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PLANE STRAIN DEFORMATIONS NEAR THE TIP OF A CONTACTING INTERFACIAL CRACK IN LOCKING MATERIALS

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Abstract—Plane strain deformations near the tip of a crack between two homogeneous and isotropic linear elastic bodies are studied on the assumptions that the two surfaces on either side of the crack contact each other and that the dilatation everywhere in the body is greater than or equal to a constant. The region where the dilatation equals the constant is called the locking region. It is found that one of the two half planes is in the locking state and the singularity index equals 1/2.

1. INTRODUCTION

SINCE the time Williams [1] obtained the characteristic oscillating stress singularity near the interface crack tip that implied interpenetration of the material, there have been several proposals (see e.g. refs [2–8]) to explain this phenomenon. One of these [8] involves the assumption that the crack surfaces contact each other, and there may be frictional forces acting between them. Here we assume that the contact surfaces are smooth but require that the dilatation at every point in the body exceeds or equals a constant greater than -1. We use Prager's [9] terminology and call the region wherein the dilatation equals the constant the locking region, and the remaining region the elastic region.

2. BASIC EQUATIONS

We concentrate on finding the deformation and stress fields in the vicinity of the tip of a crack between two linear elastic, isotropic and homogeneous bodies undergoing plane strain deformations in the x_1–x_2 plane. We assume that deformations satisfy the constraint

$$\frac{\mathrm{d}v}{\mathrm{d}V} = 1 + \epsilon_{\alpha\alpha} \geqslant 1 + \delta > 0, \tag{1}$$

where $\mathrm{d}v$ and $\mathrm{d}V$ equal, respectively, the volume of the same material element in the present and stress free reference configurations, $\epsilon_{\alpha\beta}$ is the infinitesimal strain tensor, a repeated index implies summation over the range 1, 2 of the index, and δ is a material constant. The region where $\epsilon_{\alpha\alpha} = \delta$ is called the "locking region"; in the remaining region, deformations are unconstrained.

In the locking region, stresses $\sigma_{\alpha\beta}$ are related to the strains $\epsilon_{\alpha\beta}$ by

$$\epsilon_{\alpha\alpha} = \delta \tag{2.1}$$

$$\sigma_{\alpha\beta} = -p\delta_{\alpha\beta} + 2\mu\epsilon_{\alpha\beta} + \lambda\epsilon_{\gamma\gamma}\delta_{\alpha\beta}, \tag{2.2}$$

and in the unconstrained region by

$$\sigma_{\alpha\beta} = \lambda\epsilon_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu\epsilon_{\alpha\beta}, \tag{3}$$

where λ and μ are Lamé's constants, $p(x_1, x_2) > 0$ is the hydrostatic pressure not determined by the deformation field, and $\delta_{\alpha\beta}$ is the Kronecker delta. Of course, the last term on the right-hand side of eq. (2b) could be absorbed in p. In terms of the analytic functions $\phi_e(z)$ and $\psi_e(z)$ of the complex variable $z = x_1 + ix_2$, we have in the unconstrained or the elastic region [10]

$$\sigma_{11} + \sigma_{22} = 2(\phi_e' + \overline{\phi}_e'), \tag{4.1}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2(\bar{z}\phi_e'' + \psi_e') \tag{4.2}$$

$$2\mu(u_1 + iu_2) = \kappa\phi_e - z\overline{\phi_e'} - \overline{\psi}_e, \tag{4.3}$$

where a superimposed bar indicates the complex conjugate of the variable, $\phi' = \partial\phi/\partial z$, and $\kappa = 3 - 4\nu$, ν being Poisson's ratio for the material of the body. The stress field in the elastic region must satisfy

$$\epsilon_{\alpha\alpha} = \sigma_{\alpha\alpha}/(2(\lambda + \mu)) > \delta. \tag{5}$$

Similarly, in the locking region, we have

$$\sigma_{11} + \sigma_{22} = 2(\phi_0' + \overline{\phi_0'}) \tag{6.1}$$

$$\sigma_{11} - \sigma_{22} + 2i\sigma_{12} = 2(\bar{z}\phi_0'' + \psi_0') \tag{6.2}$$

$$2\mu(u_1 + iu_2) = \phi_0 - z\overline{\phi_0'} - \overline{\psi}_0 + \mu\delta z \tag{6.3}$$

$$2(\lambda + \mu)\delta > \sigma_{\alpha\alpha}, \tag{6.4}$$

where ϕ_0 and ψ_0 are analytical functions of z, and inequality (6.4) follows from $p > 0$.

