{E}}^2 &= \eta^2 (\operatorname{tr} \mathbf{E}^{(1)})^2, \\ II_{\mathbf{E}} &= \tfrac{1}{2}\eta^2[-(\operatorname{tr} \mathbf{E}^{(1)})^2 + \operatorname{tr}(\mathbf{E}^{(1)^2})], \\ I_{\mathbf{HH}^T} &= \eta^2 \operatorname{tr}(\mathbf{H}^{(1)}\mathbf{H}^{(1)^T}). \end{aligned} \tag{10}$$

Substitution from (9) and (10) into (2) yields

$$\mathbf{T} = \sum_{n=1}^{\infty} \eta^n \mathbf{T}^{(n)}, \tag{11}$$

where

$$\mathbf{T}^{(1)} = 2\mu \mathbf{E}^{(1)} + \lambda(\operatorname{tr} \mathbf{E}^{(1)})\mathbf{1},$$

$$\begin{aligned} \mathbf{T}^{(2)} = \overline{\mathbf{T}}^{(2)} + \mu\Bigg[&\left\{\frac{\alpha_1}{2}\operatorname{tr}(\mathbf{H}^{(1)}\mathbf{H}^{(1)^T}) + 2(\operatorname{tr} \mathbf{E}^{(1)})^2\right\} \\ &+ \alpha_3(\operatorname{tr} \mathbf{E}^{(1)})^2 + \left\{\frac{\alpha_4}{2}(\operatorname{tr}(\mathbf{E}^{(1)^2}) - (\operatorname{tr} \mathbf{E}^{(1)})^2)\right\} \\ &\times \mathbf{1} + (\alpha_5 + 2)(\operatorname{tr} \mathbf{E}^{(1)})\mathbf{E}^{(1)} \\ &- \alpha_1(\operatorname{tr} \mathbf{E}^{(1)})\mathbf{H}^{(1)^T} - (\mathbf{H}^{(1)^T})^2 + \alpha_6 \mathbf{E}^{(1)^2}\Bigg], \end{aligned} \tag{12}$$

$$\overline{\mathbf{T}}^2 = 2\mu \mathbf{E}^{(2)} + \lambda \operatorname{tr} \mathbf{E}^{(2)}\mathbf{1}.$$

From (1), (7) and (11), we conclude that for $n = 1, 2, \ldots$

$$\begin{aligned} \operatorname{Div} \mathbf{T}^{(n)} &= \mathbf{0}, \quad \text{in } \mathcal{A} \times [0, \ell], \\ \mathbf{T}^{(n)}\mathbf{N} &= \mathbf{0}, \quad \text{on } \partial\mathcal{A} \times [0, \ell]. \end{aligned} \tag{13}$$

We note that $\overline{\mathbf{T}}^{(2)}$ is obtained from $\mathbf{T}^{(1)}$ when $\mathbf{E}^{(1)}$ is replaced by $\mathbf{E}^{(2)}$ in the expression for $\mathbf{T}^{(1)}$. Also, $\mathbf{T}^{(1)}$ and $\overline{\mathbf{T}}^{(2)}$ are symmetric tensors; however $\mathbf{T}^{(2)}$ is not symmetric.

Let $\mathbf{e}$ denote, in the reference configuration, a unit vector along the axis of the prismatic body. We introduce the following decompositions.

