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Analytical Solution for Radial Deformations of Functionally Graded Isotropic and Incompressible Second-Order Elastic Hollow Spheres

G.L. Iaccarino · R.C. Batra

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Abstract We analytically analyze radial expansion/contraction of a hollow sphere composed of a second-order elastic, isotropic, incompressible and inhomogeneous material to delineate differences and similarities between solutions of the first- and the second-order problems. The two elastic moduli are assumed to be either affine or power-law functions of the radial coordinate R in the undeformed reference configuration. For the affine variation of the shear modulus μ, the hoop stress for the linear elastic (or the first-order) problem at the point $R = (R_{ou} R_{in} (R_{ou} + R_{in})/2)^{1/3}$ is independent of the slope of the μ vs. R line. Here R_{in} and R_{ou} equal, respectively, the inner and the outer radius of the sphere in the reference configuration. For $\mu(R) \propto R^n$, for the linear problem, the hoop stress is constant in the sphere for $n = 1$. However, no such results are found for the second-order (i.e., materially nonlinear) problem. Whereas for the first-order problem the shear modulus influences only the radial displacement and not the stresses, for the second-order problem the two elastic constants affect both the radial displacement and the stresses. In a very thick homogeneous hollow sphere subjected only to pressure on the outer surface, the hoop stress at a point on the inner surface depends upon values of the two elastic moduli. Thus conclusions drawn from the analysis of the first-order problem do not hold for the second-order problem. Closed form solutions for the displacement and stresses for the first-order and the second-order problems provided herein can be used to verify solutions of the problem obtained by using numerical methods.

Keywords Analytical solutions · Functionally graded material · Sphere · Second-order isotropic elastic material

RCB dedicates with deep respect this work to the memory of Professor D.E. Carlson.

G.L. Iaccarino
Department of Constructions and Mathematical Methods in Architecture, University of Naples "Federico II", via Forno Vecchio, 36, 80134 Naples, Italy

R.C. Batra (✉)
Department of Engineering Science and Mechanics, M/C 0219, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA
e-mail: rbatra@vt.edu

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1 Introduction

Functionally graded materials (FGMs) are inhomogeneous and are usually comprised of two constituents whose volume fractions vary smoothly either in one or in two or in all three directions. Most works presented in the literature consider material properties varying smoothly only in one direction. Also, with very few exceptions, problems studied have been for linear elastic isotropic materials with material properties assumed to vary according to either a power law or an exponential relation. There is enormous literature on linear problems for FGMs and it is almost impossible to review it here. The reader is referred to the review paper of Byrd and Birman [1] for a summary of some of the literature.

Nonlinear problems for FGMs studied in [2–10] are for rubberlike materials that are assumed to be isotropic, incompressible and hyperelastic. Ono [11], Ikeda [12, 13] and Ikeda et al. [14] have developed rubbers with continuous spatial variation of the chemical and mechanical properties. The constitutive relation for an incompressible material involves hydrostatic pressure that cannot be determined from the deformation gradient but is found as a part of the solution of the problem. The presence of the hydrostatic pressure generally facilitates finding an analytical solution of a problem. Rivlin [15], amongst others, has shown that the analysis of problems for 2nd-order elastic materials can reveal effects of material nonlinearities that are representative of the behavior of general nonlinear elastic materials. We note that linear kinematic relations are used in deriving the constitutive relation for a 2nd-order elastic material, hence effects of geometric nonlinearities are not considered.

Here we study the radial expansion/contraction of a FG sphere made of an incompressible 2nd-order elastic isotropic material with the two material parameters smoothly varying in the radial direction. It extends our work on FG cylinders [4] to FG spheres; however, the analysis presented here for the first-order (linear elastic) inhomogeneous problem with the shear modulus varying with the radius according to power law is more extensive than that given heretofore. The approach followed here is similar to that employed in [4, 16] for analyzing problems for 2nd-order elastic materials.

We note that problems for linear elastic FG spheres with the moduli varying according to power law have been analytically studied by Tutuncu and Ozturk [17] who assumed that the quadratic characteristic equation has two real and distinct roots. They plotted through-the-thickness variation of stresses for a FG sphere normalized by the corresponding ones for a sphere made of a homogeneous material. Stresses in a FG sphere made of an isotropic linear thermoelastic material have also been studied in [18–22].

2 Problem Formulation

We study radial deformations of a sphere made of a FG isotropic and incompressible 2nd-order elastic material subjected to uniform pressures p_{in} and p_{ou} on its inner and outer surfaces. We use spherical coordinates (r, θ, ϕ) with the origin at the sphere center to describe deformations of a point with $r = R$ in the undeformed reference configuration. We assume that material properties are continuous functions of R, a material point undergoes radial displacement, $u(R) = r - R$, denote du/dR by $u'(R)$, and set

$$I^{(1)} = u' + 2u/R, \qquad I^{(2)} = (u')^2 + 2(u/R)^2. \tag{2.1}$$

For a 2nd-order incompressible material, u must satisfy

$$2I^{(1)} + I^{(1)^2} - I^{(2)} = 0. \tag{2.2}$$

For radial expansion/contraction of a sphere, the constitutive relation for a 2nd-order elastic isotropic and incompressible material is

$$\begin{aligned}
\sigma_{rr} &= -p + 2\mu u' + (\mu+\alpha)(u')^2,\\
\sigma_{\theta\theta} &= \sigma_{\phi\phi} = -p + 2\mu\frac{u}{R} + (\mu+\alpha)\left(\frac{u}{R}\right)^2,\\
\sigma_{r\theta} &= \sigma_{\theta\phi} = \sigma_{\phi r} = 0.
\end{aligned} \tag{2.3}$$

Here $\boldsymbol{\sigma}$ is the Cauchy stress tensor, p the hydrostatic pressure not determined from the deformation field, μ the shear modulus, and α the 2nd-order elastic constant of the material. Both μ and α have units of stress, and are presumed to be functions of R. Physical components of the first Piola-Kirchhoff stress tensor $\mathbf{T}$ are given by

$$\begin{aligned}
T_{RR} &= -p\big(1 - u' + (u')^2\big) + 2\mu\left(u' - \frac{1}{2}(u')^2\right) + \alpha(u')^2,\\
T_{\Theta\Theta} &= T_{\Phi\Phi} = -p\left(1 - \frac{u}{R} + \left(\frac{u}{R}\right)^2\right) + 2\mu\left(\frac{u}{R} - \frac{1}{2}\left(\frac{u}{R}\right)^2\right) + \alpha\left(\frac{u}{R}\right)^2.
\end{aligned} \tag{2.4}$$

The equilibrium equation and boundary conditions governing static radial deformations of a sphere are

$$\begin{aligned}
&T'_{RR} + \frac{2(T_{RR} - T_{\Theta\Theta})}{R} = 0, \quad R_{in} < R < R_{ou},\\
&T_{RR} = -p_{in}\frac{r}{R_{in}} \quad \text{on} \quad R = R_{in}, \quad T_{RR} = -p_{ou}\frac{r}{R_{ou}} \quad \text{on} \quad R = R_{ou}.
\end{aligned} \tag{2.5}$$

Thus T_{RR} at $R = R_{in}$ and at $R = R_{ou}$ depends, respectively, upon the radial displacements of a point on the inner and the outer surfaces of the sphere.

We non-dimensionalize stresses $\boldsymbol{\sigma}$ and $\mathbf{T}$, the pressure p, and material parameters μ and α by $\mu_{ou} = \mu(R_{ou})$, and u, r and R by R_{ou}. Henceforth we employ non-dimensional variables and use for them the same symbols as those used for dimensional variables.

