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# Linear Constitutive Relations in Isotropic Finite Viscoelasticity

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Batra [1] used a linear relationship between the second Piola-Kirchhoff stress tensor $\mathbf{S}$ and the Green-St. Venant strain tensor $\mathbf{E}$ to study finite simple shearing and finite simple extension deformations of an elastic body. In each case he found that the tangent modulus (i.e. the slope of the shear stress vs. the shear strain curve or the slope of the axial stress vs. the axial stretch curve) is a monotonically nondecreasing function of the pertinent measure of strain. This contradicts the response observed for most materials. However, a linear relation between the Cauchy stress tensor $\boldsymbol{\sigma}$ and the left Cauchy–Green tensor $\mathbf{B}$ was found to give a response similar to that observed in experiments [2] for most materials. Here we study the corresponding problem for an incompressible linear viscoelastic material. We use two constitutive relations: in one the relationship between $\mathbf{S}$ and the history of $\mathbf{E}$ is linear and in the other $\boldsymbol{\sigma}$ is linearly related to the history of the relative Green-St. Venant strain tensor $\mathbf{E}_t$. It is shown that the instantaneous elastic response given by the former constitutive relation is unrealistic in the sense described above but that obtained with the latter one agrees with the expected one.

For an incompressible linear viscoelastic material, the aforestated two constitutive relations are (e.g. see Christensen [3] for Equation (1a) and Fosdick and Yu [4] for Equation (1b))

$$\mathbf{S} = -p\mathbf{C}^{-1} + g_0\mathbf{1} + \int_0^t g_1(t-\tau)\frac{\partial \mathbf{E}(\tau)}{\partial \tau}\,\mathrm{d}\tau, \tag{1a}$$

$$\boldsymbol{\sigma} = -p\mathbf{1} + g_0\mathbf{B} + \int_0^t g_1(t-\tau)\frac{\partial \mathbf{E}_t(\tau)}{\partial \tau}\,\mathrm{d}\tau, \tag{1b}$$

where

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^T, \quad \mathbf{E} = \tfrac{1}{2}(\mathbf{C} - \mathbf{1}),$$

$$\mathbf{F} = \mathrm{Grad}\ \mathbf{x}, \quad \mathbf{F}_t(\tau) = \mathrm{Grad}_t\ \mathbf{x}(\mathbf{X}, \tau). \tag{2}$$

Here $\mathbf{x}$ gives the present position of a material particle that occupied place $\mathbf{X}$ in the reference configuration, $\mathbf{F}$ is the deformation gradient, $\mathbf{C}(\mathbf{B})$ the right (left) Cauchy–Green tensor, Grad is the gradient operator with respect to $\mathbf{X}$, $\text{Grad}_t$ is the gradient operator with respect to $\mathbf{x}(\mathbf{X}, t)$, $p$ the hydrostatic pressure undetermined from the deformation, $g_0 > 0$ the constant modulus and $g_1(t)$ the relaxation modulus of the material. Note that $g_1(\cdot)$ is a smooth, positive, monotonically decreasing function of time $t$; i.e.

$$g_1(\cdot) > 0, \quad \frac{dg_1(\tau)}{\mathrm{d}\tau} < 0. \tag{3}$$

The relative right Cauchy–Green tensor $\mathbf{C}_t$ and the relative Green-St. Venant tensor $\mathbf{E}_t$ are related to the relative deformation gradient $\mathbf{F}_t(\tau)$ in the same way as $\mathbf{C}$ and $\mathbf{E}$ are related to $\mathbf{F}$. We recall the following relationships between the first Piola-Kirchhoff (sometimes called the nominal or the engineering) stress tensor $\mathbf{T}$ and $\mathbf{S}$, and $\mathbf{T}$ and $\boldsymbol{\sigma}$:

$$\mathbf{T} = \mathbf{FS}, \quad \mathbf{T}^T = (\det \mathbf{F})\mathbf{F}^{-1}\boldsymbol{\sigma}. \tag{4}$$

We note that Coleman and Noll [5] have provided a mathematical foundation of the linear theory of viscoelasticity. Pipkin and Rogers [6] motivated the addition of successive terms in the integral series representation of the stress in terms of the history of strain till the difference between the test data and the prediction from the theory became smaller than a preassigned value. They thus derived a nonlinear theory of viscoelasticity. Equations (1a) and (1b) are special cases of the integral constitutive relations given in References 5 and 6.