$$\begin{aligned}
&\mathbf{x} = \mathbf{r} + \xi\mathbf{e},\\
&\mathbf{u}^{(n)} = w^{(n)}\mathbf{e} + \mathbf{v}^{(n)},\\
&\mathbf{E}^{(n)} = \epsilon^{(n)}\mathbf{e}\otimes\mathbf{e} + \boldsymbol{\gamma}^{(n)}\otimes\mathbf{e} + \mathbf{e}\otimes\boldsymbol{\gamma}^{(n)} + \hat{\mathbf{E}}^{(n)},\\
&\mathbf{T}^{(1)} = \sigma^{(1)}\mathbf{e}\otimes\mathbf{e} + \mathbf{t}^{(1)}\otimes\mathbf{e} + \mathbf{e}\otimes\mathbf{t}^{(1)} + \hat{\mathbf{T}}^{(1)},\\
&\overline{\mathbf{T}}^{(2)} = \overline{\sigma}^{(2)}\mathbf{e}\otimes\mathbf{e} + \bar{\mathbf{t}}^{(2)}\otimes\mathbf{e} + \mathbf{e}\otimes\bar{\mathbf{t}}^{(2)} + \hat{\overline{\mathbf{T}}}^{(2)},
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
&\hat{\mathbf{E}}^{(n)} = \text{sym grad}\,\mathbf{v}^{(n)},\\
&\epsilon^{(n)} = w^{(n)\prime} = \frac{\partial w^{(n)}}{\partial\xi},\\
&\boldsymbol{\gamma}^{(n)} = \tfrac{1}{2}(\mathbf{v}^{(n)\prime} + \text{grad}\,w^{(n)}),\\
&\text{sym grad}\,\mathbf{v} = \tfrac{1}{2}(\text{grad}\,\mathbf{v} + (\text{grad}\,\mathbf{v})^T),
\end{aligned} \tag{15}$$

and grad(div) is the two-dimensional gradient (divergence) operator with respect to coordinates in the cross-section $\mathcal{A}$. Thus w gives the displacement and ξ the coordinate of a point along the axis of the prismatic body, $\mathbf{v}$ the displacement in the plane of the body, and the tensor product $\otimes$ between two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined in terms of the Euclidean inner product as follows:

$$(\mathbf{a}\otimes\mathbf{b})\mathbf{c} = (\mathbf{b}\cdot\mathbf{c})\mathbf{a} \tag{16}$$

for every vector $\mathbf{c}$.

For the initial infinitesimal twist, we assume that

$$\mathbf{v}^{(1)} = \xi\tau(*\mathbf{r}), \qquad w^{(1)} = \tau\phi, \tag{17}$$

where τ is the angle of twist per unit length, ϕ is the Saint-Venant warping function in linear elasticity, and $(*\mathbf{r})$ equals $\mathbf{e}\times\mathbf{r}$. The displacement field (17) implies that

$$\mathbf{E}^{(1)} = \tau\ \text{sym}\left\{\left[(*\mathbf{r}) + \text{grad}\,\phi\right]\otimes\mathbf{e}\right\} =: \tau\ \text{sym}[\boldsymbol{\gamma}^{(1)}\otimes\mathbf{e}] \tag{18}$$

and thus

$$\mathrm{tr}\,\mathbf{E}^{(1)} = 0. \tag{19}$$

The warping function ϕ is a solution of

$$\begin{aligned} &\mathrm{div}\,\mathrm{grad}\,\phi = 0 \quad \text{in } \mathcal{A}, \\ &\mathrm{grad}\,\phi \cdot \mathbf{n} = -(*\mathbf{r}) \cdot \mathbf{n} \quad \text{on } \partial\mathcal{A}. \end{aligned} \tag{20}$$

We recall that (13) are satisfied for $n = 1$, and in the following we seek their solution for $n = 2$. The governing equations can be written as

$$\overline{\sigma}^{(2)\prime} + \mathrm{div}\,\bar{\mathbf{t}}^{(2)} = -2(\lambda + \mu)\xi\tau^2, \quad \text{in } \mathcal{A} \times [0, \ell], \tag{21}$$

$$\begin{aligned} \bar{\mathbf{t}}^{(2)\prime} + \mathrm{div}\,\bar{\hat{\mathbf{T}}}^{(2)} \;=\; & \tau^2(\lambda + \mu)\mathrm{grad}\left[(*\mathbf{r}) \cdot (\mathrm{grad}\,\phi)\right] - \left(\lambda - \mu\frac{\alpha_4}{2}\right)\boldsymbol{\gamma}^{(1)} \\ & -\mu\frac{\alpha_6}{4}\mathrm{div}\left(\boldsymbol{\gamma}^{(1)} \otimes \boldsymbol{\gamma}^{(1)}\right), \quad \text{in } \mathcal{A} \times [0, \ell], \end{aligned} \tag{22}$$

$$\bar{\mathbf{t}}^{(2)} \cdot \mathbf{N} = -\mu\xi(\mathbf{r} \cdot \mathbf{N})\tau^2, \quad \text{on } \partial\mathcal{A} \times [0, \ell], \tag{23}$$