Let $\varepsilon = \max(p_{in}, p_{ou}) \ll 1$. With the assumption that $p_{in} > p_{ou}$, $\varepsilon = p_{in}$. Assuming that u, p, T_{RR} and $T_{\Theta\Theta}$ are analytic functions of ε, we expand them in terms of Taylor series in ε. Recalling that for $\varepsilon = 0$, $u = p = 0$, we get

$$\begin{aligned}
u &= \varepsilon u^{(1)} + \varepsilon^2 u^{(2)}, \qquad & p &= \varepsilon p^{(1)} + \varepsilon^2 p^{(2)},\\
T_{RR} &= \varepsilon T_{RR}^{(1)} + \varepsilon^2 T_{RR}^{(2)}, \qquad & T_{\Theta\Theta} &= \varepsilon T_{\Theta\Theta}^{(1)} + \varepsilon^2 T_{\Theta\Theta}^{(2)},
\end{aligned} \tag{2.6}$$

up to 2nd-order terms in ε. Substitution from (2.6) into (2.2), (2.4) and (2.5) and equating terms of order ε and ε^2 on both sides of resulting equations, we arrive at the following set of equations:

$$u^{(1)'} + \frac{2u^{(1)}}{R} = 0, \qquad 2\left[u^{(2)'} + \frac{2u^{(2)}}{R}\right] - \left[\left(u^{(1)'}\right)^2 + 2\left(\frac{u^{(1)}}{R}\right)^2\right] = 0, \tag{2.7}$$

$$T_{RR}^{(1)'} + \frac{2(T_{RR}^{(1)} - T_{\Theta\Theta}^{(1)})}{R} = 0, \qquad T_{RR}^{(2)'} + \frac{2(T_{RR}^{(2)} - T_{\Theta\Theta}^{(2)})}{R} = 0, \tag{2.8}$$

$$\begin{aligned} T_{RR}^{(1)} &= -p_{in}^{(1)} \quad \text{on} \quad R = R_{in}, \qquad T_{RR}^{(1)} = -p_{ou}^{(1)} \quad \text{on} \quad R = R_{ou}, \\ T_{RR}^{(2)} &= -p_{in}^{(2)} \quad \text{on} \quad R = R_{in}, \qquad T_{RR}^{(2)} = -p_{ou}^{(2)} \quad \text{on} \quad R = R_{ou}, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} p_{in}^{(1)} &= 1, \qquad p_{in}^{(2)} = p_{in} \frac{u^{(1)}(R_{in})}{R_{in}}, \\ p_{ou}^{(1)} &= \frac{p_{ou}}{p_{in}}, \qquad p_{ou}^{(2)} = \frac{p_{ou}}{p_{in}} \frac{u^{(1)}(R_{ou})}{R_{ou}}. \end{aligned} \tag{2.10}$$

Equations $(2.7)_1$, $(2.8)_1$ and $(2.9)_{1,2}$ govern deformations of the 1st-order problem or equivalently the problem in the linear elasticity theory, and $(2.7)_2$, $(2.8)_2$ and $(2.9)_{3,4}$ are used to solve the 2nd-order problem. Relations in (2.10) imply that pressures to be applied on the inner and the outer surfaces for the 2nd-order problem depend upon the solution of the 1st-order problem. Thus the 2nd-order problem can only be analyzed after the 1st-order problem has been solved.

3 Sphere made of Homogeneous Material

For a sphere made of a homogeneous material μ and α equal constants b_1 and b_2 respectively. Proceeding in a way similar to that in [19, 22], we get the following for the solution of the problem.

$$\begin{aligned} u^{(1)}(R) &= \frac{(-p_{ou} + p_{in})R_{ou}^3 R_{in}^3}{4p_{in}b_1(R_{ou}^3 - R_{in}^3)} \frac{1}{R^2}, \\ p^{(1)}(R) &= \frac{p_{ou}R_{ou}^3 - p_{in}R_{in}^3}{p_{in}(R_{ou}^3 - R_{in}^3)}, \\ T_{RR}^{(1)} &= \frac{p_{in}(R^3 - R_{ou}^3)R_{in}^3 - p_{ou}(R^3 - R_{in}^3)R_{ou}^3}{R^3 p_{in}(R_{ou}^3 - R_{in}^3)}, \\ T_{\Theta\Theta}^{(1)} = T_{\Phi\Phi}^{(1)} &= \frac{p_{in}(2R^3 + R_{ou}^3)R_{in}^3 - p_{ou}(2R^3 + R_{in}^3)R_{ou}^3}{2R^3 p_{in}(R_{ou}^3 - R_{in}^3)}; \end{aligned} \tag{3.1}$$

$$u^{(2)}(R) = \frac{(p_{ou} - p_{in})^2 R_{ou}^3 R_{in}^3}{p_{in}^2 (R_{ou}^3 - R_{in}^3)^2} \left[-\frac{R_{ou}^3 R_{in}^3}{16R^5 b_1^2} + \frac{(11b_1 + b_2)(R_{ou}^3 + R_{in}^3)}{64R^2 b_1^3} \right],$$

$$p^{(2)}(R) = \frac{(p_{ou} - p_{in})^2 R_{ou}^3 R_{in}^3}{p_{in}^2 (R_{ou}^3 - R_{in}^3)^2} \left[-\frac{11b_1 + b_2}{16b_1^2} + \frac{3R_{ou}^3 R_{in}^3}{16b_1 R^6} \left(1 + \frac{b_2}{b_1}\right) \right],$$

$$T_{RR}^{(2)} = -p^{(2)}(R) + \frac{10A_1^2 b_1}{R^6} - \frac{4b_1}{R^3} A_3, \qquad T_{\Theta\Theta}^{(2)} = T_{\Phi\Phi}^{(2)} = -p^{(2)}(R) - \frac{2A_1^2 b_1}{R^6} + \frac{2b_1}{R^3} A_3,$$

$$A_1 = -\frac{(p_{ou} - p_{in})R_{ou}^3 R_{in}^3}{4b_1 p_{in}(R_{ou}^3 - R_{in}^3)}, \qquad A_3 = \frac{(11b_1 + b_2)(p_{ou} - p_{in})^2 R_{in}^3 (1 + R_{in}^3/R_{ou}^3)}{64b_1^3 p_{in}^2 (1 - R_{in}^3/R_{ou}^3)^2}. \tag{3.2}$$

The complete solution of the 2nd-order problem is obtained by substituting into (2.6) expressions for the 1st-order and the 2nd-order quantities. With our non-dimensionalization $b_1 = 1$, but we still use b_1 to indicate which quantities depend upon it. It follows from (2.6) that for the 1st-order problem, i.e., for a linear elastic sphere, values of the radial displacement u and stresses T_{RR} and $T_{\Theta\Theta}$ equal $\varepsilon = p_{in}$ times their values listed in (3.1), and agree with those given in [19]. When $p_{in} = p_{ou} = p$, i.e., the pressures on the inner and the outer surfaces are equal, we get $T_{RR} = T_{\Theta\Theta} = T_{\Phi\Phi} = -p$. Thus the state of stress at every point of the sphere is a hydrostatic pressure.

Whereas the 1st-order elastic constant b_1 affects only the 1st-order displacement, the 2nd-order elastic constant b_2 affects both the 2nd-order displacement and the 2nd-order stresses. Note that in the 1st-order theory the hydrostatic pressure is a constant but in the 2nd-order theory the hydrostatic pressure varies with the radius R.

In the limit of $\beta \equiv R_{ou}/R_{in} \gg 1$, $(2.6)_4$, (3.1) and (3.2) give

$$\begin{aligned}\lim_{\beta\to\infty} T_{\Theta\Theta} &= \lim_{\beta\to\infty} (\varepsilon T^{(1)}_{\Theta\Theta} + \varepsilon^2 T^{(2)}_{\Theta\Theta}),\\ &= -p_{ou} - (p_{ou} - p_{in})\frac{R_{in}^3}{2R^3}\left[1 - \frac{(3b_1+b_2)p_{ou} - (11b_1+b_2)p_{in}}{16b_1^2}\right]\\ &\quad - \frac{(3b_1+b_2)}{8b_1^2}(p_{ou}-p_{in})^2\frac{R_{in}^6}{R^6}.\end{aligned} \tag{3.3}$$

Thus in a very thick hollow sphere, the circumferential stress $T^{\infty}_{\Theta\Theta}(R_{in})$ at the inner surface is given by

$$\begin{aligned}T^{\infty}_{\Theta\Theta}(R_{in}) = -p_{ou} - \frac{(p_{ou}-p_{in})}{2}\left[1 - \frac{(3b_1+b_2)p_{ou} - (11b_1+b_2)p_{in}}{16b_1^2}\right]\\ - \frac{(3b_1+b_2)}{8b_1^2}(p_{ou}-p_{in})^2.\end{aligned} \tag{3.4}$$

For $p_{in} = 0$ and $p_{ou} > 0$, we get

$$T^{\infty}_{\Theta\Theta}(R_{in}) = -\frac{3}{2}p_{ou} - \frac{3(3b_1+b_2)}{32b_1^2}p_{ou}^2,$$

and the stress concentration factor at the inner surface of a very thick sphere, defined as $T^{\infty}_{\Theta\Theta}(R_{in})/p_{ou}$, increases from 1.5 by the consideration of 2nd-order effects provided that $(3b_1 + b_2) > 0$. However, for $p_{ou} < 0$, the consideration of the 2nd-order deformations decreases the stress concentration factor at the inner surface.

