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A Second-Order Solution of Saint-Venant's Problem for a Piezoelectric Circular Bar Using Signorini's Perturbation Method

The paper is dedicated with deep respect to Professor Roger Fosdick on his 60th birthday.

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Abstract. We study electromechanical deformations of a homogeneous transversely isotropic piezoelectric prismatic circular bar loaded only at the end faces. The constitutive relations for the material of the bar are taken to be quadratic in the displacement gradients and the electric field. It is found that the two end faces of the bar when twisted with no electric charge applied to them will exhibit a difference in the electric potential. Thus the piezoelectric cylinder could be used to measure the torque or the angular twist.

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1. Introduction

Poynting [1] discovered in 1909 that a wire when twisted also stretches and the stretch is proportional to the square of the angular twist. Since then there have been several attempts made to quantify this effect. Truesdell and Noll [2] and Wang and Truesdell [3] have reviewed the pertinent literature on the Poynting effect and also on the Signorini's method [4] of solving a nonlinear problem by reducing it to a series of linear problems. Green and Adkins [5] have pointed out that the compatibility conditions to be satisfied by the loads in the sequence of linear problems are automatically satisfied if the centroid of one end face is rigidly clamped in the sense that the displacements and infinitesimal rotations there vanish. Rivlin [6] and Green and Shield [7] have studied the Poynting effect in nonlinear elastic materials. Recently, dell'Isola et al. [8] used the Signorini expansion method to find a second-order solution of the Saint-Venant problem [9] for a pretwisted bar. Subsequently, they [10] extended it to a prebent bar and delineated generalized Poynting effects. Using the general theory of piezoelasticity (see e.g. [11]) we analyze here electromechanical deformations of a circular cylindrical piezoelectric bar made of a transversely isotropic material. Second order constitutive relations for a piezoelectric material have been derived by Yang and Batra [12]. It is found that the second-order Poisson effect is not of the Saint-Venant type, and even when the bar is deformed by applying pure torques and no electric charges at the end faces, the potential difference between the end faces is proportional to the square of the angular twist.

We note that Batra and Yang [13] have proved Toupin's version [14] of the Saint-Venant principle for a linear piezoelectric bar. Iesan [15–18] has studied the Saint-Venant problem for inhomogeneous and anisotropic linear elastic bodies, elastic dielectrics, and microstretch elastic solids. Dell'Isola and Rosa [19, 20] and Davi [21] have analyzed the Saint-Venant problem for linear piezoelectric bodies, and dell'Isola and Batra [22] for linear elastic porous solids.

2. Formulation of the Problem

Equations governing quasistatic deformations of a homogeneous transversely isotropic piezoelectric body Ω are

$$\mathrm{Div}(\mathbf{T}+\mathbf{T}^E)=\mathbf{0}, \quad \text{in}\,\Omega, \tag{1.1}$$

$$(\mathbf{T}+\mathbf{T}^E)\mathbf{F}^T=\mathbf{F}(\mathbf{T}+\mathbf{T}^E)^T, \quad \text{in}\,\Omega, \tag{1.2}$$

$$\mathrm{Div}(\mathbb{D})=0,\ \text{in}\,\Omega, \tag{1.3}$$

where $\mathbf{T}$ is the first Piola–Kirchhoff stress tensor, $\mathbf{T}^E$ the first Piola–Kirchhoff–Maxwell stress tensor, $\mathbb{D}$ the referential electric displacement, and Div is the divergence operator with respect to coordinates in the reference configuration. These quantities are related to their counterparts in the present configuration as follows.

$$\mathbf{T}=J\boldsymbol{\sigma}\mathbf{F}^{-1^T}, \qquad \mathbf{T}^E=J\boldsymbol{\sigma}^E\mathbf{F}^{-1^T}, \qquad \mathbb{D}=J\mathbf{F}^{-1}\mathbf{D}. \tag{2}$$

Here $J=\det\mathbf{F}$, $\mathbf{F}$ is the deformation gradient, $\boldsymbol{\sigma}$ the Cauchy stress tensor, $\boldsymbol{\sigma}^E$ the Cauchy–Maxwell stress tensor, and $\mathbf{D}$ the electric displacement in the present configuration. Equations (1.1), (1.2) and (1.3) express, respectively, the balance of linear momentum, the balance of moment of momentum, and the Maxwell law for the electric displacement with the body charge density set equal to zero. Constitutive relations for $\mathbf{T}$ and $\mathbf{T}^E$ will be chosen so that (1.2) is identically satisfied.

For a piezoelectric material, we introduce, in the present configuration, electric field $\hat{\mathbf{E}}$ and electric polarization $\mathbf{P}$ through

$$\mathbf{D}=\mathbf{P}+\hat{\mathbf{E}}. \tag{3}$$

Following Abraham, Einstein and Laub (see [11] Equation 3.6.22,23) we choose the following constitutive equation for $\boldsymbol{\sigma}^E$

$$\boldsymbol{\sigma}^E = \mathrm{Sym}(\mathbf{P} \otimes \hat{\mathbf{E}}) + \hat{\mathbf{E}} \otimes \hat{\mathbf{E}} - \tfrac{1}{2}\hat{E}^2\mathbf{1}, \tag{4}$$

where $\mathrm{Sym}(\mathbf{a} \otimes \mathbf{b}) = (\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})/2$, $\mathbf{1}$ is the identity tensor, and the tensor product $\otimes$ between two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \tag{5}$$

for every vector $\mathbf{c}$. Quantities $\mathbf{P}$ and $\hat{\mathbf{E}}$ are related to their counterparts $\boldsymbol{\Pi}$ and $\mathbf{W}$ in the reference configuration as

$$\boldsymbol{\Pi} = J\mathbf{F}^{-1}\mathbf{P}, \qquad \mathbf{W} = \mathbf{F}^T\hat{\mathbf{E}}. \tag{6}$$

Let ψ denote an electric potential field in the reference configuration so that

$$\mathbf{W} = -\mathrm{Grad}\,\psi, \tag{7}$$

where Grad is the gradient operator in the reference configuration. The existence of ψ is guaranteed by the referential Maxwell equation $\mathrm{Curl}\,\mathbf{W} = 0$.

We consider a prismatic body occupying the domain $\Omega = \mathcal{A} \times [0, \ell]$ in the stress and polarization free reference configuration with its axis aligned along the direction $\mathbf{e}$ of its transverse isotropy. Thus $\mathcal{A}$ is the cross-section and ℓ the length of the body. The mantle of the prismatic body is taken to be free of surface tractions and electric charge, the centroid of the end face $\mathcal{A}_0 := \mathcal{A} \times \{0\}$ is rigidly clamped in the sense that displacements $\mathbf{u} = \mathbf{x} - \mathbf{X}$, infinitesimal rotations $(\mathbf{H} - \mathbf{H}^T)/2$ and the electric potential ψ there vanish, and surface tractions and electric charge are prescribed on the end faces $\mathcal{A}_0$ and $\mathcal{A}_\ell := \mathcal{A} \times \{\ell\}$ such that the body is in equilibrium. Thus

$$(\mathbf{T} + \mathbf{T}^E)\mathbf{N} = \mathbf{0}, \qquad \mathbb{D} \cdot \mathbf{N} = 0 \quad \text{on } \partial\mathcal{A} \times [0, \ell], \tag{8.1}$$

$$(\mathbf{T} + \mathbf{T}^E)\mathbf{e} = \mathbf{f}, \qquad \mathbb{D} \cdot \mathbf{e} = q \quad \text{on } \mathcal{A}_0 \quad \text{and} \quad A_\ell. \tag{8.2}$$