$$\begin{aligned} \bar{\hat{\mathbf{T}}}^{(2)}\mathbf{N} \;=\; & \Bigg\{\tau^2\mu(*\mathbf{r}) \otimes \mathrm{grad}\,\phi \\ & -\Bigg[\left(\frac{1}{2}\lambda + \mu\frac{\alpha_4}{4}\right)\left(\boldsymbol{\gamma}^{(1)} \cdot \boldsymbol{\gamma}^{(1)}\right) - \tau^2\lambda(*\mathbf{r}) \\ & \cdot(\mathrm{grad}\,\phi) + \tau^2(\lambda + \mu)\xi^2\Bigg]\hat{\mathbf{I}}\Bigg\}\mathbf{N}, \quad \text{on } \partial\mathcal{A} \times [0, \ell], \end{aligned} \tag{24}$$

where $\hat{\mathbf{I}}$ is the two-dimensional identity tensor. The aforestated form of the equations exploits the geometry of the prismatic body, and in doing so we have closely followed Di Carlo [21].

3. A Saint-Venant/Almansi Solution

As shown by dell'Isola and Batra [13] the Clebsch hypothesis may not always be valid. Hence in the following analysis we use a semi-inverse method, looking for solutions of (21) through (24) having the form

$$\begin{aligned}
w^{(2)} &= \sum_{i=0}^{m} \frac{\xi^i}{i!} w_i(\mathbf{r}),\\
\mathbf{v}^{(2)} &= \sum_{i=0}^{m} \frac{\xi^i}{i!} \mathbf{v}_i(\mathbf{r}),\\
\overline{\sigma}^{(2)} &= \sum_{i=0}^{m} \frac{\xi^i}{i!} \sigma_i,\\
\bar{\mathbf{t}}^{(2)} &= \sum_{i=0}^{m} \frac{\xi^i}{i!} \mathbf{t}_i,\\
\hat{\bar{\bar{\mathbf{T}}}}^{(2)} &= \sum_{i=0}^{m} \frac{\xi^i}{i!} \hat{\bar{\bar{\mathbf{T}}}}_i.
\end{aligned} \tag{25}$$

For $i > 2$, (21)–(24) become

$$\begin{aligned}
&\sigma_{i+1} + \operatorname{div} \mathbf{t}_{i+1} = 0 \quad \text{in } \mathcal{A},\\
&\mathbf{t}_{i+1} + \operatorname{div} \hat{\bar{\bar{\mathbf{T}}}}_i = \mathbf{0} \quad \text{in } \mathcal{A},
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\mathbf{t}_i \cdot \mathbf{N} = \mathbf{0} \quad \text{on } \partial\mathcal{A},\\
&\hat{\bar{\bar{\mathbf{T}}}}_i \mathbf{N} = \mathbf{0} \quad \text{on } \partial\mathcal{A}.
\end{aligned} \tag{27}$$

Equations (26) and (27) imply that

$$\int_{\mathcal{A}} \sigma_{i+1} \mathrm{d}A = 0, \qquad \int_{\mathcal{A}} \mathbf{t}_{i+1} \mathrm{d}A = \mathbf{0}, \qquad \int_{\mathcal{A}} \mathbf{r} \times \mathbf{t}_{i+1} \mathrm{d}A = \mathbf{0}. \tag{28}$$

Substitution for σ_{i+1}, $\mathbf{t}_{i+1}$ etc. in (26) and (27) in terms of displacements yields the following.

$$\begin{aligned}
&(\lambda + 2\mu) w_{i+2} + (\lambda + \mu) \operatorname{div} \mathbf{v}_{i+1} + \mu \Delta_R w_i = 0 \quad \text{in } \mathcal{A},\\
&\mu \mathbf{v}_{i+2} + (\lambda + \mu) \operatorname{grad} w_{i+1} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{v}_i + \mu \Delta_R \mathbf{v}_i = \mathbf{0} \quad \text{in } \mathcal{A},
\end{aligned} \tag{29}$$

$$\begin{aligned}
&\mu(\mathbf{v}_{i+1} + \operatorname{grad} w_i) \cdot \mathbf{N} = 0 \quad \text{on } \partial\mathcal{A},\\
&[2\mu \operatorname{sym} \operatorname{grad} \mathbf{v}_i + \lambda(w_{i+1} + \operatorname{div} \mathbf{v}_i) \hat{\mathbf{I}}]\mathbf{N} = \mathbf{0} \quad \text{on } \partial\mathcal{A},
\end{aligned} \tag{30}$$