In terms of rectangular Cartesian coordinates, a simple shear deformation is described by

$$x_1 = X_1 + \kappa(t)X_2, \qquad x_2 = X_2, \qquad x_3 = X_3, \tag{5}$$

where $\kappa$ may be called the shear strain. For the deformation (5) Equations (1a) and (1b) when combined with (4) give

$$\begin{aligned} T_{12}(t) &= g_0\kappa(t) + \frac{1}{2}\int_0^t g_1(t-\tau)\frac{\partial\kappa(\tau)}{\partial\tau}\,\mathrm{d}\tau \\ &\quad + \frac{\kappa(t)}{2}\int_0^t g_1(t-\tau)\frac{\partial\kappa^2(\tau)}{\partial\tau}\,\mathrm{d}\tau, \end{aligned} \tag{6a}$$

$$T_{12}(t) = g_0\kappa(t) + \frac{1}{2}\int_0^t g_1(t-\tau)\frac{\partial\kappa(\tau)}{\partial\tau}\,\mathrm{d}\tau. \tag{6b}$$

For a stress-relaxation test,

$$\kappa(t) = \kappa_0 h(t), \tag{7}$$

where $h(\cdot)$ is a Heaviside unit step function, Equations (6a) and (6b) reduce to

$$T_{12}(t) = g_0\kappa_0 + \tfrac{1}{2}g_1(t)(\kappa_0 + \kappa_0^3), \tag{8a}$$

$$T_{12}(t) = (g_0 + \tfrac{1}{2}g_1(t))\kappa_0. \tag{8b}$$

For infinitesimal deformations, $\kappa_0 \ll 1$, and Equations (8a) and (8b) give the same expression for the shear relaxation function $\mu(\cdot)$, viz.

$$2\mu(\cdot) = 2g_0 + g_1(\cdot). \tag{9}$$

Both constitutive relations (1a) and (1b) imply that in simple shearing deformations (5) with $\kappa$ given by (7), the isochrone of $T_{12}$ will decrease in time, however, the rate of decay is the same for infinitesimal deformations (i.e., $\kappa_0 \ll 1$) but different for finite deformations. It follows from Equations (8a) and (8b) that the curvature, $d^2T_{12}/d\kappa_0^2$, of the isochrone is positive for the constitutive relation (1a) but equals zero for (1b). The experimentally observed response for most materials [2] indicates that $d^2T_{12}/d\kappa_0^2 \leqslant 0$ suggesting thereby that the constitutive relation (1b) describes the material response that qualitatively agrees with the observed one. The material behavior predicted by the constitutive relation (1a) qualitatively disagrees with that observed experimentally.

We now study simple extension of a prismatic bar. Recalling that an incompressible material can undergo only isochoric deformations, we have

$$x_1 = \alpha(t)X_1, \qquad x_2 = \frac{1}{\sqrt{\alpha(t)}}X_2, \qquad x_3 = \frac{1}{\sqrt{\alpha(t)}}X_3, \tag{10}$$

where $\alpha$ is the stretch in the $X_1$-direction. The pressure $p$ is determined by requiring that surface tractions vanish on the mantle of the prismatic body. The nominal stress $T_{11}$ obtained from constitutive relations (1a) and (1b) is given by

$$\begin{aligned} T_{11}(t) &= g_0\left(\alpha(t) - \frac{1}{\alpha^2(t)}\right) + \frac{\alpha(t)}{2}\int_0^t g_1(t-\tau)\frac{\partial\alpha^2(\tau)}{\partial\tau}\,\mathrm{d}\tau \\ &\quad - \frac{1}{2\alpha^2(t)}\int_0^t g_1(t-\tau)\frac{\partial\alpha^{-1}(\tau)}{\partial\tau}\,\mathrm{d}\tau, \end{aligned} \tag{11a}$$

$$\begin{aligned} T_{11}(t) &= g_0\left(\alpha(t) - \frac{1}{\alpha^2(t)}\right) + \frac{1}{2\alpha^3(t)}\int_0^t g_1(t-\tau)\frac{\partial\alpha^2(\tau)}{\partial\tau}\,\mathrm{d}\tau \\ &\quad - \frac{1}{2}\int_0^t g_1(t-\tau)\frac{\partial\alpha^{-1}(\tau)}{\partial\tau}\,\mathrm{d}\tau. \end{aligned} \tag{11b}$$