For a thin sphere we set $R_{ou} = R_{in} + t$ where t is the sphere thickness, and compute results in the limit of $t/R_{in} = 0$. For $p_{ou} = 0$, the circumferential stress $T^0_{\Theta\Theta}$ is given by

$$T^0_{\Theta\Theta} = \lim_{t/R_{in}\to 0} T_{\Theta\Theta} = \frac{p_{in}R_{in}}{2t}\left[1 + \frac{11b_1+b_2}{16b_1^2}p_{in} + \frac{p_{in}R_{in}}{6tb_1}\right]. \tag{3.5}$$

Recall that in the linear theory, $\frac{p_{in}R_{in}}{2t}$ equals the hoop stress in a thin sphere. The first term on the right hand side of (3.5) is the hoop stress according to the linear theory and the second and the third terms represent contributions from the consideration of the 2nd-order effects. Whereas the third term is always positive the sign of the second term depends upon that of $(11b_1 + b_2)$.

For $p_{ou}=0$, $(3.1)_3$ and $(3.1)_4$ simplify to

$$T_{RR}^{(1)} = p_{in}\frac{(R^3-R_{ou}^3)R_{in}^3}{R^3(R_{ou}^3-R_{in}^3)}, \qquad T_{\Theta\Theta}^{(1)} = \frac{(2R^3+R_{ou}^3)R_{in}^3}{2R^3(R_{ou}^3-R_{in}^3)}p_{in.} \tag{3.6}$$

Thus in a sphere made of a linear elastic homogeneous and isotropic material the radial stress is compressive and the hoop stress is tensile. For $R_{in}/R_{ou} < (1/4)^{1/3} \simeq 0.63$,

$$\begin{aligned} &T_{\Theta\Theta}^{(1)} > \left|T_{RR}^{(1)}\right| \quad \text{when } R > 0.63R_{ou} \quad \text{and} \quad T_{\Theta\Theta}^{(1)} = \left|T_{RR}^{(1)}\right| \quad \text{at } R \simeq 0.63R_{ou}; \\ &\text{for } 0.63 \le R_{in}/R_{ou} < 1, \quad T_{\Theta\Theta}^{(1)} > \left|T_{RR}^{(1)}\right| \text{ throughout the sphere.} \end{aligned} \tag{3.7}$$

Further discussion on the signs of $T_{RR}^{(1)}$ and $T_{\Theta\Theta}^{(1)}$ is given in Appendix A.

4 Sphere Material Functionally Graded

4.1 Affine Variation of the Two Moduli

We assume that μ and α vary affinely in the radial direction, i.e.,

$$\mu(R) = b_1(1+nR), \qquad \alpha(R) = b_2(1+mR), \tag{4.1}$$

where b_1, b_2, n and m are constants. Omitting details the solution of the problem is

$$\begin{aligned} &u^{(1)}(R) = \frac{\bar{A}_1}{R^2}, \qquad T_{RR}^{(1)} = -\left[\bar{A}_2 + 2\bar{A}_1 b_1\left(\frac{n}{R^2} + \frac{2(1+nR)}{R^3}\right)\right], \\ &T_{\Theta\Theta}^{(1)} = T_{\phi\phi}^{(1)} = -\left[\bar{A}_2 + 2\bar{A}_1 b_1\left(\frac{n}{R^2} - \frac{(1+nR)}{R^3}\right)\right], \\ &\bar{A}_1 = \frac{(p_{ou}-p_{in})R_{ou}^3 R_{in}^3}{2b_1 D}, \qquad \bar{A}_2 = \frac{p_{in}R_{in}^3(2+3nR_{ou}) - p_{ou}R_{ou}^3(2+3nR_{in})}{D}, \\ &D = p_{in}\big[R_{in}^3(2+3nR_{ou}) - R_{ou}^3(2+3nR_{in})\big]; \end{aligned} \tag{4.2}$$

$$\begin{aligned} &u^{(2)}(R) = \frac{(p_{ou}-p_{in})^2 R_{ou}^3 R_{in}^3}{4b_1^2 p_{in}^2 R^2}\left[-\frac{R_{ou}^3 R_{in}^3}{C_4 R^3} + \frac{11b_1 C_2 + b_2 C_3}{10 b_1 C_5}\right], \\ &T_{RR}^{(2)}(R) = B_1 R^{-2} + B_2 R^{-3} + B_3 R^{-5} + B_4 R^{-6} + B_5, \\ &T_{\Theta\Theta}^{(2)}(R) = -\frac{B_2}{2}R^{-3} - \frac{3}{2}B_3 R^{-5} - 2B_4 R^{-6} + B_5, \end{aligned} \tag{4.3}$$

and expressions for constants are given in Appendix B. As before, the complete solution of the 2nd-order problem is obtained by substituting into (2.6) expressions for the 1st-order and the 2nd-order quantities.

For a *linear elastic* hollow FG sphere subjected to internal pressure only, we get the following for the radial and the circumferential stresses.

$$T_{RR}^{(1)} = \frac{p_{in}}{R^3}\frac{(R-R_{ou})[3nRR_{ou}(R+R_{ou})+2(R^2+RR_{ou}+R_{ou}^2)]}{(R_{ou}-R_{in})[2(R_{ou}^2+R_{ou}R_{in}+R_{in}^2)+3nR_{ou}R_{in}(R_{ou}+R_{in})]},$$
$$T_{\Theta\Theta}^{(1)} = -\frac{p_{in}}{R^3}\frac{[R^3(2+3nR_{ou})+R_{ou}^3]R_{in}^3}{[R_{in}^3(2+3nR_{ou})-R_{ou}^3(2+3nR_{in})]}. \tag{4.4}$$

For $n \geq 0$, $(4.4)_1$ implies that $T_{RR} < 0$ and $dT_{RR}/dR > 0$. Thus the radial stress is compressive for all values of R and is a monotonically increasing function of R or, said differently, the magnitude of T_{RR} monotonically decreases from p_{in} at $R = R_{in}$ to zero at $R = R_{ou}$. For two distinct values of n, i.e., $n_1 \neq n_2$

$$T_{RR}^{(1)}(R,n_1) - T_{RR}^{(1)}(R,n_2) = \frac{6p_{in}(n_1-n_2)(R-R_{ou})(R-R_{in})(R+R_{in}+R_{ou})R_{in}^3R_{ou}^3}{R^3(R_{ou}-R_{in})E},$$
$$T_{\Theta\Theta}^{(1)}(R,n_1) - T_{\Theta\Theta}^{(1)}(R,n_2) = \frac{p_{in}}{R^3}\frac{3(n_2-n_1)(-2R^3+R_{ou}^2R_{in}+R_{ou}R_{in}^2)}{(R_{ou}-R_{in})E}, \tag{4.5}$$

where

$$E = p_{in}[2R_{in}^2+R_{ou}(2+3n_2R_{in})(R_{ou}+R_{in})][2R_{in}^2+R_{ou}(2+3n_1R_{in})(R_{ou}+R_{in})]. \tag{4.6}$$

Thus $T_{RR}^{(1)}(R,n_1) - T_{RR}^{(1)}(R,n_2) > 0$ provided that $n_1 < n_2$, and $T_{RR}^{(1)}$ is independent of n only for $R = R_{in}$ and $R = R_{ou}$, i.e., at the inner and the outer surfaces which follows from the boundary conditions. However,

$$T_{\Theta\Theta}^{(1)}(\bar{R},n_1) = T_{\Theta\Theta}^{(1)}(\bar{R},n_2) \quad \text{for } \bar{R} = \left(\frac{R_{ou}R_{in}(R_{ou}+R_{in})}{2}\right)^{1/3}. \tag{4.7}$$

That is, *the hoop stress at the point $R = \bar{R}$ is independent of the value assigned to n*. One can check that $R_{in} < \bar{R} < R_{ou}$. For $n_1 < n_2$, $T_{\Theta\Theta}^{(1)}(R,n_1) - T_{\Theta\Theta}^{(1)}(R,n_2) > (<)0$ for $R < (>)\bar{R}$.

For $n \geq 0$, it follows from $(4.4)_2$ that $T_{\Theta\Theta}^{(1)} > 0$. Since $dT_{\Theta\Theta}^{(1)}/dR < 0$, the first-order hoop stress is a monotonically decreasing function of R. For $n < 0$ we set $k = -n$, and get the following: for $1/R_{ou} < k < \bar{k}$, $T_{\Theta\Theta}^{(1)}$ is tensile for $R_{in} < R < \tilde{R}$, compressive for $\tilde{R} < R < R_{ou}$, and vanishes at $R = \tilde{R}$. When $\bar{k} < k < C/3R_{ou}$, $T_{\Theta\Theta}^{(1)}$ is tensile for $\tilde{R} < R < R_{ou}$, and compressive for $R_{in} < R < \tilde{R}$. Here $\tilde{R} = R_{ou}/(3kR_{ou}-2)^{1/3}$, and $C = (R_{ou}/R_{in})^3 + 2$. For other values of k, $T_{\Theta\Theta}^{(1)}$ is tensile. By analyzing the sign of $dT_{\Theta\Theta}^{(1)}/dR$, we find that the first-order hoop stress is a monotonically increasing function of R when $k > \bar{k}$ but is a monotonically decreasing function of R for $k < \bar{k}$ with $\bar{k} = 2(R_{ou}^2 + R_{ou}R_{in} + R_{in}^2)/3R_{ou}R_{in}(R_{in} + R_{ou})$. One can show that $1/R_{ou} < \bar{k} < 1/R_{in}$.