where Δ_R is the Laplacian operator in the plane $\mathscr{P}$ that contains $\mathscr{A}$. For $i > 3$, (29) and (30) have the solution

$$w_i = 0 \text{ and } \mathbf{v}_i = \text{rigid body motion in } \mathscr{P}, \tag{31}$$

and (28) yields that the constants appearing in $(31)_2$ must vanish. Thus the displacement fields $(25)_1$ and $(25)_2$ take the following forms:

$$\begin{aligned} w^{(2)}(\mathbf{r},\xi) &= \frac{\xi^3}{6} w_3^0 + \frac{\xi^2}{2} w_2(\mathbf{r}) + \xi w_1(\mathbf{r}) + w_0(\mathbf{r}), \\ \mathbf{v}^{(2)}(\mathbf{r},\xi) &= \frac{\xi^3}{6}(\mathbf{v}_3^0 + (*\mathbf{r})\omega_3^0) + \frac{\xi^2}{2}\mathbf{v}_2(\mathbf{r}) + \xi \mathbf{v}_1(\mathbf{r}) + \mathbf{v}_0(\mathbf{r}). \end{aligned} \tag{32}$$

In (32) and below, quantities with superscript zero denote constants.

For $i = 2$, (21)–(24) yield

$$\begin{aligned} &\mu \Delta_R w_2 = 0 \quad \text{in } \mathscr{A}, \\ &F\mathbf{v}_2 = \mathbf{0} \quad \text{in } \mathscr{A}, \end{aligned} \tag{33}$$

$$\begin{aligned} &(\text{grad } w_2)\cdot \mathbf{N} = -(\mathbf{v}_3^0 + (*\mathbf{r})\omega_3^0)\cdot \mathbf{N} \quad \text{on } \partial\mathscr{A}, \\ &(G\mathbf{v}_2 + \lambda w_3^0 \hat{\mathbf{I}})\mathbf{N} = -2\tau^2(\lambda+\mu)\mathbf{N} \quad \text{on } \partial\mathscr{A}, \end{aligned} \tag{34}$$

where

$$G = \mu \text{ sym grad} + \lambda \hat{\mathbf{I}} \text{ div}$$

and

$$F = \text{div } G = \mu\Delta_R + (\lambda+\mu) \text{ grad div} \tag{35}$$

is the Navier operator. For $i = 1$, the integral version of (21) and (22) over $\mathscr{A}$ and the global balance of torque give

$$\begin{aligned} &\lambda \int_{\mathscr{A}} (\text{div } \mathbf{v}_2)\mathrm{d}A + (\lambda + 2\mu)w_3^0\mathscr{A} + 2\tau^2\lambda\mathscr{A} = 0, \\ &\int_{\mathscr{A}} [\mathbf{v}_3^0 + (*\mathbf{r})\omega_3 + \text{grad } w_2]\mathrm{d}A = \mathbf{0}, \\ &\int_{\mathscr{A}} (*\mathbf{r})\cdot[\mathbf{v}_3^0 + (*\mathbf{r})\omega_3 + \text{grad } w_2]\mathrm{d}A = \mathbf{0}. \end{aligned} \tag{36}$$

A solution of $(33)_1$ and $(34)_1$ is

$$w_2(\mathbf{r}) = w_2^0 - \mathbf{v}_3^0\cdot\mathbf{r} + \omega_3^0\phi(\mathbf{r}), \tag{37}$$

and $(36)_3$ requires that $\omega_3^0 = 0$. However, when on the mantle surface tractions whose resultant is a linearly varying torque are applied, then ω_3^0 will not vanish and the warping function for w_2 will equal that for the linear elastic problem. A solution of $(33)_2$ and $(34)_2$ is

$$\mathbf{v}_2(\mathbf{r}) = \mathbf{v}_2^0 + (*\mathbf{r})\omega_2^0 - \left(\tau^2 + \nu w_3^0\right)\mathbf{r}, \tag{38}$$

where $\nu = \frac{\lambda}{2(\lambda+\mu)}$ is Poisson's ratio. Now $(36)_1$ yields