In order for the surface tractions and displacements to be continuous across the interface between the locking and the elastic regions, we must have

$$[\![\phi + z\overline{\phi}' + \overline{\psi}]\!] = 0 \tag{7.1}$$

$$\kappa\phi_e - \phi_0 - [\![z\overline{\phi}' + \overline{\psi}]\!] = \mu\delta z, \tag{7.2}$$

where $[\![f]\!]$ equals the difference between the values of f on the two sides of the interface betwen the elastic and locking regions. Equations (7.1) and (7.2) imply the following Hilbert condition [10]:

$$(\kappa + 1)\phi_e - 2\phi_0 = \mu\delta z, \tag{8}$$

which we assume replaces eq. (7.2).

3. CRACK-TIP ANALYSIS

We assume that the crack surface, as shown in Fig. 1, is defined by $x_2 = 0$, $x_1 < 0$ with its tip located at the origin of the rectangular Cartesian coordinate axes. In polar coordinates $z = r\,e^{i\theta}$, the cracked body occupies the domain $0 \leqslant r < \infty$, $-\pi \leqslant \theta \leqslant \pi$. We assume that the material of the body in the upper half plane is characterized by elasticities (μ_1, κ_1) and that in the lower half

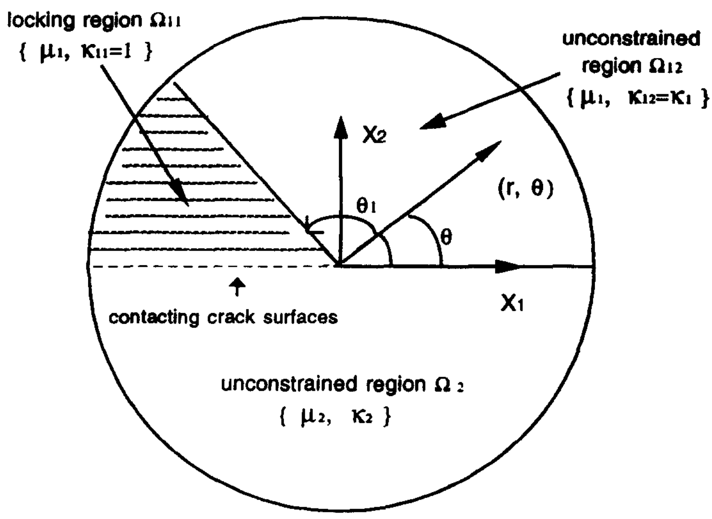

Fig. 1. A schematic sketch of the problem studied.

plane by (μ_2, κ_2). Note that $\kappa = 1$ in the locking region. Since the classical solution of the contact crack problem gives [8]

$$\epsilon_{\alpha\alpha} \sim r^{-1/2} \sin(\theta/2), \quad -\pi \leqslant \theta \leqslant \pi, \tag{9}$$

which varies monotonically from one crack surface to another, accordingly, we assume that the solution is of a single locking domain, say the upper one, adjoining one of the crack surfaces. Thus the upper half plane is divided into two parts

$$\Omega_{11}: \theta_1 \leqslant \theta \leqslant \pi \text{ with elasticities } (\mu_1, \kappa_{11}) \tag{10.1}$$

$$\Omega_{12}: 0 \leqslant \theta \leqslant \theta_1 \text{ with elasticities } (\mu_2, \kappa_{12}) \tag{10.2}$$

and the value of θ_1 is determined by conditions (5) and (6.4).