For $R_{ou}/R_{in} = 5$ and four different positive values of n we have plotted in Fig. 1 the variation with R/R_{in} of $T_{\Theta\Theta}^{(1)}$ and $T_{RR}^{(1)}$. For $R_{ou}/R_{in} = 5$, $\bar{R} = 2.47R_{in}$. With an increase in the value of n from 0 to 2.5, the maximum value of the hoop stress at the inner surface decreases from $0.5p_{in}$ to $0.14p_{in}$. For $R < \bar{R}$ the hoop stress depends noticeably upon the value of n but for $R > \bar{R}$ values of the hoop stress are insensitive to the value assigned to n. Thus the maximum hoop stress at the inner surface can be decreased by suitably grading the value of the shear modulus in the radial direction. At $R = \bar{R}$ computed values of the hoop stress $T_{\Theta\Theta}^{(1)}$ are independent of the value of n as derived analytically above. It is clear that by suitably assigning a negative value to n one can have tensile $T_{RR}^{(1)}$ and compressive $T_{\Theta\Theta}^{(1)}$ at an interior point of the sphere. Values of $T_{RR}^{(1)}$ strongly depend upon the value of

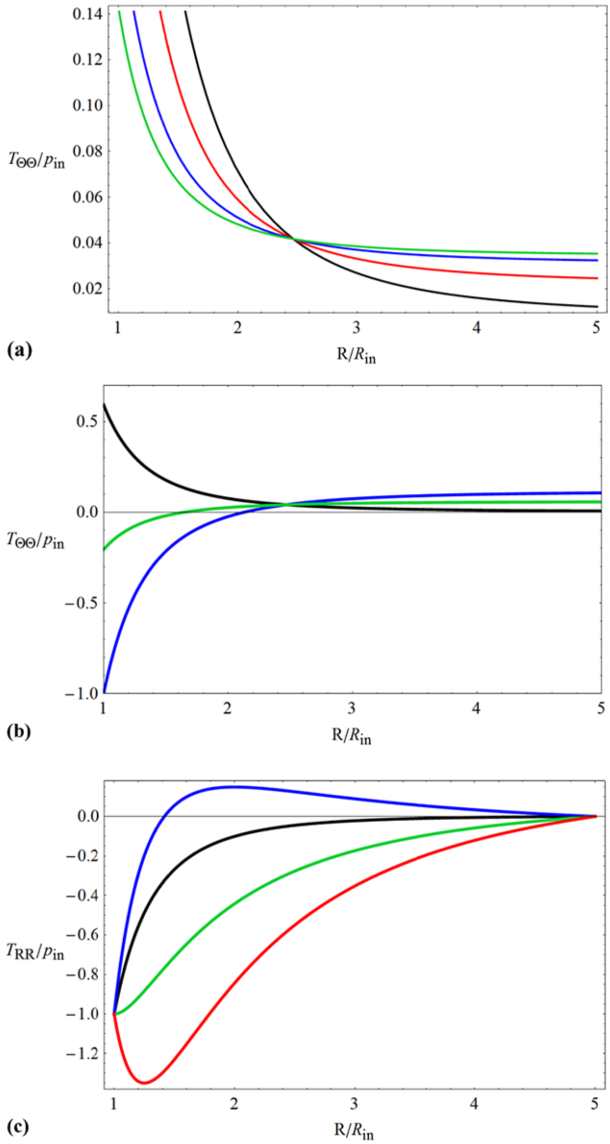


Fig. 1 (Color online) For $R_{ou}/R_{in} = 5$ and different values of n, the variation of the hoop stress and the radial stress through the thickness of the hollow sphere; (**a**) *black*, $n = 0$; *red*, $n = 0.5$; *blue*, $n = 1.5$; *green*, $n = 2.5$; (**b**) *black*, $n = -0.1$; *blue* $n = -1$; *green* $n = -2$; (**c**) *black*, $n = -0.1$; *blue*, $n = -0.5$; *red*, $n = -0.8$; *green*, $n = -1$

n throughout the sphere thickness but those of $T^{(1)}_{\Theta\Theta}$ vary rapidly with n only in the region $1 \le R/R_{in} \le 2.5$.

4.2 Power Law Variation of the two Moduli

We assume that $\mu(R)$ and $\alpha(R)$ are given by $\mu(R) = b_1 R^n, \alpha(R) = b_2 R^m$ where m and n are constants, and $b_1 = \mu(R_{in})/R^n_{in}, b_2 = \alpha(R_{in})/R^m_{in}$. Expressions for displacements and stresses for $n = 3$ are different from those for $n \neq 3$. Accordingly, we give below the solution for several integer values of n and m, and list in Appendix B expressions for various constants appearing in the solutions.

4.2.1 Solution of the 1st-order problem

The solution of the 1st-order problem for $n \neq 3$ is

$$\begin{aligned} u^{(1)}(R) &= \frac{(n-3)(p_{ou}-p_{in})R_{ou}^3 R_{in}^3}{12R^2 b_1 p_{in}(R_{ou}^3 R_{in}^n - R_{ou}^n R_{in}^3)}, \\ T_{RR}^{(1)} &= -\frac{p_{ou}}{p_{in}} + \frac{(p_{ou}-p_{in})(-R^n R_{ou}^3 + R^3 R_{ou}^n)R_{in}^3}{R^3 p_{in}(R_{ou}^n R_{in}^3 - R_{ou}^3 R_{in}^n)}, \\ T_{\Theta\Theta}^{(1)} = T_{\Phi\Phi}^{(1)} &= -\frac{p_{ou}}{p_{in}} + \frac{(p_{ou}-p_{in})((1-n)R^n R_{ou}^3 + 2R^3 R_{ou}^n)R_{in}^3}{2R^3 p_{in}(R_{ou}^n R_{in}^3 - R_{ou}^3 R_{in}^n)}, \end{aligned} \tag{4.8}$$

and that for $n = 3$ and $p_{ou} = 0$ is

$$\begin{aligned} u^{(1)}(R) &= \frac{1}{R^2}\frac{1}{12b_1 \ln(R_{ou}/R_{in})}, \qquad T_{RR}^{(1)}(R) = \frac{\ln(R/R_{ou})}{\ln(R_{ou}/R_{in})}, \\ T_{\Theta\Theta}^{(1)}(R) &= \frac{(1+2\ln(R/R_{ou}))}{2\ln(R_{ou}/R_{in})}. \end{aligned} \tag{4.9}$$

We note that the solution for $n = 3$ is not given in [19].

4.2.2 Solution of the 2nd-order problem

For the 2nd-order problem the solution for $n \neq 3, 6$; $m \neq 6$ is given below.

$$\begin{aligned} u^{(2)}(R) &= \frac{(n-3)^2(p_{ou}-p_{in})^2 R_{ou}^3 R_{in}^3}{144 b_1^2 p_{in}^2 (R_{ou}^n R_{in}^3 - R_{ou}^3 R_{in}^n)^2} \\ &\quad \times \left\{ \left[\frac{(n-3)[(n-6)b_2(-R_{ou}^m R_{in}^6 + R_{ou}^6 R_{in}^m) + 11(m-6)b_1(-R_{ou}^n R_{in}^6 + R_{ou}^6 R_{in}^n)]}{2(m-6)(n-6)b_1(R_{ou}^n R_{in}^3 - R_{ou}^3 R_{in}^n)} \right] R^{-2} \right. \\ &\quad \left. - R_{ou}^3 R_{in}^3 R^{-5} \right\}, \\ T_{RR}^{(2)}(R) &= H_1 R^{-3} + H_2 R^{-6+m} + H_3 R^{-6+n} + H_4 R^{-3+n} + H_5, \\ T_{\Theta\Theta}^{(2)}(R) &= -\frac{H_1}{2}R^{-3} + \frac{H_2(m-4)}{2}R^{-6+m} + \frac{H_3(n-4)}{2}R^{-6+n} + \frac{H_4(n-1)}{2}R^{-3+n} + H_5. \end{aligned} \tag{4.10}$$

Expressions for constants H_i's are given in Appendix B.