$$w_3^0 = 0. \tag{39}$$

For $i = 1$, (21)–(24) reduce to

$$\begin{aligned} &\Delta w_1 = 0 \quad \text{in } \mathcal{A}, \\ &F\mathbf{v}_1 = \lambda \mathbf{v}_3^0 \quad \text{in } \mathcal{A}, \\ &(\text{grad } w_1) \cdot \mathbf{N} = -\left(\mathbf{v}_2^0 + (*\mathbf{r})\omega_2^0\right) \cdot \mathbf{N} \quad \text{on } \partial\mathcal{A}, \\ &(G\mathbf{v}_1)\mathbf{N} = \lambda(\mathbf{v}_3^0 \cdot \mathbf{r} - w_2^0)\mathbf{N} \quad \text{on } \partial\mathcal{A}, \end{aligned} \tag{40}$$

and a solution of $(40)_1$ and $(40)_3$ may simply be written as

$$w_1(\mathbf{r}) = w_1^0 - \mathbf{v}_2^0 \cdot \mathbf{r} + \omega_2^0 \phi(\mathbf{r}). \tag{41}$$

The balance of torque implies that

$$\begin{aligned} &\mu \int_{\mathcal{A}} (*\mathbf{r}) \cdot (\mathbf{v}_2 + \text{grad } w_1) \mathrm{d}A - \tau^2 \mu \int_{\mathcal{A}} \mathbf{r} \cdot \text{grad } \phi = 0 \\ &\Rightarrow \omega_2^0 = \frac{\tau^2 \mu \int_{\mathcal{A}} \mathbf{r} \cdot \text{grad } \phi}{J_0 - D}, \end{aligned} \tag{42}$$

where J_0 is the polar moment of inertia and $D := \int_{\mathcal{A}} \mathbf{r} \cdot *\text{grad } \phi$ is the so called Dirichlet integral, and (38) simplifies to

$$\mathbf{v}_2(\mathbf{r}) = \mathbf{v}_2^0 + \frac{\tau^2 \mu \int_{\mathcal{A}} \mathbf{r} \cdot \text{grad } \phi}{J_0 - D} * \mathbf{r} - \tau^2 \mathbf{r}. \tag{43}$$

A solution of $(40)_2$ and $(40)_4$ is

$$\mathbf{v}_1(\mathbf{r}) = \mathbf{v}_1^0 + (*\mathbf{r})\omega_1^0 - \nu w_2^0 \mathbf{r} + \tfrac{1}{2}\nu[\mathbf{r} \otimes \mathbf{r} + (*\mathbf{r}) \otimes (*\mathbf{r})]\mathbf{v}_3^0. \tag{44}$$

In order that the axial forces be balanced, we must have

$$w_2^0 = 0. \tag{45}$$

For $i = 0$, the governing equations are

$$\Delta_R w_0 = 2\mathbf{v}_3^0 \cdot \mathbf{r} \quad \text{in } \mathcal{A},$$

$$\begin{aligned} F\mathbf{v}_0 \; = \; & \lambda\left(\mathbf{v}_2^0 - \omega_2^0 \operatorname{grad}\phi\right) + \mu\tau^2\mathbf{r} + \operatorname{div}\Bigg\{\Bigg[(\lambda+\mu)\tau^2\,(*\mathbf{r})\cdot(\operatorname{grad}\phi) \\ & - \left(\frac{1}{2}\lambda - \mu\frac{\alpha_4}{4}\right)\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(1)}\Bigg]\hat{\mathbf{I}} - \frac{\mu\alpha_6}{4}\boldsymbol{\gamma}^{(1)}\otimes\boldsymbol{\gamma}^{(1)}\Bigg\} \quad \text{in } \mathcal{A}, \end{aligned} \tag{46}$$

$$(\operatorname{grad} w_0)\cdot\mathbf{N} = -\mathbf{v}_1\cdot\mathbf{N} \quad \text{on } \partial\mathcal{A},$$

$$\begin{aligned} (G\mathbf{v}_0)\mathbf{N} \; = \; & \Bigg\{\Bigg[\lambda(\mathbf{v}_2^0\cdot\mathbf{r} - w_1^0) + \tau^2\lambda(*\mathbf{r})\cdot(\operatorname{grad}\phi) - \left(\frac{1}{2}\lambda + \frac{\mu\alpha_4}{4}\right) \\ & \boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(1)}\Bigg]\hat{\mathbf{I}} + \mu\tau^2(*\mathbf{r})\otimes(\operatorname{grad}\phi)\Bigg\}\mathbf{N} \quad \text{on } \partial\mathcal{A}. \end{aligned}$$

