



Linear Constitutive Relations in Isotropic Finite Elasticity

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Abstract. For simple shearing and simple extension deformations of a homogeneous and isotropic elastic body, it is shown that a linear relation between the second Piola–Kirchhoff stress tensor and the Green–St. Venant strain tensor does not predict a physically reasonable response of the body. This constitutive relation implies that the slope of the curve between an appropriate component of the first Piola–Kirchhoff stress tensor and a deformation measure is an increasing function of the deformation measure.

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For isotropic materials a generalization of Hooke’s law in linear elasticity to a linear stress-strain relation in finite elasticity is the neo-Hookean constitutive relation [1] which is an affine relation between the Cauchy stress σ and the left Cauchy-Green tensor $\mathbf{B}$, and is generally used for incompressible rubberlike materials. Another possibility is to assume a linear relation between the second Piola–Kirchhoff stress tensor $\mathbf{S}$ and the Green-St. Venant strain tensor $\mathbf{E}$ [2–4]. For infinitesimal deformations, these two relations reduce to Hooke’s law as they should. This note is intended to show that a linear relationship between $\mathbf{S}$ and $\mathbf{E}$ does *not* qualitatively mimic the stress-strain response of most materials deformed in simple shear or simple extension. For finite deformations of several elastic materials, the slope of the nominal (or the first Piola–Kirchhoff) stress-strain curve is generally a non-increasing function of strain when they are deformed either in simple shear or simple tension [5]. However, a linear relation between $\mathbf{S}$ and $\mathbf{E}$ for isotropic materials predicts that this slope increases with the amount of deformation. Thus the instability load for such materials, assuming that it can be found, will be unrealistically large.

We consider homogeneous and quasistatic deformations of an isotropic and homogeneous body that is stress free in the reference configuration. Thus the balance of linear momentum is identically satisfied. Let

$$\mathbf{S} = \lambda(\text{tr} \mathbf{E})\mathbf{1} + 2\mu\mathbf{E} \quad (1)$$

represent a linear constitutive relation for the elastic body. In order for (1) to reduce to Hooke's law for infinitesimal deformations, λ and μ must equal the Lamé constants and are taken to satisfy $\mu > 0$, $\lambda + (2/3)\mu > 0$.

In rectangular Cartesian coordinates, a simple shearing deformation is given by

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3, \quad (2)$$

where $\mathbf{x}$ gives the position vector in the deformed configuration of the material particle that occupied place $\mathbf{X}$ in the reference configuration, and k is a measure of the shear strain. From (1), (2) and

$$\boldsymbol{\sigma} = \mathbf{F}\mathbf{S}\mathbf{F}^T/J, \quad \mathbf{T} = \mathbf{F}\mathbf{S}, \quad (3)$$

where $\mathbf{T}$ is the first Piola–Kirchhoff stress tensor, $\mathbf{F}$ the deformation gradient, and $J = \det \mathbf{F}$, we obtain

$$T_{12} = \mu k + [\frac{1}{2}\lambda + \mu]k^3. \quad (4)$$

Thus

$$\frac{d^2 T_{12}}{dk^2} = 3(\lambda + 2\mu)k > 0 \quad (5)$$

and the tangent modulus dT_{12}/dk is an increasing function of k .

For general deformations, equations (1), (3) and the Hamilton–Cayley theorem [1] imply that

$$\boldsymbol{\sigma} = -\frac{\mu}{J}II\mathbf{I} + \frac{1}{J}[(\frac{1}{2}\lambda + \mu)(I - 3) + 2\mu]\mathbf{B} + \mu J\mathbf{B}^{-1}, \quad (6)$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, $I = \text{tr } \mathbf{B}$ and $II = (I^2 - \text{tr } \mathbf{B}^2)/2$. Thus the relation between $\boldsymbol{\sigma}$ and $\mathbf{B}$ corresponding to (1) is nonlinear.

We now study simple extension, given by

$$x_1 = \alpha X_1, \quad x_2 = \beta X_2, \quad x_3 = \beta X_3, \quad (7)$$

of a prismatic body; α and β are constants in (7). We assume that the deformation (7) is produced by surface tractions parallel to the x_1 -axis, applied only on the end faces of the prismatic body. Equations (1), (7), the requirement that the mantle of the prismatic body be traction free, and (3) give

$$T_{11} = \mu \frac{3\lambda + 2\mu}{2(\lambda + \mu)} \alpha(\alpha^2 - 1). \quad (8)$$

Thus

$$\frac{d^2 T_{11}}{d\alpha^2} = 3\mu \frac{3\lambda + 2\mu}{(\lambda + \mu)} \alpha > 0 \quad (9)$$

and the tangent modulus is an increasing function of α .

The neo-Hookean constitutive relation

$$\boldsymbol{\sigma} = \frac{1}{2}\lambda(\text{tr } \mathbf{B} - 3)\mathbf{1} + \mu(\mathbf{B} - \mathbf{1}) \quad (10)$$

gives $T_{12} = \mu k$ for simple shearing deformations (2) and

$$T_{11} = \left(\frac{3\lambda + 2\mu}{2(\lambda + \mu)} \right) \left(\frac{\mu^2}{\lambda + \mu} \right) (\alpha^2 - 1)(1 - \lambda(\alpha^2 - 3)/2\mu), \quad (11)$$

for simple extension (7) which are reasonable. From (11) one can conclude that the tangent modulus $d T_{11}/d\alpha$ is positive at $\alpha = 1$, equals zero for $\alpha^2 = (\mu + 2\lambda)/\lambda$ and is negative for $\alpha^2 > (\mu + 2\lambda)/\lambda$. However, the corresponding relation

$$\mathbf{S} = J[(\lambda \text{tr } \mathbf{E} - \mu)(\mathbf{1} + 2\mathbf{E})^{-1} + \mu\mathbf{1}], \quad (12)$$

between $\mathbf{S}$ and $\mathbf{E}$ is more involved, and is nonlinear.

In conclusion, the linear relation between $\mathbf{S}$ and $\mathbf{E}$ does *not* mimic the nominal stress vs. strain curve experimentally observed for a large class of isotropic elastic materials.

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