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FINITE ELEMENTS IN ANALYSIS AND DESIGN

Application of Zienkiewicz–Zhu's error estimate with superconvergent patch recovery to hierarchical *p*-refinement

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Abstract

The Zienkiewicz–Zhu error estimate is slightly modified for the hierarchical *p*-refinement, and is then applied to three plane elastostatic problems to demonstrate its effectiveness. In each case, the error decreases rapidly with an increase in the number of degrees of freedom. Thus Zienkiewicz–Zhu's error estimate can be used in the *hp*-refinement of finite element meshes. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Adaptive p-refinement; Error estimate; Superconvergent patch recovery

1. Introduction

One way to control the quality of a finite element solution with an optimal use of computational resources is to refine the mesh adaptively. The adaptive finite element analysis generally consists of two stages: a posteriori error estimate and the mesh refinement. The goal is to refine the mesh so that the error is within the specified tolerance and is as uniformly distributed throughout the domain as possible. Two types of a posteriori error estimates, namely the post-processing and the residual, have been employed. The post-processing type error estimate was proposed for the *h*-refinement by Zienkiewicz and Zhu [1]. They used the nodal averaging method to obtain recovered stresses and compared stresses interpolated from the recovered stresses with those computed from the finite element solution at the quadrature points to find the error in the numerical solution. Henceforth, we will refer to this error estimate as the Zienkiewicz–Zhu's (Z^2) error estimate. Many authors [2–5] have shown its effectiveness in the *h*- and *r*-refinements. The nodal averaging procedure or the L_2 projection technique for obtaining recovered stresses is valid

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only for linear elements. The residual-type error estimate has been proposed to evaluate errors for higher-order hierarchical elements [13–15]. The residual error for an hierarchical element is defined as the difference between the displacement fields over the original and a refined mesh, and is computationally more expensive than the Z^2 error estimate. We note that the Z^2 error estimate has not been applied to the *p*-refinement [6–15] because shape functions used to interpolate displacements within an element are also used to interpolate recovered stresses. The hierarchical shape functions cannot be used for interpolation since they do not satisfy the basic features of interpolation functions, e.g., they do not generally add upto 1. However, the hp-refinement [23–25], based on different error estimates for each *h*- and *p*-refinements, gives very accurate solutions. Because of the different error estimates used, the algorithm for the *hp*-refinement is complicated.

Here we use the superconvergent patch recovery method of Zienkiewicz and Zhu [19–22] to compute errors for hierarchical elements and adopt these for uniform *p*-refinement. This method computes stresses at optimal sampling points where the accuracy of stresses is an order of magnitude higher, and then recovers stresses at a node by using the least-squares method to fit a polynomial to the stresses computed at the optimal sampling points. Zienkiewicz and Zhu showed, through numerical experiments, the validity of this technique for higher-order isoparametric elements. We tacitly assume that stresses for the hierarchical element computed at these optimal sampling points are also accurate, and demonstrate the validity of our approach by analysing results for three plane elastostatic problems employing quadrilateral elements. We are limited to uniform *p*-refinement because of our inability to determine optimal sampling points in a transition element between two different order hierarchical elements.

2. $\mathbf{Z}^2$ error estimate for *p*-refinement

Shape functions for a hierarchical finite element [6–18] are different from those for conventional finite elements [7–10]. Among the many choices available for forming higher-order shape functions [10], we adopt Legendre polynomials because of their orthogonal property and yielding numerically stable stiffness matrices [9,10,16–18]. Fig. 1 shows hierarchical quadrilateral elements of order 1 – 4, and their shape functions are listed in Table 1. As indicated in Table 1, there should be a hierarchical node at the center of the element of order 4 in order to have complete polynomials in the shape functions. In terms of the hierarchical shape functions $\boldsymbol{N}$, we express displacement fields u and v as