We use William's [1] eigenexpansion method combined with Muskhelishvili's complex variable method [10] to obtain an r–θ separable asymptotic series solution near the interface crack tip [3, 4]. Without any loss of generality (see e.g. ref. [4]), we seek solutions $\phi(z)$ and $\psi(z)$ of the form

$$z^{\bar{\rho}} \quad \text{and} \quad z^{\rho}, \quad 0 < \mathrm{Re}(\rho) < 1, \tag{11}$$

in the locking and elastic regions, respectively, where ρ is to be determined from the solution of an eigenvalue problem. For complex ρ, conditions (5) and (6.4) cannot be satisfied in the limit $r \to 0$ because of the oscillatory nature of the singularity. Accordingly, we assume that ρ is a real number, and take

$$\phi = Az^{\rho}, \quad \psi = Bz^{\rho} \quad \text{on} \quad \Omega_{11} \tag{12.1}$$

$$\phi = Ez^{\rho}, \quad \psi = Fz^{\rho} \quad \text{on} \quad \Omega_{12} \tag{12.2}$$

$$\phi = Mz^{\rho}, \quad \psi = Nz^{\rho} \quad \text{on} \quad \Omega_2, \tag{12.3}$$

where z^{ρ} is the principal value $(-\pi \leqslant \theta \leqslant \pi)$ and A, B, E, F, M, and N are six undetermined complex constants.

The requirements that normal surface tractions and surface displacements at the contacting crack surfaces $\theta = \pm\pi$ are equal, the tangential tractions on $\theta = \pm\pi$ are null, and the continuity conditions across the bounding surface $\theta = \theta_1$ and the interface $\theta = 0$ give the following homogeneous equations for the determination of A, B, E, F, M, and N.

$$\bar{A} + A\rho\, \mathrm{e}^{2i\rho\pi} + B\, \mathrm{e}^{2i\rho\pi} - N - M\rho - \bar{M}\, \mathrm{e}^{2i\rho\pi} = 0 \tag{13.1}$$

$$(1-\rho)[A\, \mathrm{e}^{2i\rho\pi} - \bar{A}] - (B\, \mathrm{e}^{2i\rho\pi} - \bar{B}) = 0 \tag{13.2}$$

$$\mu_1[\kappa_2(M - \bar{M}\, \mathrm{e}^{2i\rho\pi}) - \rho\bar{M}\, \mathrm{e}^{2i\rho\pi} + M\rho - \bar{N}\, \mathrm{e}^{2i\rho\pi} + N] \\ - \mu_2[\kappa_{11}(A\, \mathrm{e}^{2i\rho\pi} - \bar{A}) - \rho\bar{A} + A\rho\, \mathrm{e}^{2i\rho\pi} - \bar{B} + B\, \mathrm{e}^{2i\rho\pi}] = 0 \tag{13.3}$$

$$E + \rho\bar{E} + \bar{F} - (M + \rho\bar{M} + \bar{N}) = 0 \tag{13.4}$$

$$\mu_1[\kappa_2 M - \rho\bar{M} - \bar{N}] - \mu_2[\kappa_{12}E - \rho\bar{E} - \bar{F}] = 0 \tag{13.5}$$

$$A(\kappa_{11} + 1) - E(\kappa_{12} + 1) = 0 \tag{13.6}$$

$$A\, \mathrm{e}^{2i\rho\theta_1} + \rho\bar{A}\, \mathrm{e}^{2i\theta_1} + \bar{B} - (E\, \mathrm{e}^{2i\rho\theta_1} + \rho\bar{E}\, \mathrm{e}^{2i\theta_1} + \bar{F}) = 0. \tag{13.7}$$

Note that conditions (13.2) and (13.3) are real and the remaining equations in the set of equations (13) are complex. Using eqs (13.1) and (13.2) eq. (13.3) can be rewritten as

$$\mu_1(1 + \kappa_2)(M - \bar{M}\, \mathrm{e}^{2i\rho\pi}) - \mu_2(1 + \kappa_{11})(A\, \mathrm{e}^{2i\rho\pi} - \bar{A}) = 0. \tag{13.8}$$

Equations (13.1), (13.2), and (13.4) through (13.8) constitute an eigenvalue problem for the determination of constants A, B, E, F, M, and N, and ρ is the eigenvalue.