The solution for $m = n = 3$ and $p_{ou} = 0$ is:

$$\begin{aligned} u^{(2)}(R) &= -\frac{1}{144R^5(b_1\ln(R_{ou}/R_{in}))^2} + \frac{(11b_1+b_2)(-R_{ou}^{-3}+R_{in}^{-3})}{864R^2(b_1\ln(R_{ou}/R_{in}))^3}, \\ T_{RR}^{(2)} &= H\big\{(11b_1+b_2)R^3\big[\ln(R/R_{ou})R_{ou}^3 - \ln(R/R_{in})R_{in}^3\big] \\ &\quad + R_{ou}^3 R_{in}^3 \ln(R_{ou}/R_{in})\big[(11+12\ln(R/R_{ou}))b_1 + b_2\big]\big\}, \\ T_{\Theta\Theta}^{(2)} &= \frac{H}{2}\big\{(11b_1+b_2)R^3\big[(1+2\ln(R/R_{ou}))R_{ou}^3 - (1+2\ln(R/R_{in}))R_{in}^3\big] \\ &\quad + R_{ou}^3 R_{in}^3 \ln(R_{ou}/R_{in})[(1-12\ln(R/R_{ou}))b_1 - b_2]\big\}, \end{aligned} \tag{4.11}$$

where

$$H = \frac{1}{72b_1^2 R^3 (R_{in} R_{ou} \ln(R_{ou}/R_{in}))^3}. \tag{4.12}$$

The complete solution of the problem for $m = n = 6$, $p_{ou} = 0$ is given below.

$$u(R) = \frac{p_{in}}{4b_1 R^2 (R_{ou}^3 - R_{in}^3)} + \frac{3\ln(R_{ou}/R_{in})(11b_1 + b_2) p_{in}^2}{32 R^2 b_1^3 (R_{ou}^3 - R_{in}^3)^3} - \frac{p_{in}^2}{16 b_1^2 R^5 (R_{ou}^3 - R_{in}^3)^2},$$

$$\begin{aligned}
T_{RR} &= \frac{p_{in}(R^3 - R_{ou}^3)}{R_{ou}^3 - R_{in}^3} + G\big[3b_2 R^3 \big\{R^3 \ln(R_{ou}/R_{in}) - R_{ou}^3 \ln(R/R_{in}) - R_{in}^3 \ln(R/R_{ou})\big\} \\
&\quad + b_1\big\{-4R_{ou}^6 + R_{ou}^3\big(R^3\big(4 - 33\ln(R/R_{in})\big) + 4R_{in}^3\big) \\
&\quad + R^3\big(33R^3 \ln(R_{ou}/R_{in}) + \big(-4 - 33\ln(R_{ou}/R)\big)R_{in}^3\big\}\big], \\
T_{\Theta\Theta} &= \frac{p_{in}(5R^3 - 2R_{ou}^3)}{2(R_{ou}^3 - R_{in}^3)} + \frac{G}{2}\big[3b_2 R^3 \big\{5R^3 \ln(R_{ou}/R_{in}) \\
&\quad - R_{ou}^3 + R_{in}^3 - 2R_{ou}^3 \ln(R/R_{in}) - 2R_{in}^3 \ln(R_{ou}/R)\big\} \\
&\quad + b_1\big\{4R_{ou}^6 - R_{ou}^3\big(R^3\big(25 + 66\ln(R/R_{in})\big) + 4R_{in}^3\big) \\
&\quad + R^3\big(165R^3 \ln(R_{ou}/R_{in}) + R_{in}^3\big(25 - 66\ln(R_{ou}/R)\big)\big)\big\}\big],
\end{aligned} \tag{4.13}$$

where

$$G = \frac{p_{in}^2}{8R^3 b_1^2 (R_{ou}^3 - R_{in}^3)}. \tag{4.14}$$

For $n = 3, m = 6$, we give below the solution of the problem.

$$\begin{aligned}
u^{(1)}(R) &= \frac{\bar{B}_1}{R^2}, \qquad T_{RR}^{(1)}(R) = -\bar{B}_2 + 4\bar{B}_1 b_1 (3\ln R - 1), \\
T_{\Theta\Theta}^{(1)}(R) &= -\bar{B}_2 + 2\bar{B}_1 b_1 (6\ln R + 1),
\end{aligned} \tag{4.15}$$

$$u^{(2)}(R) = -\frac{\bar{B}_1^2}{R^5} + \frac{\bar{B}_3}{R^2},$$

$$\begin{aligned}
T_{RR}^{(2)} &= \frac{(p_{ou} - p_{in})}{\bar{B}_4}\big\{11R^3(p_{ou} - p_{in})\big[\ln(R/R_{ou})R_{ou}^3 - \ln(R/R_{in})R_{in}^3\big] \\
&\quad + R_{ou}^3 R_{in}^3 \ln(R_{ou}/R_{in})\big[\big(11 + 12\ln(R/R_{in})\big)p_{ou} + \big(-11 - 12\ln(R/R_{ou})\big)p_{in}\big]\big\}, \\
T_{\Theta\Theta}^{(2)} &= \frac{(p_{ou} - p_{in})}{2B_4}\big\{11R^3(p_{ou} - p_{in})\big[\big(1 + 2\ln(R/R_{ou})\big)R_{ou}^3 - \big(1 + 2\ln(R/R_{in})\big)R_{in}^3\big] \\
&\quad - R_{ou}^3 R_{in}^3 \ln(R_{ou}/R_{in})\big[\big(-1 + 12\ln(R/R_{in})\big)p_{ou} + \big(1 - 12\ln(R/R_{ou})\big)p_{in}\big]\big\}.
\end{aligned} \tag{4.16}$$

Expressions for constants $\bar{B}_i$'s are listed in Appendix B.

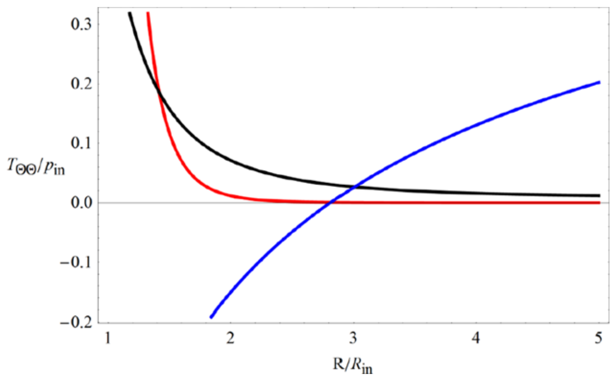


Fig. 2 (Color online) For $R_{ou}/R_{in} = 5$ and different values of n the variation of the hoop stress through the thickness of the hollow sphere. *Black* $n = 0$; *red* $n = -5$; *blue* $n = 2.5$

In [19] it was proved that the hoop stress in a FG sphere is uniform if $n = 1$. Accordingly we list below the solution for $m = n = 1$, and $p_{ou} = 0$.

$$u^{(1)}(R) = \frac{1}{6b_1 R^2}\frac{R_{ou}^2 R_{in}^2}{R_{ou}^2 - R_{in}^2}, \qquad T_{RR}^{(1)}(R) = \frac{(R^2 - R_{ou}^2)R_{in}^2}{R^2(R_{ou}^2 - R_{in}^2)}, \qquad T_{\Theta\Theta}^{(1)}(R) = \frac{R_{in}^2}{R_{ou}^2 - R_{in}^2},$$

$$u^{(2)}(R) = -\frac{R^{-2}}{6b_1}\left(D_1 + \frac{D_2 R_{ou}^2}{2} R^{-3}\right), \qquad T_{RR}^{(2)} = D_1 R^{-2} + D_2 R^{-3} + D_3 R^{-5} + D_4, \tag{4.17}$$

$$T_{\Theta\Theta}^{(2)} = -\frac{D_2}{2} R^{-3} - \frac{3}{2} D_3 R^{-5} + D_4.$$

Expressions for constants D_i's are given in Appendix B.

For a *linear elastic* sphere it follows from $(4.8)_3$ that when $n = 1$,

$$T_{\Theta\Theta}^{(1)} = T_{\Phi\Phi}^{(1)} = -\frac{p_{ou}}{p_{in}} + \frac{(p_{ou} - p_{in})}{(1 - R_{ou}^2/R_{in}^2)p_{in}}; \tag{4.18}$$

thus $T_{\Theta\Theta}$ and $T_{\Phi\Phi}$ are constants throughout the sphere; $(4.17)_3$ is a special case of (4.18) for $p_{ou} = 0$. In a very thick sphere, $R_{ou}/R_{in} \gg 1$, $T_{\Theta\Theta} \simeq -p_{ou}$. That is, in a very thick sphere with the shear modulus proportional to the radius R, the hoop stress nearly equals the negative of the pressure applied on the outer surface of the sphere. For a very thin sphere with $p_{ou} = 0$, $R_{ou} = R_{in} + t$, $t/R_{in} \ll 1$, and we recover the classical result that $T_{\Theta\Theta} = T_{\Phi\Phi} = (p_{in}R_{in})/2t$. Thus the linear gradation of material properties in the radial direction has no effect on the surface tension in a very thin sphere made of a linear elastic FG material. It follows from $(4.17)_2$ that for $n = 1$, $T_{RR}^{(1)}$ varies with R.