$$u = \sum_{i=1}^{n} N_i \bar{u}_i = \bar{\boldsymbol{N}}\bar{\boldsymbol{u}}, \qquad v = \sum_{i=1}^{n} N_i \bar{v}_i = \bar{\boldsymbol{N}}\bar{\boldsymbol{v}}, \tag{1}$$

where $\bar{u}_\mathrm{i}$ and $\bar{v}_\mathrm{i}$ are unknown variables in the global coordinates; $n = 4p$ when $p \leqslant 3$, and $n = 4p + 1$ when $p = 4$. Only for a linear element, i.e., $p = 1$, $\bar{u}_\mathrm{i}$ and $\bar{v}_\mathrm{i}$ are nodal displacements. With displacements given by (1), the element stiffness matrix can be expressed as

$$\boldsymbol{K}^\mathrm{e} = \int_{\Omega_e} \boldsymbol{B}^\mathrm{T} \boldsymbol{D} \boldsymbol{B} \, \mathrm{d}\Omega, \tag{2}$$

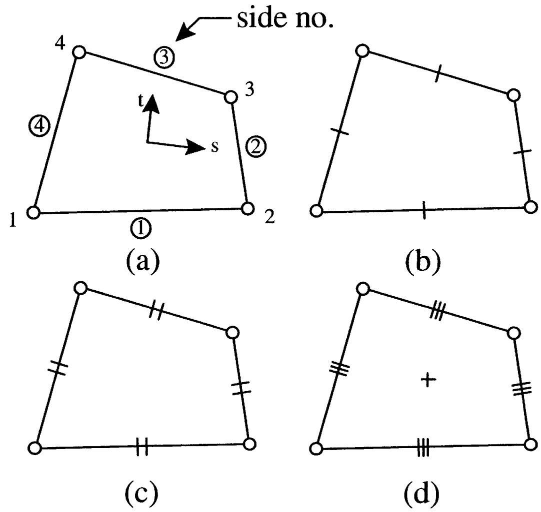

Fig. 1. Hierarchical finite elements: (a) linear; (b) quadratic; (c) cubic; (d) quartic.

Table 1
Shape functions for heirarchical elements of order p with $1 \leqslant p \leqslant 4$

Order (p)	Side	Shape function	Order (p)	Side	Shape function
Linear		$N_1 = (1-s)(1-t)/4$	Cubic	1	$N_9 = f_3(s)(1-t)/2$
		$N_2 = (1+s)(1-t)/4$		2	$N_{10} = f_3(t)(1+s)/2$
		$N_3 = (1+s)(1+t)/4$		3	$N_{11} = f_3(s)(1+t)/2$
		$N_4 = (1-s)(1+t)/4$		4	$N_{12} = f_3(t)(1-s)/2$
Quadratic	1	$N_5 = f_2(s)(1-t)/2$	Quartic	1	$N_{13} = f_4(s)(1-t)/2$
	2	$N_6 = f_2(t)(1+s)/2$		2	$N_{14} = f_4(t)(1+s)/2$
	3	$N_7 = f_2(s)(1+t)/2$		3	$N_{15} = f_4(s)(1+t)/2$
	4	$N_8 = f_2(t)(1-s)/2$		4	$N_{16} = f_4(t)(1-s)/2$
$f_2(u) = (u^2-1)/2$,				Center	$N_{17} = f_2(s)f_2(t)$
$f_3(u) = (u^3-u)/2$					
$f_4(u) = (5u^4-6u^2+1)/8$					

where $\boldsymbol{B}$ is the strain–displacement matrix, $\boldsymbol{D}$ is the elasticity matrix, and the integration is over an element Ω_e. We used $(p+1)\times(p+1)$ quadrature rule to numerically evaluate $\boldsymbol{K}^e$.

For conventional finite elements, Zienkiewicz and Zhu [21] computed stresses at the optimal sampling points, determined their nodal values by fitting curves to the sampling-point values in an element patch by the least-squares method, and then averaged contributions from all element patches meeting at a node. This technique, called superconvergent patch recovery, is detailed in Ref. [19–22]. Even though optimal sampling points have not been established for the hierarchical finite elements, we assume that they coincide with those for the corresponding conventional finite elements and depict them in Fig. 2. We use the superconvergent patch recovery technique to compute recovered stresses.