Here $\mathbf{N}$ is an outward unit normal on the mantle $\partial\mathcal{A} \times [0, \ell]$, $\mathbf{f}$ the prescribed surface traction, q the specified electric charge, $\mathbf{H} = \mathrm{Grad}\,\mathbf{u}$, $\mathbf{x}$ and $\mathbf{X}$ denote, respectively, the position of a material point in the present and reference configurations. With the origin at the centroid of the cross-section $\mathcal{A}_0$, we set

$$\mathbf{X} = \mathbf{r} + z\mathbf{e}, \qquad \mathbf{u} = w\mathbf{e} + \mathbf{v}, \qquad \mathbf{W} = -(\psi'\mathbf{e} + \mathrm{grad}\,\psi), \tag{9}$$

where a prime denotes differentiation with respect to the axial coordinate z. Thus w and $\mathbf{v}$ equal the axial and in-plane components of the displacement $\mathbf{u}$ of a point. Similarly ψ' and $\mathrm{grad}\,\psi$ equal the axial and in-plane components of $\mathbf{W}$, and grad and div signify respectively the two-dimensional gradient and divergence operators in the plane $\mathcal{A}$. The integrability conditions for the problem are

$$
\begin{aligned}
&\left(\int_{\mathcal{A}} \mathbf{f}\,\mathrm{d}A\right)' = \mathbf{0}, \qquad \left(\int_{\mathcal{A}} q\,\mathrm{d}A\right)' = 0,\\
&\left(\int_{\mathcal{A}} \mathbf{x}\wedge\mathbf{f}\,\mathrm{d}A\right)' + \mathbf{x}|'_{\mathbf{r}=\mathbf{0}} \wedge \int_{\mathcal{A}} \mathbf{f}\,\mathrm{d}A = \mathbf{0},
\end{aligned}
\tag{10}
$$

where $\mathbf{a}\wedge\mathbf{b} = (\mathbf{a}\otimes\mathbf{b} - \mathbf{b}\otimes\mathbf{a})$ for arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$. Equations (10) imply that the resultant force and the resultant charge on every cross-section is the same and every portion of the bar is in equilibrium.

3. Signorini's Expansion

In Signorini's method, we assume that the displacement $\mathbf{u}$ and the electric potential ψ have a series expansion

$$
\mathbf{u} = \eta\dot{\mathbf{u}} + \eta^2\ddot{\mathbf{u}} + \ldots, \qquad \psi = \eta\dot{\psi} + \eta^2\ddot{\psi} + \ldots, \tag{11}
$$

where η is a small, yet to be determined, parameter in the problem. Surface tractions $\mathbf{f}$ and the surface charge q are similarly expanded as a power series in η. For a second-order piezoelectric material with null stresses and polarization in the reference configuration,

$$
\mathbf{T} = \eta\dot{\mathbf{S}} + \eta^2(\ddot{\mathbf{S}} + \dot{\mathbf{H}}\dot{\mathbf{S}}), \tag{12.1}
$$

$$
\mathbf{\Pi} = \eta\dot{\mathbf{\Pi}} + \eta^2\ddot{\mathbf{\Pi}}. \tag{12.2}
$$

Here $\mathbf{S}$ is the second Piola–Kirchhoff stress tensor, $\dot{\mathbf{S}}$ and $\dot{\mathbf{\Pi}}$ are homogeneous linear forms in $\dot{\mathbf{H}}$ and $\dot{\mathbf{W}}$, and $\ddot{\mathbf{S}}$ and $\ddot{\mathbf{\Pi}}$ are homogeneous quadratic forms in $\dot{\mathbf{H}}$ and $\dot{\mathbf{W}}$, and linear forms in $\ddot{\mathbf{H}}$ and $\ddot{\mathbf{W}}$. Explicit expressions for $\dot{\mathbf{S}}$, $\ddot{\mathbf{S}}$, $\dot{\mathbf{\Pi}}$ and $\ddot{\mathbf{\Pi}}$ are given as equations (16)–(19) in Yang and Batra's [12] paper and are reproduced in the Appendix. We will adopt Yang and Batra's notations for various material parameters with the exception that $2c_2$ and c_5 will be denoted by the Lamé constants λ and μ respectively. Substituting from (11) into the constitutive relations, the result into the balance laws and boundary conditions, and equating like powers of η on both sides of these equations, we arrive at the following equations for the first and second-order problems.

$$
\begin{aligned}
&\mathrm{Div}\,\dot{\mathbf{T}} = \mathbf{0}, \quad \text{in}\,\Omega,\\
&\mathrm{Div}(\dot{\mathbf{\Pi}} + \dot{\mathbf{W}}) = 0, \quad \text{in}\,\Omega,\\
&\dot{\mathbf{T}}\mathbf{N} = 0, \quad (\dot{\mathbf{\Pi}} + \dot{\mathbf{W}})\cdot\mathbf{N} = 0, \quad \text{on}\,\partial\mathcal{A}\times[0,\ell],\\
&\dot{\mathbf{T}}\mathbf{e} = \dot{\mathbf{f}},\ (\dot{\mathbf{\Pi}} + \dot{\mathbf{W}})\cdot\mathbf{e} = \dot{q}, \quad \text{on}\,\mathcal{A}_0\ \text{and}\ \mathcal{A}_\ell,
\end{aligned}
\tag{13}
$$

$$\begin{aligned}
&\operatorname{Div}(\ddot{\mathbf{T}} + \ddot{\mathbf{T}}^E) = \mathbf{0}, \quad \text{in}\,\Omega,\\
&\operatorname{Div}(\ddot{\mathbf{\Pi}} + \ddot{\mathbf{W}} + \dot{J}\dot{\mathbf{W}} - 2(\operatorname{Sym}\dot{\mathbf{H}})\dot{\mathbf{W}}) = 0, \quad \text{in}\,\Omega,\\
&(\ddot{\mathbf{T}} + \ddot{\mathbf{T}}^E)\mathbf{N} = \mathbf{0},\ (\ddot{\mathbf{\Pi}} + \ddot{\mathbf{W}} + \dot{J}\dot{\mathbf{W}} - 2(\operatorname{Sym}\dot{\mathbf{H}})\dot{\mathbf{W}})\cdot\mathbf{N} = 0, \quad \text{on}\,\partial\mathcal{A}\times[0,\ell],\\
&(\ddot{\mathbf{T}} + \ddot{\mathbf{T}}^E)\mathbf{e} = \ddot{\mathbf{f}},\ (\ddot{\mathbf{\Pi}} + \ddot{\mathbf{W}} + \dot{J}\dot{\mathbf{W}} - 2(\operatorname{Sym}\dot{\mathbf{H}})\dot{\mathbf{W}})\cdot\mathbf{e} = \ddot{q}, \quad \text{on}\,\mathcal{A}_0 \quad \text{and} \quad \mathcal{A}_\ell.
\end{aligned} \tag{14}$$

In an attempt to express the left-hand sides of Equations (14) for $\ddot{\mathbf{u}}$ and $\ddot{\psi}$ in the same form as those of (13) for $\dot{\mathbf{u}}$ and $\dot{\psi}$, we decompose additively $\ddot{\mathbf{T}}$ and $\ddot{\mathbf{\Pi}}$ as

$$\ddot{\mathbf{T}} = \ddot{\bar{\mathbf{T}}} + \ddot{\mathbf{T}}_s, \qquad \ddot{\mathbf{\Pi}} = \ddot{\bar{\mathbf{\Pi}}} + \ddot{\mathbf{\Pi}}_s. \tag{15}$$

$\ddot{\bar{\mathbf{T}}}$ and $\ddot{\bar{\mathbf{\Pi}}}$ are related to $\ddot{\mathbf{u}}$ and $\ddot{\psi}$ in the same way as $\dot{\mathbf{T}}$ and $\dot{\mathbf{\Pi}}$ are to $\dot{\mathbf{u}}$ and $\dot{\psi}$, the relation between the former set of variables is given below.