Equation (4.18) implies that for $p_{ou}R_{ou}^2 = p_{in}R_{in}^2$, $T_{\Theta\Theta}^{(1)} = 0$ throughout the sphere.

For a hollow sphere with $p_{ou} = 0$ and $R_{in}/R_{ou} = 1/5$, we have plotted in Fig. 2, for different values of n, $T_{\Theta\Theta}^{(1)}$ vs. R/R_{in}. For $n = 0$ and -5 (which are less than 1) the hoop stress is tensile everywhere. For $n = 2.5$ (which is greater than 1) the hoop stress is compressive for $1 < R/R_{in} < \tilde{R}/R_{in}(= 2.81)$, tensile for $\tilde{R} < R < R_{ou}$, and vanishes at $R = \tilde{R}$.

It is evident from $(4.17)_6$ that even though the first-order hoop stress is constant through the sphere thickness, the 2nd-order hoop stress varies with R.

5 Numerical Results

5.1 Sphere Made of Homogeneous Material

In order to highlight differences between results for the 1st-order and the 2nd-order problems, we consider a hollow sphere with $R_{ou}/R_{in} = 5$, $p_{ou} = 0$ and made of a homogeneous 2nd-order elastic material with $\mu = 205.2$ kPa, $\alpha = -77.4$ kPa; these values of μ and α are for the rubber tested in [23]. We have plotted in Fig. 3 T_{RR}/p_{in} and $T_{\Theta\Theta}/p_{in}$ as functions of R/R_{in}. Furthermore in order to illuminate the 2nd-order effects we have plotted in Fig. 4 $|\frac{T_{RR}^{(2)}}{T_{RR}^{(1)}}|$ and $|\frac{T_{\Theta\Theta}^{(2)}}{T_{\Theta\Theta}^{(1)}}|$ vs. R/R_{in}.

It should be clear from (2.9) that $T_{RR}^{(2)}$ and hence T_{RR} depend upon the radial displacement of the inner surface. That is why T_{RR} for the 2nd-order problem does not equal p_{in} on the inner surface of the sphere.

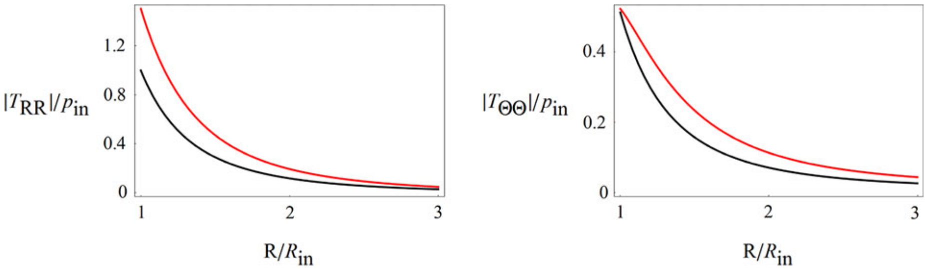
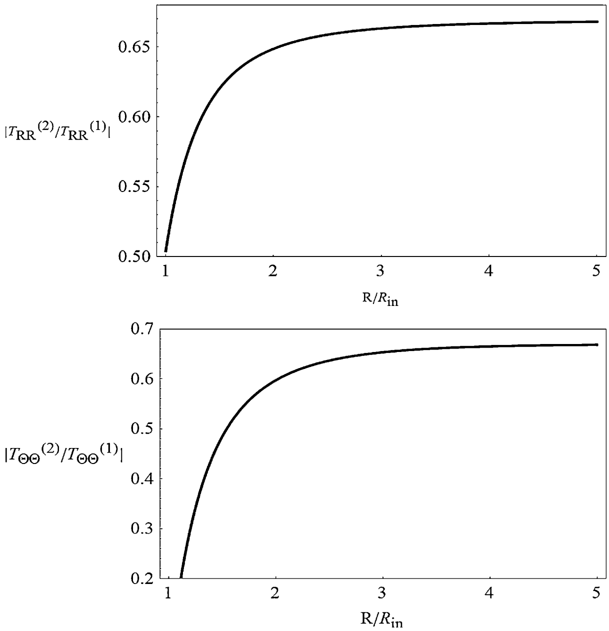


Fig. 3 (Color online) For $R_{ou}/R_{in} = 5$, variation through the sphere thickness of the radial and the hoop stress magnitude for the linear (*black curve*) and the 2nd-order elastic materials (*red curve*)

Fig. 4 For $R_{ou}/R_{in} = 5$, plots of $|T_{RR}^{(2)}/T_{RR}^{(1)}|$ and $|T_{\Theta\Theta}^{(2)}/T_{\Theta\Theta}^{(1)}|$ vs. R/R_{in}



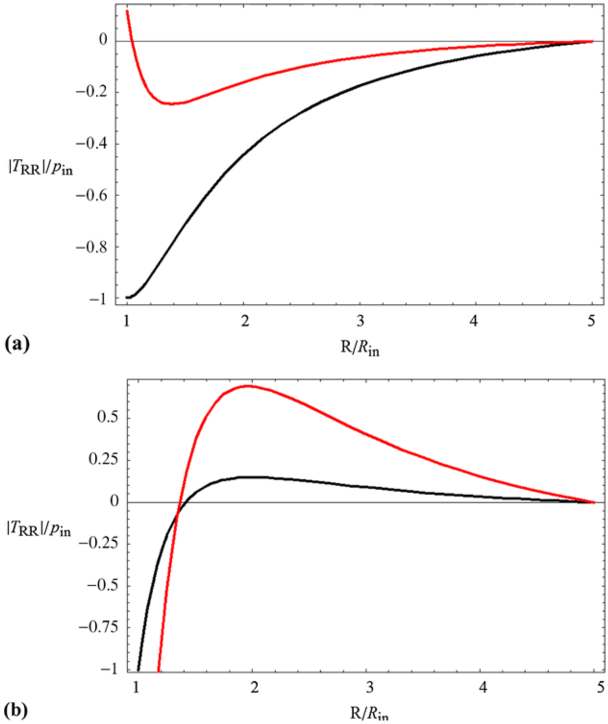


Fig. 5 (Color online) For $R_{ou}/R_{in} = 5$ through the thickness variation of the radial stress for spheres made of 1st-order (*black curve*) and 2nd-order (*red curve*) elastic materials, (**a**) $n = -1$, (**b**) $n = -0.5$

It is clear from results exhibited in Fig. 4 that the consideration of 2nd-order effects noticeably influences stress distribution in the sphere. We should add that results depend upon the relative values of the two elastic moduli μ and α.

5.2 Functionally Graded Material

5.2.1 Affine Variation of the Two Moduli

In order to study the effect of material inhomogeneity on the structural response, we have analyzed, for different values of n, the through the thickness variations of the radial and the hoop stresses. Results exhibited in Fig. 5 for a sphere with $R_{ou}/R_{in} = 5$ reveal that for $n = -0.5$, and $n = -1$, the radial stress for a 2nd-order elastic material differs noticeably from that for a linear (1st-order) elastic material.

In Fig. 6 we have displayed through the thickness variation of the hoop stress corresponding to $n = -0.5$ and $n = -1$ for $R_{ou}/R_{in} = 5$. For $n = -0.5$ the two curves intersect each other at $R \cong 2.83$ while for $n = -1$ they intersect each other at two points: $R \cong 1.22$ and $R \cong 2.11$.

Power Law Variation of the Two Moduli For the case of a power law variation of material moduli the stress fields for spheres made of linear elastic materials have the same shape as those for 2nd-order elastic materials and differ only quantitatively. For this reason we omit their graphs.

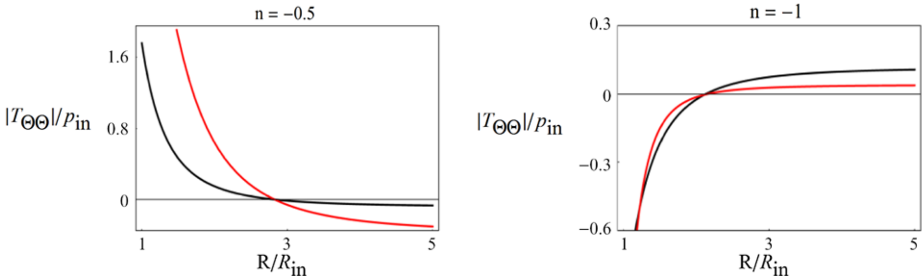


Fig. 6 (Color online) Through the thickness variation of the hoop stress for spheres made of 1st-order (*black curve*) and 2nd-order elastic materials (*red curve*)

6 Conclusions

We have studied radial expansion/contraction of a hollow sphere with the inner and the outer surfaces subjected to pressures, and the sphere material modeled as isotropic, incompressible and second-order elastic with the two material moduli smoothly varying in the radial direction either according to a power-law relation or an affine relation. Whereas for a linear variation of the shear modulus the hoop stress is uniform through the sphere thickness for a linear elastic material, it is not so for a second-order elastic material. Thus results for the 2nd-order problem cannot be deduced from those for the first-order problem. For the affine variation of the shear modulus, there exists a point in the sphere where the hoop stress is independent of the slope of the shear modulus vs. the radius curve. By suitably varying the material moduli in the radial direction, one can control the sign of the radial and the hoop stresses.