For the stress-relaxation test, $\alpha(t) = (\alpha_0 - 1)h(t) + 1$, Equations (11a) and (11b) simplify to

$$T_{11}(t) = g_0\left(\alpha_0 - \frac{1}{\alpha_0^2}\right) + \tfrac{1}{2}g_1(t)\left(\alpha_0^3 - \alpha_0 + \frac{1}{\alpha_0^2} - \frac{1}{\alpha_0^3}\right), \tag{12a}$$

$$T_{11}(t) \;=\; (g_0\alpha_0 + \tfrac{1}{2}g_1(t))\left(1 - \frac{1}{\alpha_0^3}\right). \tag{12b}$$

For infinitesimal deformations $\alpha_0 = 1 + \varepsilon$, $\varepsilon \ll 1$, and Equations (12a) and (12b) give the same value

$$2E(\cdot) = 3(2g_0 + g_1(\cdot)), \tag{13}$$

of the longitudinal relaxation function. However, for finite deformations, the values of the slope, $\mathrm{d}T_{11}(t)/\mathrm{d}\alpha_0$, of the isochrone are quite different. Furthermore,

$$\frac{d^2T_{11}(t)}{d\alpha_0^2} \;=\; -\frac{6g_0}{\alpha_0^4} + 3g_1(t)\left(\alpha_0 + \frac{1}{\alpha_0^4} - \frac{2}{\alpha_0^5}\right), \tag{14a}$$

$$\frac{d^2T_{11}(t)}{d\alpha_0^2} \;=\; -\frac{6(\alpha_0 g_0 + g_1(t))}{\alpha_0^5} < 0, \tag{14b}$$

for the constitutive relations (1a) and (1b) respectively. Thus, the slope of the longitudinal isochrone is a decreasing function of $\alpha_0$ for the constitutive relation (1b). However, for the constitutive relation (1a) it will be an increasing function of $\alpha_0$ whenever

$$\frac{g_1}{g_0} > \left(\frac{1}{2}\alpha_0^5 + \frac{1}{2} - \frac{1}{\alpha_0}\right)^{-1}, \tag{15}$$

which is likely to be satisfied for large values of $\alpha_0$.

In summary, we have shown that the linear constitutive relations (1a) and (1b) for incompressible viscoelastic materials give identical response for infinitesimal simple shearing and infinitesimal simple extension deformations. However, for finite deformations, they predict different response. In stress-relaxation tests, the slope of the stress isochrone (i.e. the slope of the pertinent nominal stress vs. a measure of deformation curves for fixed $t$) is an increasing function of the deformation measure for the constitutive relation (1a) but a non-increasing function of the deformation measure for the constitutive relation (1b). The latter behavior is in accord with experimental observations for most materials.

The aforestated analysis can be carried out for unconstrained linear viscoelastic materials and similar conclusions drawn.

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## References

1. R.C. Batra, Linear constitutive relations in isotropic finite elasticity. *J. Elasticity* **51** (1998) 243–245.
2. J.F. Bell, The experimental foundation of solid mechanics. In: C. Truesdell (ed.), *Handbuch der Physik*, VIa/1, Springer-Verlag, Berlin, Heidelberg, New York (1973) pp. 1–813.
3. R.M. Christensen, A nonlinear theory of viscoelasticity for applications to elastomers. *J. Appl. Mech.* **47** (1980) 762–768.
4. R.L. Fosdick and J.H. Yu, Thermodynamics, stability and non-linear oscillations of viscoelastic solids–II. history type solids. *Int. J. Nonlinear Mechanics* **33**(1) (1998) 165–188.
5. B.D. Coleman and W. Noll, Foundations of linear viscoelasticity, *Rev. Modern Phys.* **33** (1961) 239–249.
6. A.C. Pipkin and T.G. Rogers, A nonlinear integral representation for viscoelastic behavior. *J. Mech. Phys. Solids* **16** (1968) 59–72.