$$\begin{aligned}
\ddot{\bar{\mathbf{T}}} \;=\;& 2\mu\operatorname{Sym}\operatorname{grad}\ddot{\mathbf{v}} + [(c_3+\lambda)\ddot{w}' + \lambda\operatorname{div}\ddot{\mathbf{v}} - e_2\ddot{\psi}']\hat{\mathbf{I}}\\
&+\operatorname{Sym}\{[\tilde{\mu}(\ddot{\mathbf{v}}' + \operatorname{grad}\ddot{w}) - e_3\operatorname{grad}\ddot{\psi}]\otimes\mathbf{e}\}\\
&+[2(c_1 + \tfrac{1}{2}\lambda + c_3 + c_4 + \mu)\ddot{w}' + (c_3+\lambda)\operatorname{div}\ddot{\mathbf{v}}\\
&\quad - (e_1 + e_2 + 2e_3)\ddot{\psi}']\mathbf{e}\otimes\mathbf{e},
\end{aligned} \tag{16.1}$$

$$\begin{aligned}
\ddot{\bar{\mathbf{\Pi}}} \;=\;& 2\varepsilon_2\operatorname{grad}\ddot{\psi} - e_3(\ddot{\mathbf{v}}' + \operatorname{grad}\ddot{w})\\
&+[2(\varepsilon_1+\varepsilon_2)\ddot{\psi}' - (e_1+e_2+2e_3)\ddot{w}' - e_2\operatorname{div}\ddot{\mathbf{v}}]\mathbf{e}.
\end{aligned} \tag{16.2}$$

Here $c_1, c_3, c_4, e_1, e_2, e_3, \varepsilon_1$ and ε_2 are material constants, $\tilde{\mu} = (c_4+2\mu)/2$, and $\hat{\mathbf{I}}$ is the two-dimensional identity operator. Equations (16.1) and (16.2) are constitutive relations for a linear transversely isotropic piezoelectric material. We presume that the piezoelastic constants $\lambda, \mu, c_1, c_3, c_4, e_1, e_2, e_3, \varepsilon_1$ and ε_2 are such that the strain energy density is positive definite so that the solution of a traction boundary value problem for a linear piezoelectric body is unique to within a rigid body motion. Substitution from (15) into (14) and the integrability conditions (10) yields

$$\begin{aligned}
&\operatorname{Div}\ddot{\bar{\mathbf{T}}} = \mathbf{b}_s, \quad \text{in}\,\Omega,\\
&\operatorname{Div}(\ddot{\bar{\mathbf{\Pi}}} + \ddot{\mathbf{W}}) = c_s, \quad \text{in}\,\Omega,\\
&\ddot{\bar{\mathbf{T}}}\mathbf{N} = \mathbf{f}_{ms},\ (\ddot{\bar{\mathbf{\Pi}}} + \ddot{\mathbf{W}})\cdot\mathbf{N} = q_{ms}, \quad \text{on}\,\partial\mathcal{A}\times[0,\ell],\\
&\int_{\mathcal{A}}\ddot{\bar{\mathbf{T}}}\mathbf{e}\,\mathrm{d}A = \int_{\mathcal{A}}\ddot{\mathbf{f}}\,\mathrm{d}A + \mathbf{R}_{Fs},
\end{aligned}$$

$$\int_{\mathcal{A}} \mathbf{X} \wedge (\ddot{\ddot{\mathbf{T}}}\mathbf{e})\,\mathrm{d}A = \int_{\mathcal{A}} \mathbf{X} \wedge \ddot{\mathbf{f}}\,\mathrm{d}A + \mathbf{R}_{Ms}, \tag{17}$$

$$\int_{\mathcal{A}} (\ddot{\ddot{\boldsymbol{\Pi}}} + \ddot{\mathbf{W}}) \cdot \mathbf{e}\,\mathrm{d}A = \int_{\mathcal{A}} \ddot{q}\,\mathrm{d}A + R_{Qs},$$

$$\int_{\mathcal{A}} \ddot{\ddot{\mathbf{T}}}{}'\mathbf{e}\,\mathrm{d}A = \mathbf{h}_s, \int_{\mathcal{A}} (\mathbf{X} \wedge \ddot{\ddot{\mathbf{T}}}\mathbf{e})'\,\mathrm{d}A + \mathbf{e} \wedge \int_{\mathcal{A}} \ddot{\ddot{\mathbf{T}}}\mathbf{e}\,\mathrm{d}A = \mathbf{g}_s,$$

$$\int_{\mathcal{A}} (\ddot{\ddot{\boldsymbol{\Pi}}} + \ddot{\mathbf{W}})' \cdot \mathbf{e}\,\mathrm{d}A = i_s,$$

where

$$\begin{aligned}
\mathbf{b}_s &= -\mathrm{Div}(\ddot{\mathbf{T}}_s + \ddot{\mathbf{T}}^E),\ c_s = -\mathrm{Div}(\ddot{\boldsymbol{\Pi}}_s + \dot{J}\dot{\mathbf{W}} - 2(\mathrm{Sym}\,\dot{\mathbf{H}})\dot{\mathbf{W}}), \\
\mathbf{f}_{ms} &= -(\ddot{\mathbf{T}}_s + \ddot{\mathbf{T}}^E)\mathbf{N},\ q_{ms} = -(\ddot{\boldsymbol{\Pi}}_s + \dot{J}\dot{\mathbf{W}} - 2(\mathrm{Sym}\,\dot{\mathbf{H}})\dot{\mathbf{W}}) \cdot \mathbf{N}, \\
\mathbf{R}_{Fs} &= -\int_{\mathcal{A}} (\ddot{\mathbf{T}}_s + \ddot{\mathbf{T}}^E)\mathbf{e}\,\mathrm{d}A,\ \mathbf{R}_{Ms} = -\int_{\mathcal{A}} \mathbf{X} \wedge (\ddot{\mathbf{T}}_s + \ddot{\mathbf{T}}^E)\mathbf{e}\,\mathrm{d}A, \\
R_{Qs} &= -\int_{\mathcal{A}} (\ddot{\boldsymbol{\Pi}}_s + \dot{J}\dot{\mathbf{W}} - 2(\mathrm{Sym}\,\dot{\mathbf{H}})\dot{\mathbf{W}}) \cdot \mathbf{e}\,\mathrm{d}A, \\
\mathbf{h}_s &= -\int_{\mathcal{A}} (\ddot{\mathbf{T}}_s + \ddot{\mathbf{T}}^E)'\mathbf{e}\,\mathrm{d}A, \\
\mathbf{g}_s &= -\int_{\mathcal{A}} [\mathbf{X} \wedge (\ddot{\mathbf{T}}_s + \ddot{\mathbf{T}}^E)\mathbf{e} + \dot{\mathbf{u}} \wedge \dot{\mathbf{T}}\mathbf{e}]'\,\mathrm{d}A \\
&\quad -\mathbf{e} \wedge \int (\ddot{\mathbf{T}}_s + \ddot{\mathbf{T}}^E)\mathbf{e}\,\mathrm{d}A - \dot{\mathbf{u}}'|_{\mathbf{r}=\mathbf{0}} \wedge \int_{\mathcal{A}} \dot{\mathbf{T}}\mathbf{e}\,\mathrm{d}A, \\
i_s &= -\int_{\mathcal{A}} (\ddot{\boldsymbol{\Pi}}_s + \dot{J}\dot{\mathbf{W}} - 2(\mathrm{Sym}\,\dot{\mathbf{H}})\dot{\mathbf{W}})' \cdot \mathbf{e}\,\mathrm{d}A.
\end{aligned} \tag{18}$$