The challenging inverse problem of finding the spatial distribution of material moduli (i.e., material tailoring) to achieve a desired through-the-thickness distribution of either the circumferential stress or the in-plane shear stress is addressed in [3, 24–27]. The radial expansion as well as inversion of a functionally graded cylinder made of a Mooney-Rivlin material has been analyzed in [28].

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Appendix A: Discussion of signs of $T^{(1)}_{RR}$ and $T^{(1)}_{\Theta\Theta}$

Signs of Stresses for the Affine Variation of the Moduli when $n < 0$ We now analyze stresses when $n < 0$, and set $k = -n$ so that $k > 0$. Let

$$\begin{aligned} f(R,k) &= (R - R_{ou})P(R,k), \\ P(R,k) &= (2 - 3kR_{ou})R(R + R_{ou}) + 2R_{ou}^2. \end{aligned} \tag{A.1}$$

We note that $f(R,k)$ determines the sign of the numerator for $T^{(1)}_{RR}$ in (4.4)$_1$. The expression of $P(R,k)$ implies that $f(R,k) \le 0$ for $k \le 2/(3R_{ou})$ and $f(R,k) = 0$ for $R = R_{ou}$. We first consider the case of $k > 2/(3R_{ou})$. Using the Descartes rule of signs the quadratic equation

$P(R,k)=0$ has one positive root R^+ and one negative root R^-. Since R cannot be negative, we discard the negative root, and study further the case of the positive root:

$$R^+ = \frac{R_{ou}[\sqrt{3A(kR_{ou}+2)} - A]}{2A}, \quad A = 3kR_{ou} - 2 > 0. \tag{A.2}$$

The only case of interest here is $R_{in} \le R^+ \le R_{ou}$. For $R^+ \ge R_{in}$, one can show that $2/(3R_{ou}) < k \le \bar{k}$ with

$$\bar{k} = \frac{2\left(R_{ou}^2 + R_{ou}R_{in} + R_{in}^2\right)}{3R_{ou}R_{in}\left(R_{in}+R_{ou}\right)}, \tag{A.3}$$

and $1/R_{ou} < \bar{k} < 1/R_{in}$. One can similarly prove that for $R^+ \le R_{ou}, k \ge 1/R_{ou}$. Thus for $1/R_{ou} \le k \le \bar{k}$, we get

$$\begin{aligned} f(R,k) &\ge 0 \quad \text{when } R^+ \le R \le R_{ou}, \\ f(R,k) &< 0 \quad \text{when } R_{in} < R < R^+. \end{aligned} \tag{A.4}$$

For $k < \bar{k}$, the denominator in the expression $(4.4)_1$ for $T_{RR}^{(1)}$ is positive.

By combining the afore-stated discussion of the signs of the numerator and the denominator in the expression $(4.4)_1$ for $T_{RR}^{(1)}$ we conclude that for $1/R_{ou} \le k < \bar{k}$, $T_{RR}^{(1)}$ is tensile for $R^+ < R < R_{ou}$, compressive for $R_{in} < R < R^+$, and equals zero for $R = R^+$. For other values of k, $T_{RR}^{(1)}$ is compressive.

Differentiation of both sides of $(4.4)_1$ with respect to R gives

$$\frac{dT_{RR}^{(1)}}{dR} = \frac{6(1-kR)p_{in}R_{in}^3R_{ou}^3}{R^4\left[R_{ou}^3(2-3kR_{in}) - R_{in}^3(2-3kR_{ou})\right]}. \tag{A.5}$$

Since the denominator is positive for $k < \bar{k}$, we conclude that

$$\frac{dT_{RR}^{(1)}}{dR} > 0 \quad \text{for } k > \frac{1}{R_{ou}}, \tag{A.6}$$

and $T_{RR}^{(1)}$ increases for $R < 1/k$, decreases for $R > 1/k$ and is the maximum at $R = 1/k$.

For $R_{ou}/R_{in} = 5$ and different negative values of n we have plotted in Fig. 1(b), (c) the variation with R/R_{in} of $T_{RR}^{(1)}$.

Signs of Stresses for the Power Law Variation of the Moduli We investigate the sign of $T_{RR}^{(1)}$ and $T_{\Theta\Theta}^{(1)}$ for $n \ne 1, 3, 6$ when the sphere is loaded on the inner surface only ($p_{ou} = 0$). We first note that the function

$$f(R) = R^{n-3}, \quad R_{in} \le R \le R_{ou}, \tag{A.7}$$

is always positive. It is a monotonically increasing (decreasing) function of R for $n > (<)3$. Values of R for which $f(R)$ is maximum and minimum, and the sign of the term $(R_{ou}^{n-3} - R_{in}^{n-3})$ are summarized in Table 1.

Thus $T_{RR}^{(1)} < 0$ throughout the thickness of the hollow sphere and $T_{RR}^{(1)}(R_{ou}) = 0$ as required by the boundary condition at $R = R_{ou}$. Furthermore, since

$$\frac{dT_{RR}^{(1)}}{dR} = \frac{(n-3)p_{in}}{(R_{ou}^{n-3} - R_{in}^{n-3})}R^{n-4}, \tag{A.8}$$

Table 1 Values of R where $f(R)$ is either maximum or minimum, and the sign of $R_{ou}^{n-3} - R_{in}^{n-3}$

	R for $f(R)$ maximum	R for $f(R)$ minimum	Sign of $R_{ou}^{n-3} - R_{in}^{n-3}$
$n > 3$	R_{ou}	R_{in}	positive
$n < 3$	R_{in}	R_{ou}	negative

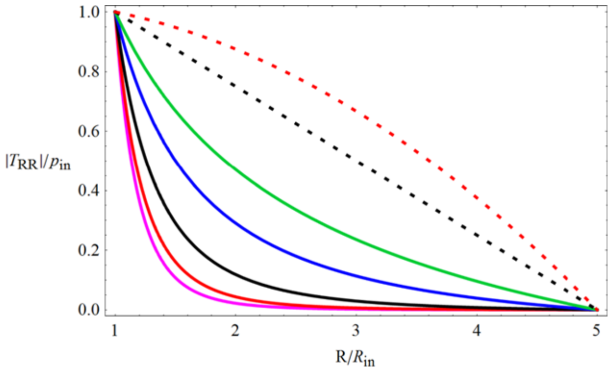

Fig. 7 (Color online) For $R_{ou}/R_{in} = 5$ and different values of n, variation of the radial stress through the thickness of the hollow sphere. *Fuchsia line*: $n = -2.5$; *red line*: $n = -1.5$; *black line*: $n = 0$; *blue line*: $n = 1.5$; *green line*: $n = 2.5$; *dashed black line*: $n = 5$; *dashed red line*: $n = 6$

$|T_{RR}^{(1)}|$ is a monotonically decreasing function of R through the sphere thickness. For $R_{ou}/R_{in} = 5$ we have plotted in Fig. 7 $|T_{RR}^{(1)}|$ vs. R for different values of n. It is clear that for a given value of R, $|T_{RR}^{(1)}|$ increases with an increase in n.

When $n > 3$ and $p_{ou} = 0$, we derive from (4.8) the following result

$$T_{\Theta\Theta}^{(1)} \geq 0 \quad \text{if and only if } ((n-1)R^{n-3} - 2R_{ou}^{n-3}) \geq 0. \tag{A.9}$$

Thus $T_{\Theta\Theta}^{(1)} \geq 0$ if and only if $R \geq \tilde{R}$ where

$$\tilde{R} = \beta(n) R_{ou}, \qquad \beta(n) = \left(\frac{2}{n-1}\right)^{\frac{1}{n-3}}. \tag{A.10}$$

We note that $\tilde{R} < R_{ou}$, and $R_{in} < \tilde{R}$ if and only if $R_{in}/R_{ou} < \beta(n)$. For $R_{in}/R_{ou} < \beta(n)$,

$$\begin{aligned} T_{\Theta\Theta}^{(1)} &\geq 0 \quad \text{when } \tilde{R} \leq R < R_{ou}, \\ T_{\Theta\Theta}^{(1)} &\leq 0 \quad \text{when } R_{in} < R \leq \tilde{R}. \end{aligned} \tag{A.11}$$

If $R_{in}/R_{ou} > \beta(n)$, then $T_{\Theta\Theta}^{(1)} > 0$ for $R_{in} \leq R \leq R_{ou}$.