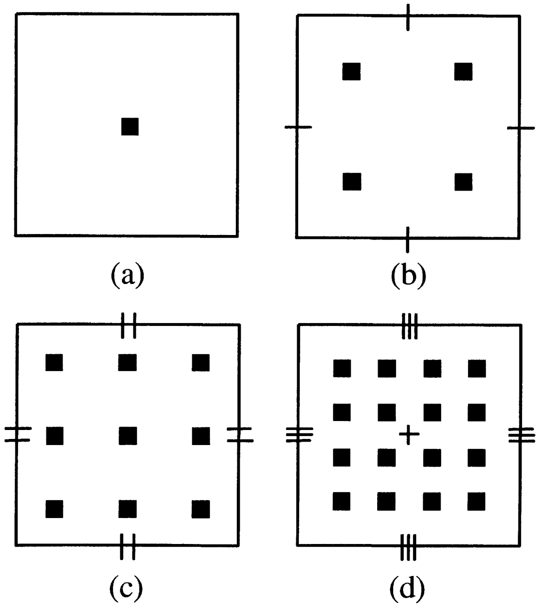

Fig. 2. Optimal sampling points: (a) linear; (b) quadratic; (c) cubic; (d) quartic elements.

Following Zienkiewicz and Zhu [1] we define the error $\boldsymbol{e}$ by

$$e = \boldsymbol{\sigma}^* - \hat{\boldsymbol{\sigma}} = \boldsymbol{N}^p \bar{\boldsymbol{\sigma}}^* - \boldsymbol{D}\boldsymbol{B}\bar{u}, \tag{3}$$

where $\bar{\sigma}^*$ is the recovered nodal stress vector, p is the order of shape functions in a hierarchical element, and $\boldsymbol{N}^p$ is a matrix containing pth order Lagrange interpolation functions [10]. In order to interpolate the recovered stresses for an element with the pth order interpolation functions, we need $(p + 1)^2$ interpolation points. Fig. 3 shows an element patch and interpolation points for various hierarchical elements. If values at an interpolation point within an element are available from several element patches, then we average these values.

3. Numerical examples

The use of the Z^2 error estimate (3) for uniform p-refinement is illustrated by analysing the following three plane linear elastostatic problems: an L-shaped plate, a plate with a circular hole and a cantilever plate. Fig. 4 depicts the initial mesh, boundary conditions and the relative error energy norm, $\|\boldsymbol{e}\|/\|\boldsymbol{u}\|$ vs. the number of degrees of freedom for the L-shaped plate. For comparison, Gago et al.'s [15] results obtained by using hierarchical elements and residual-type error estimate, and the relative error determined by Gago et al. [15] from Richardson's extrapolation are also shown. We note that the present results agree well with those of the other two studies. In each case, the relative energy norm decreases rapidly with an increase in the number of degrees

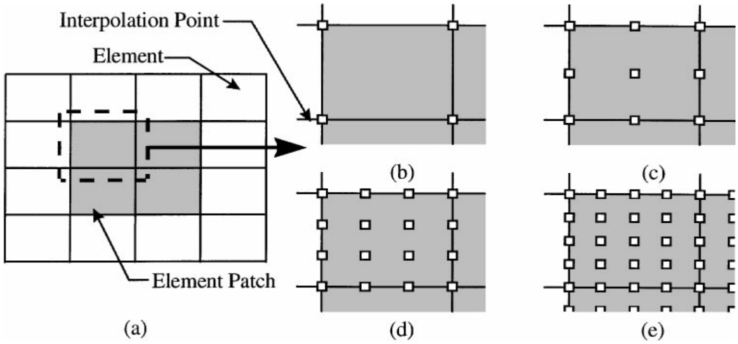

Fig. 3. Element patch and interpolation points for hierarchical elements: (a) element patch; (b)–(e) interpolation points for linear, quadratic, cubic and quartic elements.

