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PLANE STRAIN DEFORMATIONS OF LOCKING MATERIALS NEAR A CRACK TIP

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Abstract—Plane-strain deformations of an isotropic and homogeneous Hookean body containing a crack are studied and it is required that the dilatation everywhere in the body be greater than or equal to a constant. Following Prager, the region where the dilatation always equals the constant is identified as the locking region. For the case when the deformations of the body are symmetrical about the plane containing the crack, equations are derived that delimit the size of the locking region. It is shown that for a series type r,θ separable solution of the problem, the order of the singularity is essentially unchanged by the consideration of the higher-order terms in the constraint equation.

1. INTRODUCTION

In 1959 Williams [1] analyzed the deformation and stress fields near an interfacial crack tip in a linear-elastic body and showed that the oscillating stress singularity implied interpenetration of the material. Since then, several investigators (see e.g. Cherepanov [2], England [3], Erdogan [4], Rice and Shih [5], Park and Earmme [6], Shih and Asaro [7], Hutchinson *et al.* [8], and Rice [9]) have studied the problem and provided interpretations of the elastic solution. Here we impose the constraint that the dilatation at every point must equal, or exceed, a constant. Regions where the dilatation equals the constant are called locking (see e.g. Prager [10]).

2. FORMULATION OF THE PROBLEM

We study plane strain deformations of a homogeneous, isotropic and linear elastic body containing a crack along the plane $\theta = \pi$. Thus the origin of the cylindrical coordinate system is located at the crack tip (e.g. see Fig. 1). We presume that the deformations of the body satisfy the constraint

$$\frac{dv}{dV} > 0, \tag{1}$$

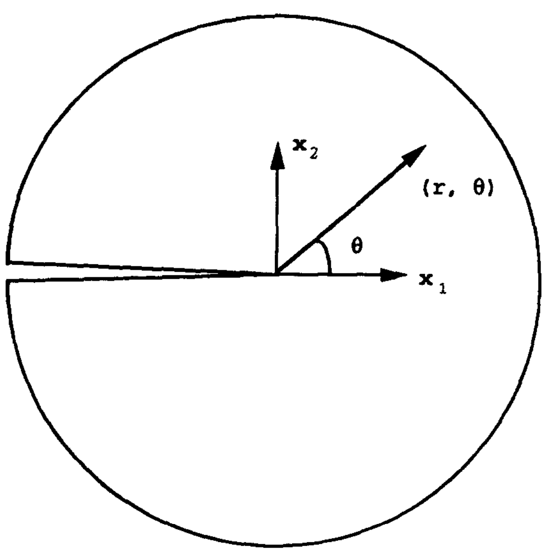

Fig. 1. A schematic sketch of the problem studied.

where dv and dV equal, respectively, the volume of the same material element in the deformed and undeformed stress-free reference configuration. For plane-strain deformations, as shown in the Appendix, the constraint (1) can be replaced by the more restrictive requirement

$$\epsilon_{rr} + \epsilon_{\theta\theta} + (\epsilon_{rr}\epsilon_{\theta\theta} - \epsilon_{r\theta}^2) \geqslant \delta > -1, \tag{2}$$

where ϵ_{rr}, $\epsilon_{\theta\theta}$ and $\epsilon_{r\theta}$ are components of the infinitesimal strain tensor, and δ is a material constant. The region where strict inequality holds in (2) is unconstrained, and that where equality holds is the locking region. In a previous paper we [11] imposed the condition

$$\epsilon_{rr} + \epsilon_{\theta\theta} \geqslant \delta \tag{3}$$

on the deformation field. However, near a crack tip the quadratic terms in (2) dominate. Thus, herein we require that deformations satisfy

$$(\epsilon_{rr}\epsilon_{\theta\theta} - \epsilon_{r\theta}^2) \geqslant \delta > -1. \tag{4}$$

For the crack-tip analysis, we may set $\delta = 0$ without any loss of generality. Then the set of strains that satisfies the constraint (4) is the union of two distinct convex sets characterized by

$$\epsilon_{rr} + \epsilon_{\theta\theta} \geqslant 0 \text{ and } \epsilon_{rr} + \epsilon_{\theta\theta} \leqslant 0 \tag{5}$$

with the origin as the only common point between the two sets.