A solution of equations (46) is

$$\begin{aligned} & w_0(\mathbf{r}) = w_0^0 + \tilde{w}_0(\mathbf{r}), \\ & \mathbf{v}_0(\mathbf{r}) = \mathbf{v}_0^0 + (*\mathbf{r})\omega_0 - \nu w_1^0\mathbf{r} \\ & \qquad + \tfrac{1}{2}\nu[\mathbf{r}\otimes\mathbf{r} - (*\mathbf{r})\otimes(*\mathbf{r})]\mathbf{v}_2^0 + \tilde{\mathbf{v}}_0(\mathbf{r}), \end{aligned} \tag{47}$$

with

$$\tilde{w}_0(\mathbf{0}) = 0,\; \tilde{\mathbf{v}}_0(\mathbf{0}) = \mathbf{0},\; \operatorname{grad}\tilde{\mathbf{v}}_0\big|_{\mathbf{0}} = (\operatorname{grad}\tilde{\mathbf{v}}_0\big|_{\mathbf{0}})^T. \tag{48}$$

Functions $\tilde{w}_0(\mathbf{r})$ and $\tilde{\mathbf{v}}_0(\mathbf{r})$ are solutions of

$$\Delta_R \tilde{w}_0 = 0 \quad \text{in } \mathcal{A},$$

$$\begin{aligned} F\tilde{\mathbf{v}}_0 \; = \; & \operatorname{div}\Bigg\{\Bigg[(\lambda+\mu)\tau^2(*\mathbf{r})\cdot(\operatorname{grad}\phi) - \left(\frac{1}{2}\lambda - \frac{\mu\alpha_4}{4}\right)\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(1)}\Bigg]\hat{\mathbf{I}} \\ & - \frac{\mu\alpha_6}{4}\boldsymbol{\gamma}^{(1)}\otimes\boldsymbol{\gamma}^{(1)}\Bigg\} \quad \text{in } \mathcal{A}, \end{aligned}$$

$$(\operatorname{grad}\tilde{w}_0)\cdot\mathbf{N} = -\mathbf{v}_1\cdot\mathbf{N} \quad \text{on } \partial\mathcal{A}, \tag{49}$$

$$(G\tilde{\mathbf{v}}_0)\mathbf{N} = \left\{\left[\lambda\tau^2(*\mathbf{r})\cdot(\operatorname{grad}\phi) - \left(\frac{1}{2}\lambda + \frac{\mu\alpha_4}{4}\right)\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(1)}\right]\hat{\mathbf{I}} + \mu\tau^2(*\mathbf{r})\otimes\operatorname{grad}\phi\right\}\mathbf{N} \quad \text{on } \partial\mathcal{A}.$$

Equations (6) require that

$$w_0^0 = 0, \qquad \mathbf{v}_0^0 = \mathbf{0}, \qquad \mathbf{v}_1^0 = \operatorname{grad}\tilde{w}_0(\mathbf{0}), \qquad \operatorname{grad}\mathbf{v}_0\big|_{\mathbf{0}} = (\operatorname{grad}\mathbf{v}_0\big|_{\mathbf{0}})^T. \tag{50}$$

Thus we have six non-zero scalar constants, namely, w_1^0, ω_1^0, and the components of the vectors $\mathbf{v}_3^0$, $\mathbf{v}_2^0$, which characterize respectively the (second-order) flexure, bending, extension and torsion. The corresponding resultant forces and moments on the face $\mathcal{A}_\xi$ are

$$N_f = Ew_1^0\mathcal{A} + \lambda\int_{\mathcal{A}}\operatorname{div}\tilde{\mathbf{v}}_0\,\mathrm{d}A + (\lambda+2\mu)\,\omega_2^0\int_{\mathcal{A}}\phi\,\mathrm{d}A + \tau^2\left(\frac{1}{2}\lambda - \frac{\mu\alpha_4}{4} - \frac{\mu\alpha_6}{4}\right)(J_0 - D) + \tau^2(\lambda+\mu)D, \tag{51}$$