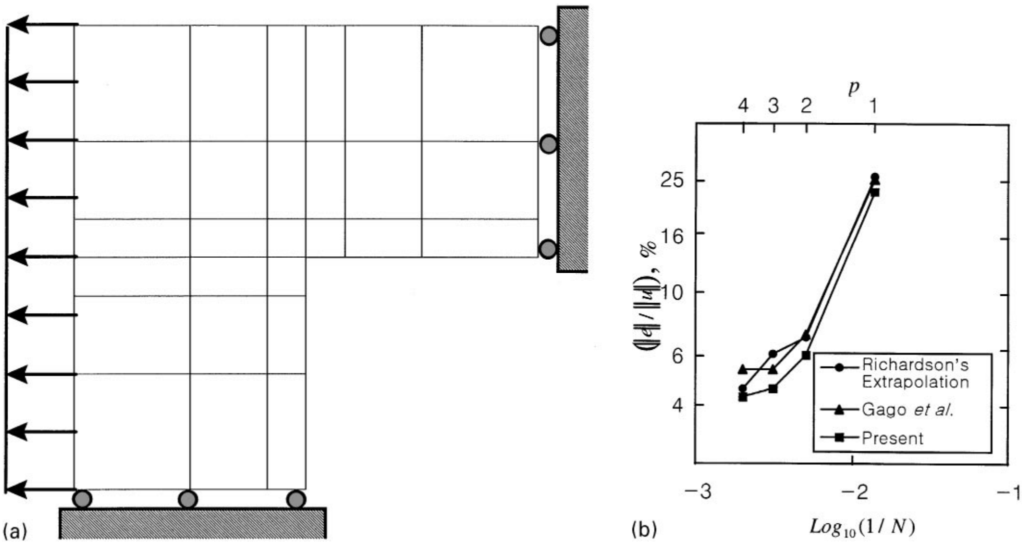

Fig. 4. An L-shaped plate: (a) initial mesh and boundary conditions; (b) convergence rates of relative percentage error for uniform p-refinement.

of freedom till around 5% error in the relative error energy norm, and then decreases extremely slowly.

Fig. 5 shows results for the plate with a circular hole. The stresses at point A, computed with the superconvergent patch recovery method, converge rapidly to the analytical value with an increase in the order of the polynomial. It suggests that the optimal sampling points[1] for conventional finite elements can be used for hierarchical finite elements also. Results for a cantilever beam exhibited in Fig. 6 are self-explanatory.

[1] Oh and Batra [26] have shown that, for higher-order conventional finite elements, the coordinates of sampling points slightly differ from those of the reduced integration points.