In the unconstrained region, hereafter identified as region 1, we have the classical constitutive relations

$$\sigma_{rr} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta}) + 2\mu\epsilon_{rr}, \tag{6.1}$$

$$\sigma_{\theta\theta} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta}) + 2\mu\epsilon_{\theta\theta}, \tag{6.2}$$

$$\sigma_{r\theta} = 2\mu\epsilon_{r\theta}, \tag{6.3}$$

and in the locking region, henceforth called region 2, we have the pressure field $p(r, \theta) > 0$ that is undetermined from the deformation field, and the following constitutive relations:

$$\sigma_{rr} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta}) + 2\mu\epsilon_{rr} - p\epsilon_{\theta\theta}, \tag{7.1}$$

$$\sigma_{\theta\theta} = \lambda(\epsilon_{rr} + \epsilon_{\theta\theta}) + 2\mu\epsilon_{\theta\theta} - p\epsilon_{rr}, \tag{7.2}$$

$$\sigma_{r\theta} = 2\mu\epsilon_{r\theta} + p\epsilon_{r\theta}. \tag{7.3}$$

Equations (7) are derived by using variational methods given in refs [2, 10]. In eqs (6) and (7), λ and μ are Lamé constants, and σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ are the components of the Cauchy stress tensor. With the definitions

$$a(r, \theta) = \frac{(\lambda + 2\mu)}{(\lambda + 2\mu)^2 - (\lambda - p(r, \theta))^2}, \tag{8.1}$$

$$b(r, \theta) = \frac{\lambda - p(r, \theta)}{(\lambda + 2\mu)^2 - (\lambda - p(r, \theta))^2}, \tag{8.2}$$

eqs (7) can be written as

$$\epsilon_{rr} = a\sigma_{rr} - b\sigma_{\theta\theta}, \tag{9.1}$$

$$\epsilon_{\theta\theta} = a\sigma_{\theta\theta} - b\sigma_{rr}, \tag{9.2}$$

$$\epsilon_{r\theta} = (a + b)\sigma_{r\theta}. \tag{9.3}$$

The constraint condition $(\epsilon_{rr}\epsilon_{\theta\theta} - \epsilon_{r\theta}^2) = 0$ takes the form

$$(a - b)^2(\sigma_{rr} + \sigma_{\theta\theta})^2 = (a + b)^2[(\sigma_{rr} - \sigma_{\theta\theta})^2 + 4\sigma_{r\theta}^2]. \tag{10}$$

In terms of the Airy stress function $F_2(r, \theta)$ for the locking domain 2, we have

$$\sigma_{\theta\theta} = \frac{\partial^2 F_2}{\partial r^2}, \quad \sigma_{rr} = \frac{1}{r}\frac{\partial F_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F_2}{\partial \theta^2}, \quad \sigma_{r\theta} = -\frac{\partial^2(F_2/r)}{\partial r\, \partial \theta} \tag{11}$$

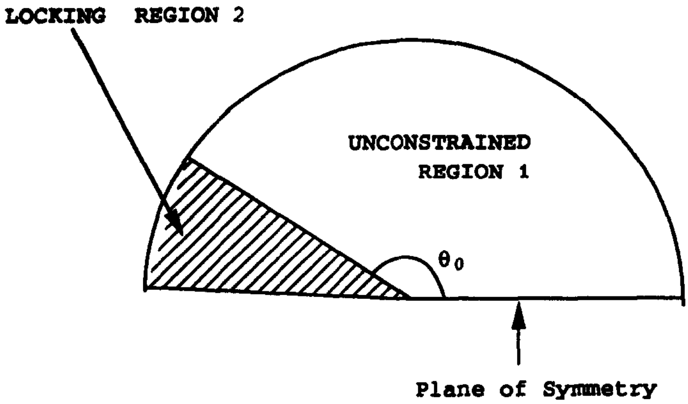

Fig. 2. A schematic sketch of the locking and unconstrained regions for the case when the deformations of the body are assumed to be symmetric about the plane $\theta = 0$.

and a similar relation with F_2 replaced by F_1 holds in the unconstrained region 1. We note that F_1 satisfies the biharmonic equation

$$\nabla^4 F_1 = 0, \tag{12}$$

and the functions $p(r, \theta)$ and $F_2(r, \theta)$ are solutions of the equations obtained by substituting from eqs (9) and (11) into eq. (10) and the following compatibility condition.