$$\mathbf{S}_f = \mu(\operatorname{grad}\tilde{w}_0\big|_{\mathbf{0}})\mathcal{A} + \mu\int_{\mathcal{A}}(\operatorname{grad}\tilde{w}_0)\,\mathrm{d}A + \mu\int_{\mathcal{A}}\frac{1}{2}\nu[\mathbf{r}\otimes\mathbf{r} - (*\mathbf{r})\otimes(*\mathbf{r})]\mathbf{v}_3^0\,\mathrm{d}A, \tag{52}$$

$$\mathbf{M}_f = E\mathbf{J}\mathbf{v}_2^0 - \lambda\int_{\mathcal{A}}(*\mathbf{r})\operatorname{div}\tilde{\mathbf{v}}_0\,\mathrm{d}A + \mu\xi(*\mathbf{S}_f) + \tau^2(\lambda+\mu) \times\int_{\mathcal{A}}(*\mathbf{r})\otimes(*\mathbf{r})(\operatorname{grad}\phi)\,\mathrm{d}A - \left(\frac{1}{2}\lambda - \frac{\mu\alpha_4}{4} - \frac{\mu\alpha_6}{4}\right) \times\int_{\mathcal{A}}\|\boldsymbol{\gamma}\|^2 * \mathbf{r}\,\mathrm{d}A, \tag{53}$$

$$T_f = \mu\left\{\omega_1^0\int_{\mathcal{A}}(*\mathbf{r})\cdot\left(*\mathbf{r} + \operatorname{grad}\frac{\tilde{w}_0}{\omega_1^0}\right)\mathrm{d}A + \int_{\mathcal{A}}[(\mathbf{r}\cdot\mathbf{r})(*\mathbf{r})\cdot\mathbf{v}_3^0]\,\mathrm{d}A\right\}, \tag{54}$$

where

$$E = \mu(3\lambda+2\mu)/(\lambda+\mu), \qquad \mathbf{J} = \int_{\mathcal{A}}(*\mathbf{r})\otimes(*\mathbf{r})\,\mathrm{d}A. \tag{55}$$

E is Young's modulus and $\mathbf{J}$ a tensor of inertia.

It is clear from the aforestated analysis that the small parameter η in (8) can be identified with the infinitesimal twist per unit length, τ, included in (17). The problem of determining the additional displacement field $\mathbf{u}^{(2)}$ caused by the loads superimposed upon the twisted bar has been reduced to that of solving two boundary value problems (cf. (49)) in the plane.

4. Extension Superimposed upon Infinitesimal Twist for a Circular Bar

For this case

$$\begin{aligned} &\mathbf{v}_3^0 = \mathbf{v}_2^0 = \mathbf{0}, \qquad w_2 = w_3 = 0, \qquad \omega_1 = 0, \qquad \phi = 0, \\ &w_1 = w_1^0, \qquad \mathbf{v}_2 = -\tau^2 \mathbf{r}, \qquad \mathbf{v}_1 = \mathbf{v}_1^0, \end{aligned} \tag{56}$$

and $(46)_1$ and $(46)_3$ reduce to

$$\begin{aligned} &\Delta_R w_0 = 0 \quad \text{in } \mathcal{A}, \\ &\text{grad } w_0 \cdot \mathbf{N} = -\mathbf{v}_1^0 \cdot \mathbf{N} \quad \text{on } \partial\mathcal{A}. \end{aligned} \tag{57}$$