When $1 < n < 3$ and $p_{ou} = 0$, we again conclude from $(4.8)_3$ that

$$T_{\Theta\Theta}^{(1)} \geq 0 \Leftrightarrow \left((n-1)R^{n-3} - 2R_{ou}^{n-3}\right) \leq 0, \tag{A.12}$$

which implies the following. If $R_{in}/R_{ou} < \beta(n)$, then $T_{\Theta\Theta}^{(1)} \geq 0$ for $\tilde{R} \leq R < R_{ou}$, and $T_{\Theta\Theta}^{(1)} \leq 0$ for $R_{in} < R \leq \tilde{R}$. For $R_{in}/R_{ou} \geq \beta(n)$, $T_{\Theta\Theta}^{(1)} > 0$ throughout the sphere thickness.

For $n < 1$ and $p_{ou} = 0$, $(4.8)_3$ implies that $T_{\Theta\Theta}^{(1)} > 0$ for $R_{in} \leq R \leq R_{ou}$. Differentiation of $(4.8)_3$ gives

$$\frac{dT_{\Theta\Theta}^{(1)}}{dR} = \frac{p_{in}}{2} \frac{(n-1)(n-3)}{(R_{ou}^{n-3} - R_{in}^{n-3})} R^{n-4} \tag{A.13}$$

which is always negative for $n < 1$. Thus $T_{\Theta\Theta}^{(1)}$ is a monotonically decreasing function of R.

Appendix B

Expressions for constants in (4.3):

$$B_1 = -\frac{3n(p_{ou}-p_{in})^2 R_{ou}^3 R_{in}^3 (11b_1C_2 + b_2C_3)}{20p_{in}^2 C_5 b_1^2},$$

$$B_2 = \frac{(p_{ou}-p_{in}) R_{ou}^3 R_{in}^3 [b_1 C_6 - b_2 (p_{ou}-p_{in}) C_3]}{10 p_{in}^2 C_5 b_1^2},$$

$$B_3 = \frac{3(nb_1 + mb_2)(p_{ou}-p_{in})^2 R_{ou}^6 R_{in}^6}{10 p_{in}^2 C_4 b_1^2},$$

$$B_4 = \frac{(3b_1 + b_2)(p_{ou}-p_{in})^2 R_{ou}^6 R_{in}^6}{4 p_{in}^2 C_4 b_1^2}, \qquad B_5 = \frac{(p_{ou}-p_{in})^2 R_{ou}^3 R_{in}^3 C_7}{20 p_{in}^2 C_5 b_1^2}.$$

Expressions for constants in (4.3):

$$C_1 = R_{ou}^5 + R_{ou}^4 R_{in} + R_{ou}^3 R_{in}^2 + R_{ou}^2 R_{in}^3 + R_{ou} R_{in}^4, \qquad C_2 = 5R_{in}^5 + C_1 (5 + 6nR_{in}),$$

$$C_3 = 5R_{in}^5 + C_1 (5 + 6mR_{in}), \qquad C_4 = \left[R_{in}^3 (2 + 3nR_{ou}) - R_{ou}^3 (2 + 3nR_{in})\right]^2,$$

$$C_5 = (R_{ou} - R_{in})^2 \left[2R_{in}^2 + R_{ou} (R_{in} + R_{ou}) (2 + 3nR_{in})\right]^3,$$

$$\begin{aligned} C_6 = &\left\{55R_{ou}^5 + 11R_{ou}^3 R_{in} (5 + 6nR_{ou}) (R_{ou} + R_{in}) + 3R_{ou}^2 R_{in}^3 (5 + 2nR_{ou})\right. \\ &\left. - 3 (R_{ou} + R_{in}) R_{in}^4 [-5 + 6nR_{ou} (3 + 5nR_{ou})]\right\} p_{in} \\ &+ \left\{-55R_{in}^5 - 3R_{ou}^3 R_{in}^2 (5 + 2nR_{in}) - 11R_{ou} R_{in}^3 (5 + 6nR_{in}) (R_{ou} + R_{in})\right. \\ &\left. + 3 (R_{ou} + R_{in}) R_{ou}^4 [-5 + 6nR_{in} (3 + 5nR_{in})]\right\} p_{ou}, \end{aligned}$$

$$\begin{aligned} C_7 = &\,5 (11b_1 + b_2) (2 + 3nR_{ou}) R_{ou}^2 + [b_2 (5 + 6mR_{ou}) + 11b_1 (5 + 6nR_{ou})] \\ &\times \left[(2 + 3nR_{ou}) (R_{ou} + R_{in}) + 3nR_{in}^2\right] R_{in}. \end{aligned}$$

Expressions for constants in (4.10):

$$H_1 = -\frac{(n-3)(p_{ou}-p_{in}) R_{ou}^3 R_{in}^3 (-p_{in} R_{ou}^n R_{in}^3 + p_{ou} R_{ou}^3 R_{in}^n)}{6b_1 p_{in}^2 (R_{ou}^n R_{in}^3 - R_{ou}^3 R_{in}^n)^2},$$

$$H_2 = -\frac{(n-3)^2 b_2 (p_{ou}-p_{in})^2 R_{ou}^6 R_{in}^6}{24b_1^2 p_{in}^2 (m-6)(R_{ou}^n R_{in}^3 - R_{ou}^3 R_{in}^n)^2},$$

$$H_3 = -\frac{(n-3)(7n-9)(p_{ou}-p_{in})^2 R_{ou}^6 R_{in}^6}{24(n-6) b_1 p_{in}^2 (R_{ou}^n R_{in}^3 - R_{ou}^3 R_{in}^n)^2},$$

$$\begin{aligned} H_4 = &\,\frac{(n-3)^2 (p_{ou}-p_{in})^2 R_{ou}^3 R_{in}^3}{24b_1^2 p_{in}^2 (m-6)(n-6)(-R_{ou}^n R_{in}^3 + R_{ou}^3 R_{in}^n)^3} \\ &\times [(n-6) b_2 (-R_{ou}^m R_{in}^6 + R_{ou}^6 R_{in}^m) + 11(m-6) b_1 (-R_{ou}^n R_{in}^6 + R_{ou}^6 R_{in}^n)] \end{aligned}$$

$$\begin{aligned} H_5 = &-\frac{(n-3)^2 (p_{ou}-p_{in})^2 R_{ou}^3 R_{in}^3}{24b_1^2 p_{in}^2 (m-6)(n-6)(-R_{ou}^n R_{in}^3 + R_{ou}^3 R_{in}^n)^3} \\ &\times [11(m-6) b_1 R_{ou}^n R_{in}^n (R_{ou}^3 - R_{in}^3) + (n-6) b_2 (R_{ou}^{3+n} R_{in}^m - R_{ou}^m R_{in}^{3+n})]. \end{aligned}$$

Expressions for constants in (4.15) and (4.16):

$$\bar{B}_1 = \frac{(p_{ou} - p_{in})}{12 b_1 p_{in} \ln(R_{in}/R_{ou})}, \qquad \bar{B}_3 = \frac{(p_{ou} - p_{in})^2}{864 b_1^3 p_{in}^2}(3b_2 \ln(R_{ou}/R_{in}) + 11 b_1 (R_{in}^{-3} - R_{ou}^{-3})),$$

$$\bar{B}_2 = \frac{(p_{ou} - p_{in}) + 3(p_{in} \ln R_{ou} - p_{ou} \ln R_{in})}{3 p_{in} \ln(R_{ou}/R_{in})}, \qquad \bar{B}_4 = 72 R^3 R_{ou}^3 R_{in}^3 b_1 p_{in}^2 (\ln(R_{ou}/R_{in}))^3.$$

Expressions for constants in (4.17):

$$D_1 = -\frac{(11b_1 + b_2) R_{ou} R_{in} (R_{ou}^4 + R_{ou}^3 R_{in} + R_{ou}^2 R_{in}^2 + R_{ou} R_{in}^3 + R_{in}^4)}{30 b_1^2 (R_{ou} - R_{in})^2 (R_{ou} + R_{in})^3},$$

$$D_2 = \frac{R_{ou}^2 R_{in}^4}{3 b_1 (R_{ou}^2 - R_{in}^2)^2}, \qquad D_3 = \frac{(b_1 + b_2) R_{ou}^4 R_{in}^4}{30 b_1^2 (R_{ou}^2 - R_{in}^2)^2},$$

$$D_4 = -\frac{(11b_1 + b_2) R_{ou} R_{in} (R_{ou}^2 + R_{ou} R_{in} + R_{in}^2)}{30 b_1^2 (R_{ou} - R_{in})^2 (R_{ou} + R_{in})^3}.$$

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