We assume that the bar is initially twisted by an infinitesimal amount τ and carries a small electric field $(-\omega)\mathbf{e}$. Its deformations are given by

$$\dot{\mathbf{u}} = -\nu a\omega\mathbf{r} + z\tau(*\mathbf{r}) + za\omega\mathbf{e},\ \dot{\psi} = z\omega, \tag{19.1}$$

where

$$a = \frac{e_2}{c_3}, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}, \qquad *\mathbf{r} = \mathbf{e} \times \mathbf{r}. \tag{19.2}$$

Note that the Saint-Venant warping function is zero for a circular cross-section. In order for the deformations caused by the electric field and the twist to be of the same order of magnitude, $a\omega$ and $R\tau$ should be about the same. Here R is the radius of the circular bar. Thus the small parameter η in (11) can be identified with either $a\omega/R$ or τ. Terms $\mathbf{b}_s$, c_s, q_{ms}, $\mathbf{f}_{ms}$, $\mathbf{R}_{Fs}$, $\mathbf{R}_{ms}$ and R_{Qs} in (17) and (18) are homogeneous quadratic forms in ω and τ and are given below.

$$\begin{aligned}
\mathbf{b}_s &= \chi_1\tau^2\mathbf{r} + \chi_2\tau^2 z\mathbf{e},\\
c_s &= \chi_3\tau^2 z,\\
\mathbf{f}_{ms} &= (\chi_4\tau^2 r^2 + \chi_5\omega^2 + \chi_6\tau^2 z^2)\mathbf{N} + \chi_7\tau^2(\mathbf{r}\otimes\mathbf{r})\mathbf{N}\\
&\quad +(\chi_8\tau^2 z\mathbf{r}\cdot\mathbf{N} + \chi_9\tau\omega(*\mathbf{r})\cdot\mathbf{N})\mathbf{e},\\
q_{ms} &= \chi_{10}\tau^2 z\mathbf{r}\cdot\mathbf{N} + \chi_{11}\tau\omega(*\mathbf{r})\cdot\mathbf{N},\\
\mathbf{R}_{Fs} &= (\chi_{12}\tau^2 J_{\mathcal{A}} + (\chi_{13}\tau^2 z^2 + \chi_{14}\omega^2)A)\mathbf{e},\\
\mathbf{R}_{Ms} &= \chi_{15}\tau\omega J_{\mathcal{A}}(\mathbf{e}_1\wedge\mathbf{e}_2),\\
R_{Qs} &= \chi_{16}\tau^2 J_{\mathcal{A}} + (\chi_{17}\tau^2 z^2 + \chi_{18}\omega^2)A.
\end{aligned}\tag{20}$$

Expressions for $\chi_1, \chi_2 \ldots \chi_{18}$ in terms of the elastic constants used in the constitutive relation are given in the Appendix. A equals the area of cross-section of the bar, $J_{\mathcal{A}}$ is the polar moment of inertia, and $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthonormal vectors in $\mathcal{A}$.

Substitution for $\ddot{\bar{\mathbf{T}}}$ and $\ddot{\bar{\mathbf{\Pi}}}$ from (16.1) and (16.2) into (17), and recalling (9), we arrive at the following field equations for the determination of $\ddot{\mathbf{u}}$ and $\ddot{\psi}$.

$$\begin{aligned}
&F(\ddot{\mathbf{v}}) + (c_3+\lambda+\tilde{\mu})\mathrm{grad}\,\ddot{w}' - (e_2+e_3)\mathrm{grad}\,\ddot{\psi}' + \tilde{\mu}\ddot{\mathbf{v}}' = \mathbf{b}_{s\mathcal{A}}, \quad \text{in } \mathcal{A},\\
&\Delta_R\ddot{\tilde{w}} + (c_3+\lambda+\tilde{\mu})\mathrm{div}\,\ddot{\mathbf{v}}' + 2(c_1+\tfrac{\lambda}{2}+c_3+c_4+\mu)\ddot{w}''\\
&\qquad -(e_1+e_2+2e_3)\ddot{\psi}'' = b_{se}, \text{ in } \mathcal{A},\\
&\Delta_R\ddot{\tilde{\psi}} - (e_2+e_3)\mathrm{div}\,\ddot{\mathbf{v}}' + 2(\varepsilon_1+\varepsilon_2-1/2)\ddot{\psi}''\\
&\qquad -(e_1+e_2+2e_3)\ddot{w}'' = c_s, \quad \text{in } \mathcal{A},\\
&G(\ddot{\mathbf{v}})\mathbf{N} + [(c_3+\lambda)\ddot{w}' - e_2\ddot{\psi}']\mathbf{N} = \mathbf{f}_{ms\mathcal{A}}, \quad \text{on } \partial\mathcal{A},\\
&\mathrm{grad}\,\ddot{\tilde{w}}\cdot\mathbf{N} + \tilde{\mu}\ddot{\mathbf{v}}'\cdot\mathbf{N} = f_{mse}, \quad \text{on } \partial\mathcal{A},\\
&\mathrm{grad}\,\ddot{\tilde{\psi}}\cdot\mathbf{N} - e_3\ddot{\mathbf{v}}'\cdot\mathbf{N} = q_{ms}, \quad \text{on } \partial\mathcal{A},
\end{aligned}\tag{21}$$

where

$$\begin{aligned}
&F(\mathbf{v}) = \mu\Delta_R\mathbf{v} + (\lambda+\mu)\mathrm{grad}\,\mathrm{div}\,\mathbf{v},\\
&G(\mathbf{v}) = 2\mu\,\mathrm{Sym}\,\mathrm{grad}\,\mathbf{v} + \lambda(\mathrm{div}\,\mathbf{v})\hat{\mathbf{I}},\\
&\ddot{\tilde{w}} = \tilde{\mu}\ddot{w} - e_3\ddot{\psi},\ \ddot{\tilde{\psi}} = -e_3\ddot{w} + (2\varepsilon_2-1)\ddot{\psi},\\
&\mathbf{b}_s = \mathbf{b}_{s\mathcal{A}} + b_{se}\mathbf{e},\ \mathbf{f}_{ms} = \mathbf{f}_{ms\mathcal{A}} + f_{mse}\mathbf{e},
\end{aligned}\tag{22}$$

Δ_R is the Laplacian operator, and $F = \mathrm{div}\,G$ is the Navier operator in the plane $\mathcal{A}$.

A Saint-Venant/Almansi Solution
We seek a solution of (21) of the form

$$\ddot{w} = \sum_{i=0}^{m} \frac{z^i}{i!}\ddot{w}_i(\mathbf{r}), \qquad \ddot{\mathbf{v}} = \sum_{i=0}^{m} \frac{z^i}{i!}\ddot{\mathbf{v}}_i(\mathbf{r}), \qquad \ddot{\psi} = \sum_{i=0}^{m} \frac{z^i}{i!}\ddot{\psi}_i(\mathbf{r}). \tag{23}$$

Substituting from (23) into (21), recalling (20), and equating like powers of $z^i/i!$ on both sides, we obtain partial differential equations, boundary conditions and integrability conditions to determine $\ddot{w}_0, \ddot{w}_1, \ldots, \ddot{\mathbf{v}}_0, \ddot{\mathbf{v}}_1, \ldots, \ddot{\psi}_0, \ddot{\psi}_1, \ldots$. For $i > 3$, these boundary value problems have null solutions. Denoting constants by a superscript zero, for $i = 3$, the solution is

$$\ddot{\mathbf{v}}_3 = \mathbf{v}_3^0 + \theta_3^0(*\mathbf{r}), \qquad \ddot{w}_3 = w_3^0, \qquad \ddot{\psi}_3 = \psi_3^0. \tag{24}$$