Equations (57) have the solution

$$w_0 = w_0^0 - \mathbf{v}_1^0 \cdot \mathbf{r}. \tag{58}$$

The clamping conditions (6) require that

$$w_0^0 = 0, \qquad \mathbf{v}_1^0 = \text{grad } w_0\big|_{\mathbf{0}}, \tag{59}$$

and thus

$$\mathbf{v}_1^0 = \mathbf{0}, \qquad w_0 = 0. \tag{60}$$

That is, there is no second-order warping, as is to be expected. The solution of $(46)_2$ and $(46)_4$ is

$$\begin{aligned} \mathbf{v}_0(\mathbf{r}) \;=\; &-\nu w_1^0 \mathbf{r} + \frac{\tau^2}{8(\lambda + 2\mu)} \\ &\left\{ \mu(k_1 - \lambda k_2)(\mathbf{r}\cdot\mathbf{r}) - \frac{\mu R^2}{\lambda + \mu}[(2\lambda + 3\mu)k_1 + \lambda k_2] \right\} \mathbf{r}, \end{aligned} \tag{61}$$

where $k_1 = 1 + \frac{\alpha_6}{4}$, $k_2 = 1 - \frac{\alpha_4}{2\alpha_1}$ and R is the radius of the cross-section of the bar. We note that our computed $\mathbf{v}_0(\mathbf{r})$ differs from Rivlin's assumed displacement field ((6.4) of [14]) in some of the terms cubic in the in-plane coordinates. Since the correction terms found by Rivlin are not included in his paper, our result cannot be compared with his. However, the in-plane displacement given by (61) coincides with that found by Wang and Truesdell [20]. For an incompressible, isotropic, homogeneous nonlinear elastic cylinder, Green and Adkins [16] considered a displacement field akin to that given by (61). From (61) we conclude that the in-plane radial displacements have two components, one proportional to the radial distance from the centroid and the other proportional to the cube of this distance; both depend upon second-order elasticities and are proportional to τ^2. Also, the in-plane tangential displacements are proportional to $\tau^2 R^3$ and will probably play a significant role in the deformations of thin-walled tubular specimens. The in-plane displacements given by (61) are independent of the axial location of the cross-section.

Equations (51) through (54) yield that the second-order resultant forces and moments, except for the axial force, at the end face $\mathcal{A}_\ell$ vanish identically. The resultant axial force, N, is given by

$$N = E\mathcal{A}w_1^0 + \frac{\tau^2 \mu \pi R^4}{8(\lambda + \mu)} \left[(\lambda + 2\mu)\frac{\alpha_6}{2} - \mu\alpha_4 \right]. \tag{62}$$

For $N = 0$, (62) yields

$$w_1^0 = -\frac{\tau^2 R^2}{16} \frac{(\lambda + 2\mu)\alpha_6 - 2\mu\alpha_4}{3\lambda + 2\mu}. \tag{63}$$

Thus a cylinder made of a second-order elastic material and twisted by applying only torques at its end faces may elongate or contract depending upon the values of second-order elasticities. This effect, first discovered by Poynting [19], has been studied by Rivlin [14] and others; *e.g.* see Truesdell and Noll [18] for additional references. Equation (63) is the same as that in Wang and Truesdell [20].

5. Conclusions

We have analysed the Saint-Venant problem for an isotropic and homogeneous prismatic body made of a second-order elastic material. It is assumed that its deformations from the stress-free state consist of an infinitesimal twist and those caused by the application of loads at its end faces only. The centroid of one end face is taken to be rigidly clamped in the sense that the displacements and infinitesimal rotations there vanish. The displacements superimposed upon the infinitesimal twist are determined by the Signorini's method, and this problem is reduced to that of solving two linear elliptic problems in the plane; the corresponding resultant loads on an end face are given by (51) through (54).

The right-hand side of (51) gives the resultant traction applied on the end faces. For zero resultant axial force, w_1^0 need not vanish implying a change in the length of the prismatic body. This effect has been delineated for a solid circular cylinder in Section 4.

Equations (37), (38), and the constitutive relation for $\hat{\bar{\mathbf{T}}}^{(2)}$ imply that $\hat{\bar{\mathbf{T}}}_2$ is a nonzero spherical tensor. From $(47)_2$ we conclude that $\hat{\bar{\mathbf{T}}}_0$ does not vanish in general. Hence Clebsch's hypothesis does not hold in this case.

Equation (38) yields Poisson's effect linear in the in-plane position vector **r** but quadratic in the axial coordinate ξ. Additionally, $(46)_2$ gives Poisson's effect, not of Saint-Venant's type, which is independent of ξ.

We note that constants $\mathbf{v}_3^0$, $\mathbf{v}_2^0$, w_1^0 and ω_1^0 characterizing the flexure, bending, extension and torsion should be proportional to τ^2 in order for Signorini's expansion to be valid.

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