$$\frac{1}{r}\frac{\partial^2(r\epsilon_{\theta\theta})}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \epsilon_{rr}}{\partial \theta^2} - \frac{1}{r}\frac{\partial \epsilon_{rr}}{\partial r} - \frac{2}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial \epsilon_{r\theta}}{\partial \theta}\right) = 0. \tag{13}$$

3. CRACK-TIP ANALYSIS

We consider the case when the deformations of the body are symmetric about the plane $\theta = 0$, and study deformations of the upper-half of the body shown in Fig. 2. We assume that the locking domain is defined by $0 \leqslant r < \infty$, $\theta_O \leqslant \theta \leqslant \pi$ and the remainder is the unconstrained domain. Furthermore, we seek a solution that is separable in r and θ. Thus in the locking region 2, we have

$$F_2(r, \theta) = r^{\rho+1} f(\theta). \tag{14}$$

Equilibrium equations or equations expressing the balance of linear momentum require that the pressure field $p(r, \theta)$ and hence $a(r, \theta)$ and $b(r, \theta)$ be functions of θ only. Equations (11) and (14) yield

$$\sigma_{rr} = r^{\rho-1}[(\rho+1)f + f''], \quad \sigma_{\theta\theta} = (\rho+1)\rho r^{\rho-1} f, \quad \sigma_{r\theta} = -\rho r^{\rho-1} f' \tag{15}$$

where $f' = \mathrm{d}f/\mathrm{d}\theta$. Substituting from (15) and eqs (9.1) through (9.3) and integrating the result, we obtain

$$u_r = r^\rho[af'' + (\rho+1)(a - b\rho)f]/\rho, \tag{16.1}$$

$$u_\theta = r^\rho[2\rho^2(a+b)f' + (af'' + (\rho+1)(a-b\rho)f)']/\rho(1-\rho). \tag{16.2}$$

The compatibility equation (13) and the locking condition $(\epsilon_{rr}\epsilon_{\theta\theta} - \epsilon_{r\theta}^2) = 0$ reduce to

$$-a[f'''' + 2(\rho^2+1)f'' + (1-\rho^2)^2 f]$$
$$= 2a'f''' + a''f'' + 2[a' + \rho a' + \rho^2 a' - b'\rho]f' + (1+\rho)(a'' - b''\rho)f \tag{17.1}$$

$$(a-b)^2[(\rho+1)^2 f + f'']^2 = (a+b)^2[(f'' + (1-\rho^2)f)^2 + 4\rho^2 f'^2]. \tag{17.2}$$

For the elastic domain, we solve the biharmonic equation (12) by the method of complex variables (e.g. see Muskhelishvili [12]) and assume that the analytical functions

$$\varphi = Az^\rho, \quad \psi = Bz^\rho, \tag{18}$$

where $A = R + iI$ and $B = M + iN$ are undetermined complex numbers. Following Muskhelishvili [12], we obtain the following expressions for the three components of the stress tensor and two components of the displacement field.

$$\sigma_{\theta\theta} = \rho r^{\rho-1}[(1+\rho)(R\cos(\rho-1)\theta - I\sin(\rho-1)\theta) + M\cos(\rho+1)\theta - N\sin(\rho+1)\theta] \quad (19.1)$$

$$\sigma_{rr} = \rho r^{\rho-1}[(3-\rho)(R\cos(\rho-1)\theta - I\sin(\rho-1)\theta) - M\cos(\rho+1)\theta + N\sin(\rho+1)\theta] \quad (19.2)$$

$$\sigma_{r\theta} = \rho r^{\rho-1}[(\rho-1)(R\sin(\rho-1)\theta + I\cos(\rho-1)\theta) + M\sin(\rho+1)\theta + N\cos(\rho+1)\theta] \quad (19.3)$$

$$2\mu u_r = r^{\rho}[(\kappa-\rho)(R\cos(1-\rho)\theta + I\sin(1-\rho)\theta) - M\cos(\rho+1)\theta + N\sin(\rho+1)\theta] \quad (19.4)$$

$$2\mu u_\theta = r^{\rho}[(\kappa+\rho)(R\sin(\rho-1)\theta + I\cos(\rho-1)\theta) + M\sin(\rho+1)\theta + N\cos(\rho+1)\theta]. \quad (19.5)$$

Here, $\kappa = 3 - 4\nu$, ν being the Poisson's ratio for the material of the body.