The integrability conditions for the torque, axial force and the charge require that

$$\theta_3^0 = 0, \qquad w_3^0 = 0, \qquad \psi_3^0 = 0. \tag{25}$$

Using (24) and (25), equations for the determination of $\ddot{\mathbf{v}}_2$, $\ddot{w}_2$ and $\ddot{\psi}_2$ are

$$\begin{aligned} &F(\ddot{\mathbf{v}}_2) = \mathbf{0}, \qquad \Delta_R \ddot{\tilde{w}}_2 = 0, \qquad \Delta_R \ddot{\tilde{\psi}}_2 = 0,\\ &G(\ddot{\mathbf{v}}_2)\mathbf{N} = -2(\lambda + \mu)\tau^2\mathbf{N}, \qquad \operatorname{grad} \ddot{\tilde{w}}_2 \cdot \mathbf{N} = -\tilde{\mu}\mathbf{v}_3^0 \cdot \mathbf{N},\\ &\operatorname{grad} \ddot{\tilde{\psi}}_2 \cdot \mathbf{N} = e_3\mathbf{v}_3^0 \cdot \mathbf{N} \end{aligned} \tag{26}$$

and their solution is

$$\ddot{\mathbf{v}}_2 = \mathbf{v}_2^0 + \theta_2^0(*\mathbf{r}) - \tau^2\mathbf{r}, \qquad \ddot{w}_2 = w_2^0 - \mathbf{v}_3^0 \cdot \mathbf{r}, \qquad \ddot{\psi}_2 = \psi_2^0. \tag{27}$$

The integrability conditions for the torque, axial force and the electric charge require that

$$\theta_2^0 = 0, \qquad w_2^0 = 0, \qquad \psi_2^0 = 0. \tag{28}$$

Field equations for $\ddot{\mathbf{v}}_1$, $\ddot{w}_1$ and $\ddot{\psi}_1$ are

$$\begin{aligned} &F(\ddot{\mathbf{v}}_1) = (c_3 + \lambda)\mathbf{v}_3^0, \ \Delta_R \ddot{\tilde{w}}_1 = 0, \qquad \Delta_R \ddot{\tilde{\psi}}_1 = 0,\\ &G(\ddot{\mathbf{v}}_1)\mathbf{N} = (c_3 + \lambda)(\mathbf{v}_3^0 \cdot \mathbf{r})\mathbf{N}, \qquad \operatorname{grad} \ddot{\tilde{w}}_1 \cdot \mathbf{N} = -\tilde{\mu}\mathbf{v}_2^0 \cdot \mathbf{N},\\ &\operatorname{grad}\ddot{\psi}_1 \cdot \mathbf{N} = e_3\mathbf{v}_2^0 \cdot \mathbf{N}, \end{aligned} \tag{29}$$

and have the solution

$$\begin{aligned} &\ddot{\mathbf{v}}_1 = \mathbf{v}_1^0 + \theta_1^0(*\mathbf{r}) + \frac{c_3 + \lambda}{2(\lambda + \mu)}\operatorname{Sym}(\mathbf{r} \otimes (*\mathbf{r}))(*\mathbf{v}_3^0),\\ &\ddot{w}_1 = w_1^0 - \mathbf{v}_2^0 \cdot \mathbf{r}, \ \ddot{\psi}_1 = \psi_1^0. \end{aligned} \tag{30}$$

Equations for finding fields $\ddot{\mathbf{v}}_0$, $\ddot{w}_0$ and $\ddot{\psi}_0$ can now be written as

$$
\begin{aligned}
&F(\ddot{\mathbf{v}}_0) = (c_3+\lambda)\mathbf{v}_2^0 + (\tilde{\mu}+\chi_1)\tau^2\mathbf{r},\ \Delta_R \tilde{w}_0 = \xi_1 \mathbf{v}_3^0\cdot\mathbf{r},\ \Delta_R\tilde{\psi}_0 = \xi_3\mathbf{v}_3^0\cdot\mathbf{r},\\
&G(\ddot{\mathbf{v}}_0)\mathbf{N} = [(c_3+\lambda)(\mathbf{v}_2^0\cdot\mathbf{r}) + (e_2\psi_1^0 - (c_3+\lambda)w_1^0)\\
&\qquad + (\chi_4\tau^2 r^2 + \chi_5\omega^2)]\mathbf{N} + \chi_7\tau^2(\mathbf{r}\otimes\mathbf{r})\mathbf{N}, \qquad (31)\\
&(\operatorname{grad}\tilde{w}_0)\cdot\mathbf{N} = -\tilde{\mu}\mathbf{v}_1^0\cdot\mathbf{N} + \xi_2[\operatorname{Sym}(\mathbf{r}\otimes(*\mathbf{r}))(*\mathbf{v}_3^0)]\cdot\mathbf{N},\\
&(\operatorname{grad}\tilde{\psi}_0)\cdot\mathbf{N} = e_3\mathbf{v}_1^0\cdot\mathbf{N} + +\xi_4[\operatorname{Sym}(\mathbf{r}\otimes(*\mathbf{r}))(*\mathbf{v}_3^0)]\cdot\mathbf{N},
\end{aligned}
$$

where expressions for ξ_1, ξ_2, ξ_3, ξ_4 and other ξ's introduced below in terms of other material parameters are given in the Appendix. The solution of (31) is

$$
\begin{aligned}
\ddot{\mathbf{v}}_0 &= \mathbf{v}_0^0 + \theta_0^0(*\mathbf{r}) + \frac{(c_3+\lambda)}{2(\lambda+\mu)}\operatorname{Sym}(\mathbf{r}\otimes(*\mathbf{r}))(*\mathbf{v}_2^0)\\
&\quad + \frac{(\chi_5\omega^2 + \xi_0 R^2\tau^2 + e_2\psi_1^0 - (c_3+\lambda)w_1^0)}{2(\lambda+\mu)}\mathbf{r} + \xi_5\tau^2 r^2\mathbf{r}, \qquad (32)\\
\ddot{w}_0 &= w_0^0 - \mathbf{v}_1^0\cdot\mathbf{r} + \xi_6\Phi + \xi_7\Psi,\\
\ddot{\psi}_0 &= \psi_0^0 + \xi_8\Phi + \xi_9\Psi,
\end{aligned}
$$

where functions Φ and Ψ are given by

$$
\begin{aligned}
\Phi &= \tfrac{1}{8}[(4\xi_2 - 3\xi_1)R^2 + \xi_1 r^2]\mathbf{v}_3^0\cdot\mathbf{r},\\
\Psi &= \tfrac{1}{8}[(4\xi_4 - 3\xi_3)R^2 + \xi_3 r^2]\mathbf{v}_3^0\cdot\mathbf{r}.
\end{aligned} \qquad (33)
$$

The clamping conditions $\mathbf{u} = 0$, $\mathbf{H} - \mathbf{H}^T = \mathbf{0}$, $\psi = 0$ at the centroid of $\mathcal{A}_0$ require that

$$
\mathbf{v}_0^0 = \mathbf{0}, \quad w_0^0 = 0, \quad \theta_0^0 = 0, \quad \psi_0^0 = 0, \quad \mathbf{v}_1^0 = \mathbf{0}. \qquad (34)
$$

The second-order solution is characterized by seven constants $\mathbf{v}_3^0$, $\mathbf{v}_2^0$, θ_1^0, w_1^0 and ψ_1^0 representing second-order flexure, bending, torsion, elongation and electric potential respectively. However, these effects are coupled in the sense that if a piezoelectric circular bar is twisted by applying equal and opposite torques at the end faces, then there is also second-order torsion, elongation and electric field.