We note that when $\theta_1 = 0$, four equations (13.4) through (13.7) reduce to the following two equations:

$$A + \rho\bar{A} + \bar{B} - (M + \rho\bar{M} + \bar{N}) = 0 \tag{14.1}$$

$$\mu_1(\kappa_2 M - \rho\bar{M} - \bar{N}) - \mu_2(\kappa_{11}A - \rho\bar{A} - \bar{B}) = 0, \tag{14.2}$$

and constants E and F are not involved.

We add that the normal stress at the contact surfaces $\theta = \pm\pi$ must be compressive. The value of θ_1 is determined by

$$\phi_e' + \bar{\phi}_e' > 0 \text{ in the elastic domain} \tag{15.1}$$

$$\phi_0' + \bar{\phi}_0' < 0 \text{ in the locking domain.} \tag{15.2}$$

For the classical solution $\theta_1 = \pi$, $\rho = 1/2$ was obtained [8] for any set of material parameters (μ_1, κ_1) and (μ_2, κ_2).

4. THE SIMPLE CASE WITH $\mu_1 = \mu_2$

Recalling that in the locking region $\kappa_{11} = 1$, we set $\kappa_{12} = \kappa_1$. Equations (13.4), (13.5), and (13.6) yield

$$2A = M(\kappa_2 + 1) = E(1 + \kappa_1). \tag{16}$$

In view of

$$\sigma_{\alpha\alpha} \sim \mathrm{Re}(\phi') \sim \mathrm{Re}[A\,e^{i(\rho-1)\theta}] \quad \text{for } -\pi \leqslant \theta \leqslant \pi, \tag{17.1}$$

$$A = R + iI, \quad tg\omega = I/R \tag{17.2}$$

conditions (15.1) and (15.2) reduce to

$$\cos[(\rho - 1)\theta + \omega] > 0 \quad \text{for } -\pi \leqslant \theta < \theta_1 \tag{18.1}$$

$$\cos[(\rho - 1)\theta + \omega] < 0 \quad \text{for } \theta_1 < \theta \leqslant \pi, \tag{18.2}$$

which imply that

$$|(\rho - 1)(\theta - \theta_1)| \leqslant \pi \quad \text{for } -\pi \leqslant \theta \leqslant \pi. \tag{18.3}$$

From eqs (17), (18.1) and (18.2) we obtain

$$\bar{A} = -A\,e^{2i(\rho-1)\theta_1}. \tag{19}$$

Equations (13.8) and (16) give

$$(A + \bar{A})(1 - e^{2i\rho\pi}) = 0, \tag{20}$$

which is satisfied only if

$$A = -\bar{A}, \tag{21}$$

or A is pure imaginary. From eqs (19) and (21) we conclude that

$$\sin(\rho - 1)\theta_1 = 0. \tag{22}$$

From eqs (13.1), (13.4), and (13.7) we obtain

$$B = 2\bar{A}/(1 + \kappa_2) - \rho A, \tag{23}$$

which when substituted into eq. (13.2) yields the eigenvalue problem

$$A(2/(1 + \kappa_2) + e^{2i\rho\pi}) - \bar{A}(1 + 2\,e^{2i\rho\pi}/(1 + \kappa_2)) = 0 \tag{24}$$

for the determination of ρ. Recalling eq. (21) we conclude from eq. (24) that

$$\rho = \tfrac{1}{2} \tag{25}$$

and then from eq. (22) that

$$\theta_1 = 0. \tag{26}$$

Thus the entire half of the body above the crack surface must be in the locking state.