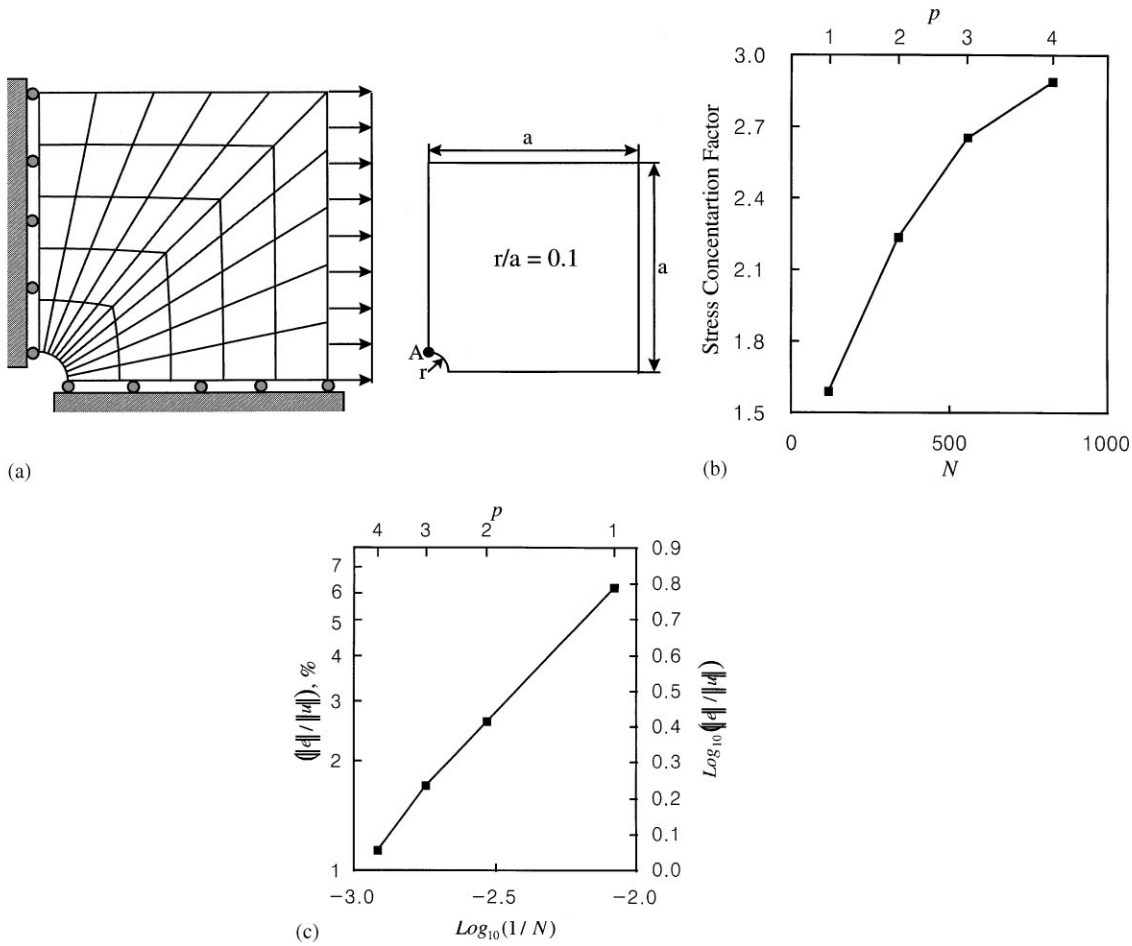

Fig. 5. Plate with a circular hole: (a) finite element mesh and boundary conditions; (b) convergence of stress concentration factor according to uniform p-refinement; (c) convergence rates of relative percentage error for uniform p-refinement.

4. Conclusions

It has been shown that the Z^2 error estimate, previously used for h-refinement, can also be used for the hierarchical p-refinement. For three typical plane elastostatic problems, stresses recovered with the superconvergent patch recovery method converge rapidly to the analytical value with an increase in the order of the polynomial and thus in the number of degrees of freedom. Hence the existing h-refinement codes can be easily modified to the hp-refinement by using the same a posteriori error estimate.

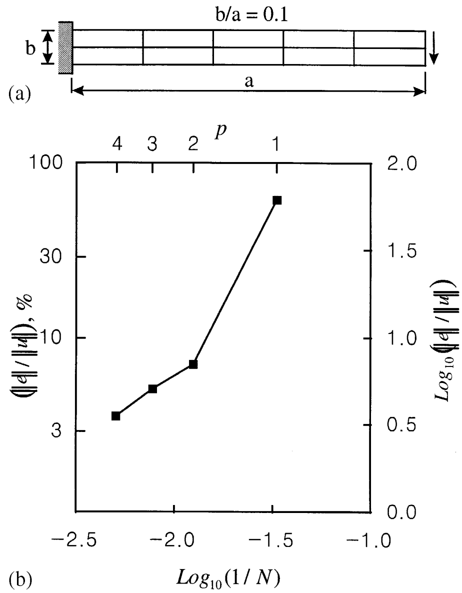

Fig. 6. Cantilever beam: (a) finite element mesh and boundary conditions; (b) convergence rates of relative percentage error for uniform p-refinement.

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