Let us consider deformations of the bar under the following resultant loads.

$$
\mathbf{R}_F = \mathbf{0}, \qquad \mathbf{R}_M = T\mathbf{e}_1\wedge\mathbf{e}_2, \qquad R_Q = Q.
$$

Here T is the torque and Q the total charge. The surface tractions $\mathbf{f}$ at the end faces $\mathcal{A}_0$ and $\mathcal{A}_\ell$ have zero resultant force, and their resultant moment equals T about the axis $\mathbf{e}$ of the bar. The solution of the second-order problem is

$$\begin{aligned}
\mathbf{v} &= -\nu a\mathbf{r}\omega + z(*\mathbf{r})\tau + [\xi_{12}\omega^2 + (\xi_{13}R^2 + \xi_5 r^2)\tau^2]\mathbf{r} + \tfrac{\chi_{15}}{\tilde{\mu}} z(*\mathbf{r})\tau\omega - \tfrac{1}{2}z^2\mathbf{r}\tau^2,\\
w &= za\omega + \xi_{10}[(2\xi_5(\lambda+\mu) + \xi_0)R^2\tau^2 + \chi_5\omega^2]z,\\
\psi &= z\omega + \xi_{11}[(2\xi_5(\lambda+\mu) + \xi_0)R^2\tau^2 + \chi_5\omega^2]z,\\
\omega &= Q/\xi_{14}A,\ \tau = T/\tilde{\mu}J_{\mathcal{A}}
\end{aligned} \tag{35}$$

Thus the angle of twist/length equals $\tau + (\chi_{15}/\tilde{\mu})\tau\omega$ implying thereby that an electric field alters the angle of twist/length and this change is proportional to the charge/area. Also there is a second-order Poisson effect with one part proportional to r and another one proportional to r^3; the part varying as r^3 depends upon the piezoelectric constants.

One part of the axial strain w' is proportional to τ^2 and ω^2 as expected and is a generalization of the Poynting effect to transversely isotropic piezoelectric materials. When $\tau = 0$, the term $\chi_5\xi_{10}\omega^2$ represents the correction to the axial strain caused by the nonlinear response of the piezoelectric cylinder to the applied electric field.

Equation $(35)_3$ indicates that the difference of the electric potential at the two end faces of the piezoelectric cylinder depends upon the square of the angular twist. Even when there is no charge applied at the end faces, twisting of the piezoelectric cylinder will induce a measurable difference in the electric potential between the end faces. Hence a piezoelectric cylinder can be used to measure the angular twist.

4. Conclusions

We have studied the electromechanical deformations of a second-order, transversely isotropic homogeneous circular cylindrical bar with mechanical loads and/or electric charges applied to its end faces only. The constitutive relations are taken to be quadratic in the displacement gradients and the electric field. The centroid of one end cross-section is rigidly clamped in the sense that displacements, infinitesimal rotations and the electric potential vanish there.

It is found that there is a second-order Poisson's effect not of the Saint-Venant type; this is proportional to r^3 where r is the distance of a point from the centroidal axis. Also, when the end faces are subjected to a pure torque and no electric charge, there may be a potential difference, proportional to the square of the angular twist, present between the end faces.

Appendix

Using the notations

$$
\begin{aligned}
&\dot{\mathbf{E}} = (\dot{\mathbf{H}} + \dot{\mathbf{H}})^T/2, \qquad \ddot{\mathbf{E}} = \dot{\mathbf{H}}^T\dot{\mathbf{H}}/2,\\
&\dot{I}_1 = \mathbf{e}\cdot(\dot{\mathbf{E}}\mathbf{e}), \qquad \ddot{I}_1 = \mathbf{e}\cdot(\ddot{\mathbf{E}}\mathbf{e}), \qquad \dot{I}_2 = \operatorname{tr}\dot{\mathbf{H}},\ \ddot{I}_2 = \operatorname{tr}\ddot{\mathbf{E}},\\
&\dot{I}_3 = \dot{\mathbf{W}}\cdot\mathbf{e}, \qquad \ddot{\Pi}_1 = \mathbf{e}\cdot(\dot{\mathbf{E}}^2\mathbf{e}), \qquad \ddot{\Pi}_2 = (\operatorname{tr}\dot{\mathbf{E}})^2,\\
&\ddot{\Pi}_3 = \dot{\mathbf{W}}\cdot\dot{\mathbf{W}}, \qquad \ddot{\Pi}_4 = \mathbf{e}\cdot(\dot{\mathbf{E}}\dot{\mathbf{W}}) + \dot{\mathbf{W}}\cdot(\dot{\mathbf{E}}\mathbf{e}), \qquad \ddot{I}_3 = \ddot{\mathbf{W}}\cdot\mathbf{e},
\end{aligned}
$$

we find that the constitutive relations for a second-order transversely isotropic material with the axis of transverse isotropy along the unit vector $\mathbf{e}$ are as follows:

$$
\begin{aligned}
\dot{\mathbf{S}} \;=\;& (2c_1\dot{I}_1 + c_3\dot{I}_2 + e_1\dot{I}_3)\mathbf{e}\otimes\mathbf{e} + (2c_2\dot{I}_2 + c_3\dot{I}_1 + e_2\dot{I}_3)\mathbf{1}\\
&+c_4\operatorname{Sym}(\mathbf{e}\otimes\dot{\mathbf{E}}\mathbf{e}) + 2c_5\dot{\mathbf{E}} + e_3\operatorname{Sym}(\mathbf{e}\otimes\dot{\mathbf{W}}),\\
\ddot{\mathbf{S}} \;=\;& [2c_1\ddot{I}_1 + c_3\ddot{I}_2 + 3\lambda_1\dot{I}_1^2 + 2\lambda_3\dot{I}_1\dot{I}_2 + \lambda_4\dot{I}_2^2 + \lambda_5\ddot{\Pi}_1\\
&+\lambda_7\ddot{\Pi}_2 + 2\nu_1\dot{I}_1\dot{I}_3 + \nu_2\dot{I}_3^2 + \nu_7\ddot{\Pi}_3 + \nu_9\ddot{\Pi}_4 + \nu_{14}\dot{I}_2\dot{I}_3]\mathbf{e}\otimes\mathbf{e}\\
&+[2c_2\ddot{I}_2 + c_3\ddot{I}_1 + 3\lambda_2\dot{I}_2^2 + \lambda_3\dot{I}_1^2 + 2\lambda_4\dot{I}_1\dot{I}_2 + \lambda_6\ddot{\Pi}_1 + \lambda_8\ddot{\Pi}_2\\
&\quad + 2\nu_3\dot{I}_2\dot{I}_3 + \nu_4\dot{I}_3^2 + \nu_8\ddot{\Pi}_3 + \nu_{10}\ddot{\Pi}_4 + \nu_{14}\dot{I}_1\dot{I}_3]\mathbf{1}\\
&+2c_4\operatorname{Sym}(\mathbf{e}\otimes\ddot{\mathbf{E}}\mathbf{e}) + 2(\lambda_5\dot{I}_1 + \lambda_6\dot{I}_2 + \nu_5\dot{I}_3)\operatorname{Sym}(\mathbf{e}\otimes\dot{\mathbf{E}}\mathbf{e})\\
&+2c_5\ddot{\mathbf{E}} + 2(\lambda_7\dot{I}_1 + \lambda_8\dot{I}_2 + \nu_6\dot{I}_3)\dot{\mathbf{E}}\\
&+2(\nu_9\dot{I}_1 + \nu_{10}\dot{I}_2 + \nu_{11}\dot{I}_3)\operatorname{Sym}(\mathbf{e}\otimes\dot{\mathbf{W}})\\
&+3\lambda_9(\dot{\mathbf{E}})^2 + \nu_{12}\dot{\mathbf{W}}\otimes\dot{\mathbf{W}} + 2\nu_{13}\operatorname{Sym}(\mathbf{e}\otimes\dot{\mathbf{E}}\dot{\mathbf{W}} + \dot{\mathbf{W}}\otimes\dot{\mathbf{E}}\mathbf{e}),\\
\dot{\boldsymbol{\Pi}} \;=\;& -(2\varepsilon_1\dot{I}_3 + e_1\dot{I}_1 + e_2\dot{I}_2)\mathbf{e} - 2\varepsilon_2\dot{\mathbf{W}} - 2e_3\dot{\mathbf{E}}\mathbf{e},\\
\ddot{\boldsymbol{\Pi}} \;=\;& -[e_1\ddot{I}_1 + e_2\ddot{I}_2 + 3\mu_1\dot{I}_3^2 + \mu_2\ddot{\Pi}_3 + \nu_1\dot{I}_1^2 + 2\nu_2\dot{I}_3\dot{I}_1 + \nu_3\dot{I}_2^2 + 2\varepsilon_1\ddot{I}_3\\
&\quad + 2\nu_4\dot{I}_3\dot{I}_2 + \nu_5\ddot{\Pi}_1 + \nu_6\ddot{\Pi}_2 + \nu_{11}\ddot{\Pi}_4 + \nu_{14}\dot{I}_1\dot{I}_2]\mathbf{e}\\
&-2[\mu_2\dot{I}_3 + \nu_7\dot{I}_1 + \nu_8\dot{I}_2]\dot{\mathbf{W}} - 2e_3\ddot{\mathbf{E}}\mathbf{e} - 2\varepsilon_2\ddot{\mathbf{W}}\\
&-2(\nu_9\dot{I}_1 + \nu_{10}\dot{I}_2 + \nu_{11}\dot{I}_3)\dot{\mathbf{E}}\mathbf{e} - 2\nu_{12}\dot{\mathbf{E}}\dot{\mathbf{W}} - 2\nu_{13}\dot{\mathbf{E}}^2\mathbf{e}.
\end{aligned}
$$