We now examine the boundary conditions. On the plane $\theta = 0$, the assumed symmetry of deformations requires that

$$\sigma_{r\theta} = 0, \quad u_\theta = 0, \quad \text{on } \theta = 0. \quad (20.1)$$

The surface $\theta = \pi$ should be traction free. Thus,

$$\sigma_{\theta\theta} = 0 \quad \text{and} \quad \sigma_{r\theta} = 0 \quad \text{on} \quad \theta = \pi. \quad (20.2)$$

Equations (20.1) are satisfied if

$$I = N = 0 \quad (21)$$

and (20.2) require that

$$f'(\pi) = 0, \quad f(\pi) = 0. \quad (22)$$

On the intersurface $\theta = \theta_o$ between the locking and unconstrained regions, the continuity of surface tractions and displacements gives the following four conditions.

$$(\rho+1)f(\theta_o) = (1+\rho)R\cos(\rho-1)\theta_o + M\cos(\rho+1)\theta_o, \quad (23.1)$$

$$-f'(\theta_o) = (\rho-1)R\sin(\rho-1)\theta_o + M\sin(\rho+1)\theta_o, \quad (23.2)$$

$$2\mu\{af''' + a'f'' + [(1+\rho+2\rho^2)a + \rho(\rho-1)b]f' + (\rho+1)(a'-b'\rho)f\}$$
$$= \rho(1-\rho)[(\kappa+\rho)R\sin(\rho-1)\theta_o + M\sin(\rho+1)\theta_o] \quad \text{at } \theta = \theta_o, \quad (23.3)$$

$$2\mu[af'' + (\rho+1)(a-b\rho)f] = \rho[(\kappa-\rho)R\cos(1-\rho)\theta_o - M\cos(\rho+1)\theta_o] \quad \text{at } \theta = \theta_o. \quad (23.4)$$

Equations (22) and (23) determine R, M and the four constants of integration in the fourth-order ordinary differential equation (17.1) for $f(\theta)$. The value of θ_o is determined from

$$\epsilon_{rr}\epsilon_{\theta\theta} - \epsilon_{r\theta}^2 > 0 \text{ in region 1}, \quad (24.1)$$

$$p(\theta) > 0 \text{ in region 2}, \quad (24.2)$$

and it can be shown to be a solution of

$$\epsilon_{rr}\epsilon_{\theta\theta} - \epsilon_{r\theta}^2 = 0 \quad \text{at } \theta = \theta_o. \quad (25.1)$$

Equation (25.1) is equivalent to

$$4(1-2\nu)^2R^2\cos^2(\rho-1)\theta_o + 2MR(1-\rho)\cos 2\theta_o = M^2 + (\rho-1)^2R^2. \quad (25.2)$$

3.1. *Power-series solution*

We solve equations (17.1) and (17.2) by the power-series method. We recall that the classical solution gives

$$(\epsilon_{rr}\epsilon_{\theta\theta} - \epsilon_{r\theta}^2) \sim [4\nu(\nu-1) + \cos^2(\theta/2)]\cos^2(\theta/2), \quad (26)$$

and it satisfies the constraint (25.1) with $\rho = 1/2$ and $\theta_o = \pi$ if and only if $v = 0$. For small v, the angle $(\pi - \theta_o)$ of the locking region 2 should be expected to be small. We thus expand the unknown functions $f(\theta)$ and $p(\theta)$ as power series in terms of $\Theta \equiv (\theta - \pi)$ over region 2, i.e. for $\theta_o \leqslant \theta \leqslant \pi$. That is,

$$f(\theta) = f_o + f_1\Theta + f_2\Theta^2 + f_3\Theta^3 + \cdots \tag{27.1}$$

$$p(\theta) = p_o + p_1\Theta + p_2\Theta^2 + p_3\Theta^3 + \cdots, \tag{27.2}$$

where $f_i, p_i (i = 0, 1, 2 \cdots)$ are yet-to-be-determined real constants. Equations (8.1), (8.2), and (27.2) give

$$a + b = \frac{1}{(p_o + 2\mu)} + \frac{-p_1\Theta}{(p_o + 2\mu)^2} + \frac{[p_1^2 - p_2(2\mu + p_0)]\Theta^2}{(p_0 + 2\mu)^3} + \cdots \tag{28.1}$$

$$a - b = \frac{1}{(2\lambda + 2\mu - p_0)} + \frac{p_1\Theta}{(2\lambda + 2\mu - p_o)^2} + \frac{[p_1^2 + p_2(2\lambda + 2\mu - p_o)]\Theta^2}{(2\lambda + 2\mu - p_o)^3} + \cdots. \tag{28.2}$$