5. THE GENERAL CASE WITH $\mu_1 \neq \mu_2$

From eqs (13.1), (13.4)–(13.7), (13.2) and (13.3) we find that

$$A(\kappa_1 - 1)(e^{2i\rho\theta_1} - 1)/(\kappa_1 + 1) + (1 - e^{-2i\rho\pi})(\bar{B} - M)$$
$$+ \rho\bar{A}[(\kappa_1 - 1)e^{2i\theta_1}/(\kappa_1 + 1) + 2/(\kappa_2 + 1) - e^{-2i\rho\pi}] = 0 \quad (27)$$

$$(A\, e^{2i\rho\pi} - \bar{A})[1 - \rho + 2\mu_2/[\mu_1(\kappa_2 + 1)]] = (B - \bar{M})e^{2i\rho\pi} - (\bar{B} - M). \quad (28)$$

Henceforth we restrict ourselves to the case when $\rho = 1/2$. Then eqs (27) and (28) yield

$$1 - e^{3i\theta_1} + \alpha\, e^{i\theta_1} - \alpha\, e^{2i\theta_1} = 0, \quad (29)$$

where

$$\alpha = \{8\mu_2 + \mu_1(7 - \kappa_1 + 2\kappa_2)\}/(\mu_1(\kappa_1 - 1)) > 0. \quad (30)$$

Thus,

$$e^{i\theta_1} = 1 \quad \text{and} \quad \theta_1 = 0,$$

and the entire region above the crack surface is in the locking state.

6. DISPLACEMENTS AND STRESSES NEAR THE CRACK TIP

Because of eq. (21), we set

$$A = iI, \quad (31)$$

where I is a real number. Also with $\rho = 1/2$, $\theta_1 = 0$, we obtain from eqs (13.1), (13.2), and (13.4) through (13.8) the following.

$$B = -i(\beta + 1/2)I, \quad M = i\beta I, \quad N = -i(1 + \beta/2)I, \quad (32)$$

where

$$\beta = [1 + (\mu_2/\mu_1)]/[\kappa_2 + \mu_2/\mu_1], \quad 0 \leqslant \beta \leqslant 1 \quad (33)$$

and

$$\sigma_{\alpha\alpha} \sim \mathrm{Re}[A\, e^{-i\theta/2}] \sim -\sin(\theta/2) \quad \text{for } -\pi \leqslant \theta \leqslant \pi. \quad (34)$$

One can verify that conditions (15.1) and (15.2) are satisfied. Recall that for $\theta_1 = 0$ the solution of the problem is independent of constants E and F, and the remaining constants are expressible in terms of I through eqs (31) and (32). Using eqs (4), (6), (12), (31), and (32) we obtain the following expressions for the displacements and stresses.

(a) In the upper-half plane:

$$\sigma_{\theta\theta} = \left(\frac{I}{2\sqrt{r}}\right)[\tfrac{3}{2}\sin(\theta/2) + (\beta + \tfrac{1}{2})\sin(3\theta/2)] \quad (35.1)$$

$$\sigma_{rr} = \left(\frac{I}{2\sqrt{r}}\right)[\tfrac{5}{2}\sin(\theta/2) - (\beta + \tfrac{1}{2})\sin(3\theta/2)] \quad (35.2)$$

$$\sigma_{r\theta} = \left(\frac{I}{2\sqrt{r}}\right)[-\tfrac{1}{2}\cos(\theta/2) - (\beta + \tfrac{1}{2})\cos(3\theta/2)] \quad (35.3)$$

$$2\mu_1 u_r = \left(\frac{I}{\sqrt{r}}\right)[\tfrac{1}{2}\sin(\theta/2) - (\beta + \tfrac{1}{2})\sin(3\theta/2)] \quad (35.4)$$

$$2\mu_1 u_\theta = \left(\frac{I}{\sqrt{r}}\right)[\tfrac{3}{2}\cos(\theta/2) - (\beta + \tfrac{1}{2})\cos(3\theta/2)]. \quad (35.5)$$