Here $c_1, c_2, \ldots, e_1, e_2, \ldots, \lambda_1, \lambda_2, \ldots, \nu_1, \nu_2, \ldots, \varepsilon_1, \varepsilon_2, \ldots$, and $\mu_1, \mu_2, \ldots$ are material parameters. Expressions for other material parameters used in the text are given below.

$$
\begin{aligned}
&\chi_1 = -c_3 + \tfrac{1}{2}c_4 - \lambda - \tfrac{1}{2}\lambda_6 - \lambda_8 + \tfrac{1}{4}3\lambda_9 + \mu,\\
&\chi_2 = -2c_3 - c_4 - 2\lambda - 2\mu,
\end{aligned}
$$

$$\chi_3 = 2(e_2 + e_3),$$

$$\chi_4 = -\tfrac{1}{2}c_3 - \tfrac{1}{2}c_4 - \tfrac{1}{2}\lambda - \tfrac{1}{4}\lambda_6 - \tfrac{1}{2}\lambda_8 - \tfrac{1}{4}3\lambda_9 - \mu,$$

$$\begin{aligned}\chi_5 = -\frac{1}{4{c_3}^2(\lambda+\mu)^2}&({e_2}^2(3\lambda^3 + 4(3\lambda_2 + \lambda_3 + 2\lambda_4 + \lambda_6 + \lambda_8)\mu^2\\&+2\lambda\mu(4\lambda_3 + 4\lambda_4 + 4\lambda_6 - 2\lambda_7 + 2\lambda_8 + \mu)\\&+\lambda^2(4\lambda_3 + 4\lambda_6 - 4\lambda_7 + 6\lambda_8 + 3\lambda_9 + 5\mu))\\&+2c_3e_2(\lambda+\mu)(e_2(\lambda+\mu)\\&-2(\mu(2\nu_{10} + \nu_{14} + 2\nu_3) + \lambda(2\nu_{10} + \nu_{14} - \nu_6)))\\&+2c_3^2(\lambda+\mu)^2(-1 + 2\nu_4 + 2\nu_8)),\end{aligned}$$

$$\chi_6 = -(\lambda+\mu),$$

$$\chi_7 = \tfrac{1}{2}c_4 + \tfrac{1}{4}3\lambda_9 + \mu,$$

$$\chi_8 = -\tfrac{1}{2}(c_4 + 2\mu),$$

$$\begin{aligned}\chi_9 = \frac{1}{4c_3(\lambda+\mu)}&(c_4e_2(\lambda+2\mu) + e_2(2\lambda\lambda_5 + 4\lambda\lambda_7\\&+3\lambda\lambda_9 + 2\lambda\mu + 2\lambda_5\mu + 2\lambda_6\mu\\&+4\lambda_7\mu + 4\lambda_8\mu + 6\lambda_9\mu + 4\mu^2)\\&+2c_3(\lambda+\mu)(e_3 - 2\nu_{13} - \nu_5 - 2\nu`_6)),\end{aligned}$$

$$\chi_{10} = e_3,$$

$$\begin{aligned}\chi_{11} = \frac{1}{2c_3(\lambda+\mu)}&(c_3(\lambda+\mu)(2 + \nu_{11} + 2\nu_{12})\\&+e_2(e_3\lambda - \mu\nu_{10} - \lambda\nu_{13} - 2\mu\nu_{13} - \lambda\nu_9 - \mu\nu_9)),\end{aligned}$$

$$\chi_{12} = -c_1 - c_3 - c_4 - \tfrac{1}{2}\lambda - \tfrac{1}{4}\lambda_5 - \tfrac{1}{4}\lambda_6 - \tfrac{1}{2}\lambda_7 - \tfrac{1}{2}\lambda_8 - \tfrac{1}{4}3\lambda_9 - \mu,$$

$$\chi_{13} = -(c_3 + \lambda),$$

$$\begin{aligned}\chi_{14} = \frac{1}{4{c_3}^2(\lambda+\mu)^2}&(-{e_2}^2(12c_4\lambda^2 + 3\lambda^3 + 12\lambda^2\lambda_1\\&+4\lambda^2\lambda_3 + 12\lambda^2\lambda_5 + 4\lambda^2\lambda_6 + 14\lambda^2\lambda_7\\&+6\lambda^2\lambda_8 + 12\lambda^2\lambda_9 + 24c_4\lambda\mu + 20\lambda^2\mu\\&+24\lambda\lambda_1\mu + 16\lambda\lambda_3\mu + 8\lambda\lambda_4\mu + 24\lambda\lambda_5\mu\end{aligned}$$

$$
\begin{aligned}
&+16\lambda\lambda_6\mu + 24\lambda\lambda_7\mu + 16\lambda\lambda_8\mu\\
&+24\lambda\lambda_9\mu + 12c_4\mu^2 + 30\lambda\mu^2 + 12\lambda_1\mu^2 + 12\lambda_2\mu^2\\
&+12\lambda_3\mu^2 + 12\lambda_4\mu^2 + 12\lambda_5\mu^2 + 12\lambda_6\mu^2\\
&+12\lambda_7\mu^2 + 12\lambda_8\mu^2 + 12\lambda_9\mu^2 + 12\mu^3\\
&+12c_1(\lambda+\mu)^2) + 2{c_3}^2(\lambda+\mu)^2\\
&\times(-1 + 4\varepsilon_1 + 4\varepsilon_2 - 4\nu_{11} - 2\nu_{12} - 2\nu_2 - 2\nu_4 - 2\nu_7 - 2\nu_8)\\
&+c_3e_2(-e_2(5\lambda^2 + 16\lambda\mu + 12\mu^2)\\
&+4(\lambda+\mu)(2\lambda\nu_1 + 2\mu\nu_1 + 2\lambda\nu_{11} + 4\mu\nu_{10} + 4\lambda\nu_{13}\\
&+4\mu\nu_{13} + \lambda\nu_{14} + 2\mu\nu_{14} + 2\mu\nu_3 + 2\lambda\nu_5 + 2\mu\nu_5\\
&+2\lambda\nu_6 + 2\mu\nu_6 + 4\lambda\nu_9 + 4\mu\nu_9))),
\end{aligned}
$$