Substituting from equation (27.1) into equations (23) and (17.2), and equating coefficients of Θ^o, Θ^1, and Θ^2 on both sides, we get

$$f_o = 0, \quad f_1 = 0, \tag{29.1}$$

$$p_o = \lambda > 0, \quad p_1 = 0, \quad p_2 = \rho(\rho - 1)(\lambda + 2\mu)/2 < 0, \tag{29.2}$$

thus

$$a(\theta) = 1/(\lambda + 2\mu) + O(\Theta^3), \tag{30.1}$$

$$b(\theta) = \frac{-p_2\Theta^2}{(\lambda + 2\mu)^2} + O(\Theta^3). \tag{30.2}$$

By comparing coefficients of Θ^o in equation (17.1), we obtain

$$f_4 = -\tfrac{1}{6}(1 + \rho^2)f_2. \tag{31}$$

The consideration of higher-order terms in equations (17.1) and (17.2) determines higher-order coefficients (f_5, p_3), (f_6, p_4), etc. For example, we found that

$$p_3 = \rho(1 - 2\rho)f_3(\lambda + 2\mu)/f_2, \tag{32.1}$$

$$p_4 = [(\lambda + 2\mu)/4][\rho(1 - \rho)(\rho^2 + 3\rho - 2)/3 + f_3^2(33\rho^2 - 12\rho)/f_2^2], \tag{32.2}$$

$$f_5 = -(1 + \rho^2)f_3/10, \tag{32.3}$$

$$f_6 = (3 + 9\rho^2 + 8\rho^3 - 4\rho^4)f_2/360, \tag{32.4}$$

and thus

$$a = \frac{1}{(\lambda + 2\mu)} + \frac{p_2^2\Theta^4}{(\lambda + 2\mu)^3} + \cdots \tag{33.1}$$

$$b = \frac{-p_2\Theta^2}{(\lambda + 2\mu)^2} - \frac{p_3\Theta^3}{(\lambda + 2\mu)^2} - \frac{p_4\Theta^4}{(\lambda + 2\mu)^2}\cdots. \tag{33.2}$$

In particular, p_4 term must be considered when checking the condition $p(\theta) > 0$ for sufficiently small v, even though it does not influence the calculation of the lowest-order eigenvalue problem.

Substitution from (27.1), (28.1), (28.2), (29) and (31) into eqs (23.1) through (23.4) gives an eigenvalue problem for the determination of f_2, f_3, R, and M; and the value of θ_o is determined from eq. (25.2).

3.1.1. *Lowest-order solution of the eigenvalue problem.* In order to illustrate that the aforestated nonlinear problem does have a nontrivial solution, we seek a series type solution for sufficiently small ν. Accordingly, we adopt the following lowest-order solution.

$$f(\theta)=f_2\Theta^2+f_3\Theta^3-(1+\rho^2)f_2\Theta^4/6, \tag{34.1}$$

$$p(\theta)=\lambda+\rho(\rho-1)(\lambda+2\mu)\Theta^2/2, \tag{34.2}$$

$$a(\theta)=1/(\lambda+2\mu), \tag{34.3}$$

$$b(\theta)=\rho(1-\rho)\Theta^2/[2(\lambda+2\mu)]. \tag{34.4}$$

Solving eqs (21.1), (21.2), and (34) for R and M, and substituting the result in eqs (21.3) and (21.4), we obtain the following eigenvalue problem for f_2 and f_3.