(b) In the lower-half plane:

$$\sigma_{\theta\theta}=\left(\frac{I}{2\sqrt{r}}\right)\left[\tfrac{3}{2}\beta\sin(\theta/2)+\left(1+\frac{\beta}{2}\right)\sin(3\theta/2)\right] \tag{36.1}$$

$$\sigma_{rr}=\left(\frac{I}{2\sqrt{r}}\right)\left[\tfrac{5}{2}\beta\sin(\theta/2)-\left(1+\frac{\beta}{2}\right)\sin(3\theta/2)\right] \tag{36.2}$$

$$\sigma_{r\theta}=\left(\frac{I}{2\sqrt{r}}\right)\left[-\tfrac{1}{2}\beta\cos(\theta/2)-\left(1+\frac{\beta}{2}\right)\cos(3\theta/2)\right] \tag{36.3}$$

$$2\mu_2u_r=\left(\frac{I}{\sqrt{r}}\right)\left[(\kappa_2-\tfrac{1}{2})\beta\sin(\theta/2)-\left(1+\frac{\beta}{2}\right)\sin(3\theta/2)\right] \tag{36.4}$$

$$2\mu_2u_\theta=\left(\frac{I}{\sqrt{r}}\right)\left[(\kappa_2+\tfrac{1}{2})\beta\cos(\theta/2)-\left(1+\frac{\beta}{2}\right)\cos(3\theta/2)\right]. \tag{36.5}$$

Thus, on the crack surface

$$\sigma_{\theta\theta}(-\pi)=\sigma_{\theta\theta}(\pi)=\frac{I(1-\beta)}{2\sqrt{r}}, \tag{37}$$

and in order that the normal tractions on the crack surface be compressive, we must have

$$I\leqslant 0. \tag{38}$$

The pressure field in the locking region computed from eq. (2.2) is found to be

$$p\approx-\frac{\sigma_{\alpha\alpha}}{2}=-\frac{I\sin(\theta/2)}{\sqrt{r}},\quad 0\leqslant\theta\leqslant\pi. \tag{39}$$

7. CONCLUSIONS

We have studied plane strain deformations near the tip of a crack in the interface between two isotropic and homogeneous linear elastic bodies under the assumptions that the crack surfaces are in contact with each other and the dilatation everywhere must be greater than or equal to a constant. The region wherein the dilatation equals the constant is identified as the locking region, and the remaining region as the elastic region. After having formulated the problem for the general case of two different materials and on the assumption that the locking region occurs in the upper half of the body, it is shown that when their shear moduli are equal, the entire half of the upper domain must be in the locking state and the singularity index $\rho = 1/2$. For the general case of unequal shear moduli, the same result is derived with the assumption that $\rho = 1/2$. Explicit expressions for displacements and stress components near the crack tip have been obtained.

REFERENCES

[1] M. L. Williams, The stresses around a fault or crack in dissimilar media. *Bull. seism. Soc. Am.* **49,** 199–204 (1959).
[2] A. H. England, A crack between dissimilar media. *J. appl. Mech.* **32,** 400–402 (1965).
[3] J. R. Rice and G. C Sih, Plane problems of cracks in dissimilar media. *J. appl. Mech.* **32,** 418–423 (1965).
[4] J. R. Rice, Elastic fracture mechanics concepts for interface cracks. *J. appl. Mech.* **55,** 98–103 (1988).
[5] O. L. Bowie and C. E. Freese, On the "overlapping" problem in crack analysis. *Engng Fracture Mech.* **8,** 373–379 (1976).
[6] O. Aksogan, Partial closure of a Griffith crack under a general loading. *Int. J. Fracture* **11,** 659–669 (1975).
[7] P. S. Theocaris, D. Pazis and B. D. Konstantellos, The exact shape of a deformed internal slant crack under biaxial loading. *Int. J. Fracture* **30,** 135–153 (1986).
[8] M. Comninou, The interface crack. *J. appl. Mech.* **44,** 631–636 (1977).
[9] W. Prager, On ideal locking materials. *Trans. Soc. Rheol.* **1,** 169–175 (1957).
[10] N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity.* Noordhoff, Groningen (1953).

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