$$
\begin{aligned}
\chi_{15} = \frac{1}{4c_3(\lambda+\mu)}&(-e_2(6c_4\lambda + 2\lambda\lambda_5 + 4\lambda\lambda_7\\
&+3\lambda\lambda_9 + 8c_4\mu + 8\lambda\mu + 2\lambda_5\mu + 2\lambda_6\mu\\
&+4\lambda_7\mu + 4\lambda_8\mu + 6\lambda_9\mu + 8\mu^2 + 8c_1(\lambda+\mu))\\
&+2c_3(3e_3\lambda - 2e_2\mu + 3e_3\mu + 2e_1(\lambda+\mu)\\
&+2\lambda\nu_{13} + 2\mu\nu_{13} + \lambda\nu_5 + \mu\nu_5 + 2\lambda\nu_6 + 2\mu\nu_6)),
\end{aligned}
$$

$$
\chi_{16} = \tfrac{1}{2}e_1 + \tfrac{1}{2}e_2 + e_3 + \tfrac{1}{2}\nu_{13} + \tfrac{1}{4}\nu_5 + \tfrac{1}{2}\nu_6,
$$

$$
\chi_{17} = e_2,
$$

$$
\begin{aligned}
\chi_{18} = \frac{1}{4c_3^2(\lambda+\mu)^2}&(12{c_3}^2(\lambda+\mu)^2(\mu_1+\mu_2)\\
&-4c_3e_2(\lambda+\mu)(\lambda(2 + 3\nu_{11} + 2\nu_{12} + 2\nu_2 + 2\nu_7)\\
&+\mu(1 + 3\nu_{11} + 2\nu_{12} + 2\nu_2 + 2\nu_4 + 2\nu_7 + 2\nu_8))\\
&+{e_2}^2(3e_2\lambda^2 + 4e_3\lambda^2 + 4e_2\lambda\mu + 8e_3\lambda\mu\\
&+2e_2\mu^2 + 4e_3\mu^2 + 2e_1(\lambda+\mu)^2\\
&+4\lambda^2\nu_1 + 8\lambda\mu\nu_1 + 4\mu^2\nu_1\\
&+4\lambda\mu\nu_{10} + 4\mu^2\nu_{10} + 8\lambda^2\nu_{13} + 16\lambda\mu\nu_{13} + 8\mu^2\nu_{13}\\
&+4\lambda\mu\nu_{14} + 4\mu^2\nu_{14} + 4\mu^2\nu_3\\
&+4\lambda^2\nu_5 + 8\lambda\mu\nu_5 + 4\mu^2\nu_5 + 6\lambda^2\nu_6 + 8\lambda\mu\nu_6\\
&+4\mu^2\nu_6 + 4\lambda^2\nu_9 + 8\lambda\mu\nu_9 + 4\mu^2\nu_9)),
\end{aligned}
$$

$$\chi_{19} = -2(c_3 + \lambda),$$

$$\chi_{20} = 2e_2,$$

$$\xi_0 = \chi_4 + \chi_7 - \frac{2\lambda + 3\mu}{4(\lambda + 2\mu)},$$

$$\xi_1 = 2c_1 + \tfrac{1}{2}\lambda + c_3 + c_4 + \mu - c_3 + \lambda + \tfrac{1}{2}c_4 + \mu \left(\frac{c_3 + \lambda}{\lambda + \mu}\right),$$

$$\xi_2 = -(\tfrac{1}{2}c_4 + \mu)\frac{(c_3 + \lambda)}{(2\lambda + 2\mu)},$$

$$\xi_3 = \frac{(e_2 + e_3)(c_3 + \lambda)}{(\lambda + \mu)} - (e_1 + e_2 + 2e_3),$$

$$\xi_4 = \frac{e_3(c_3 + \lambda)}{(2\lambda + 2\mu)},$$

$$\xi_5 = \frac{-4c_3 + 2c_4 - 4\lambda - 2\lambda_6 - 4\lambda_8 + 3\lambda_9 + 4\mu}{32(\lambda + 2\mu)},$$

$$\xi_6 = \frac{1 - 2\varepsilon_2}{e_3^2 + \tilde{\mu} - 2\varepsilon_2\tilde{\mu}},$$

$$\xi_7 = -\frac{e_3}{e_3^2 + \tilde{\mu} - 2\varepsilon_2\tilde{\mu}},$$

$$\xi_8 = -\frac{e_3}{e_3^2 + \tilde{\mu} - 2\varepsilon_2\tilde{\mu}},$$

$$\xi_9 = -\frac{\tilde{\mu}}{e_3^2 + \tilde{\mu} - 2\varepsilon_2\tilde{\mu}},$$

$$\begin{aligned}
\xi_{10} = (&-c_3 - e_1e_2 - {e_2}^2 - 2e_2e_3 + 2c_3\varepsilon_1 \\
&+2c_3\varepsilon_2 - \lambda + 2\varepsilon_1\lambda + 2\varepsilon_2\lambda)/(2c_4{e_2}^2 \\
&+{c_3}^2(-1 + 2\epsilon_1 + 2\epsilon_2) + 2c_4\lambda \\
&+{e_1}^2\lambda + 4e_1e_3\lambda + 4{e_3}^2\lambda - 4c_4\varepsilon_1\lambda - 4c_4\varepsilon_2\lambda \\
&+2c_4\mu + {e_1}^2\mu + 2e_1e_2\mu + 3{e_2}^2\mu \\
&+4e_1e_3\mu + 4e_2e_3\mu + 4{e_3}^2\mu - 4c_4\varepsilon_1\mu \\
&-4c_4\varepsilon_2\mu + 3\lambda\mu - 6\varepsilon_1\lambda\mu - 6\varepsilon_2\lambda\mu + 2\mu^2 - 4\varepsilon_1\mu^2 - 4\varepsilon_2\mu^2 \\
&+2c_1({e_2}^2 + \lambda - 2\varepsilon_1\lambda - 2\varepsilon_2\lambda + \mu \\
&-2\varepsilon_1\mu - 2\varepsilon_2\mu) - 2c_3(e_1e_2 + 2e_2e_3 - \mu + 2\varepsilon_1\mu + 2\varepsilon_2\mu)),
\end{aligned}$$

$$\xi_{11} = (-2c_1e_2 - 2c_4e_2 + c_3(e_1 - e_2 + 2e_3) + e_1\lambda + 2e_3\lambda - 2e_2\mu)/$$
$$\times \left((\lambda+\mu) \left(\left((e_1 + e_2 + 2e_3) - \frac{e_2(c_3+\lambda)}{\lambda+\mu} \right)^2 \right.\right.$$
$$- \left(-1 + 2\varepsilon_1 + 2\varepsilon_2 - \frac{e_2^2}{\lambda+\mu} \right)$$
$$\left.\left.\times \left(2(c_1 + c_3 + c_4 + \tfrac{1}{2}\lambda + \mu) - \frac{(c_3+\lambda)^2}{\lambda+\mu} \right) \right) \right),$$

$$\xi_{12} = \frac{\chi_5(1 + e_2\xi_{11} - (c_3+\lambda)\xi_{10})}{2(\lambda+\mu)},$$

$$\xi_{13} = \xi_5(e_2\xi_{11} - (c_3+\lambda)\xi_{10}) + \frac{e_2\xi_0\xi_{10}}{2(\lambda+\mu)} - \frac{(c_3+\lambda)\xi_0\xi_{11}}{2(\lambda+\mu)},$$

$$\xi_{14} = \frac{(2\varepsilon_1 + 2\varepsilon_2 - 1)c_3(\lambda+\mu) - e_2(e_2\mu + e_1(\lambda+\mu) + 2e_3(\lambda+\mu))}{c_3(\lambda+\mu)}.$$

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