$$f_2H_{11}(\rho,\theta_o,\nu)+f_3H_{12}(\rho,\theta_o,\nu)=0, \tag{35.1}$$

$$f_2H_{21}(\rho,\theta_o,\nu)+f_3H_{22}(\rho,\theta_o,\nu)=0, \tag{35.2}$$

where

$$\begin{aligned}H_{11}=\;&4a\mu(\sin 2\rho\theta_o+\rho\sin 2\theta_o)[(1-\rho)(\theta_o-\pi)+(1+\rho+2\rho^2)(1+\rho^2)(\theta_o-\pi)^3/3]\\&+\rho(1-\rho^2)(\kappa+1)[(\theta_o-\pi)^2-(1+\rho^2)(\theta_o-\pi)^4/6]\sin(\rho+1)\theta_o\sin(\rho-1)\theta_o\\&+\mu(1-\rho)\rho^2(\sin 2\rho\theta_o+\rho\sin 2\theta_o)[12(\theta_o-\pi)^3+(\rho-3\rho^2+\rho^3-3)(\theta_o-\pi)^5]/3(\lambda+2\mu)\\&+2\rho(1-\rho)[(\theta_o-\pi)-(1+\rho^2)(\theta_o-\pi)^3/3][\kappa\cos(\rho+1)\theta_o\sin(\rho-1)\theta_o\\&-\cos(\rho-1)\theta_o\sin(1+\rho)\theta_o-\rho\sin 2\theta_o],\end{aligned} \tag{36.1}$$

$$\begin{aligned}H_{12}=\;&3\rho(1-\rho)(\theta_o-\pi)^2[\kappa\cos(\rho+1)\theta_o\sin(\rho-1)\theta_o-\cos(\rho-1)\theta_o\sin(1+\rho)\theta_o-\rho\sin 2\theta_o]\\&-2a\mu(\sin 2\rho\theta_o+\rho\sin 2\theta_o)(6+3(1+\rho+2\rho^2)(\theta_o-\pi)^2)\\&+(1-\rho^2)(\kappa+1)\rho(\theta_o-\pi)^3\sin(\rho+1)\theta_o\sin(\rho-1)\theta_o\\&+\mu(1-\rho)\rho^2(\sin 2\rho\theta_o+\rho\sin 2\theta_o)(5-\rho)(\theta_o-\pi)^4/(\lambda+2\mu),\end{aligned} \tag{36.2}$$

$$\begin{aligned}H_{21}=\;&2a\mu(\sin 2\rho\theta_o+\rho\sin 2\theta_o)[(1-\rho+2\rho^2)(\theta_o-\pi)^2+(1+\rho)(1+\rho^2)(\theta_o-\pi)^4/6-2]\\&+\rho(1+\rho)[(\theta_o-\pi)^2-(1+\rho^2)(\theta_o-\pi)^4/6][\kappa\cos(\rho-1)\theta_o\sin(\rho+1)\theta_o\\&+\cos(\rho+1)\theta_o\sin(1-\rho)\theta_o-\rho\sin 2\theta_o]\\&+2\rho[(\theta_o-\pi)-(1+\rho^2)(\theta_o-\pi)^3/3](\kappa+1)\cos(\rho+1)\theta_o\cos(\rho-1)\theta_o\\&+\mu(1-\rho^2)\rho^2(\sin 2\rho\theta_o+\rho\sin 2\theta_o)(\theta_o-\pi)^4\{6-(1+\rho^2)(\theta_o-\pi)^2\}/6(\lambda+2\mu),\end{aligned} \tag{36.3}$$

$$\begin{aligned}H_{22}=\;&3\rho(\kappa+1)(\theta_o-\pi)^2\cos(\rho+1)\theta_o\cos(\rho-1)\theta_o\\&+\rho(1+\rho)(\theta_o-\pi)^3[\kappa\cos(\rho-1)\theta_o\sin(\rho+1)\theta_o+\cos(\rho+1)\theta_o\sin(1-\rho)\theta_o-\rho\sin 2\theta_o]\\&-2a\mu(\sin 2\rho\theta_o+\rho\sin 2\theta_o)\{6(\theta_o-\pi)+(1+\rho)(\theta_o-\pi)^3\}\\&+\mu(1-\rho^2)\rho^2(\sin 2\rho\theta_o+\rho\sin 2\theta_o)(\theta_o-\pi)^5/(\lambda+2\mu),\end{aligned} \tag{36.4}$$

and

$$a\mu=(\kappa-1)/(\kappa+1). \tag{36.5}$$

For eqs (35.1) and (35.2) to have a nontrivial solution,

$$H_{11}H_{22}=H_{12}H_{21}, \tag{37}$$

which determines ρ. The value of θ_o is then determined by eq. (25.2).

Let

$$\theta_o-\pi=\epsilon,\quad \rho=\tfrac{1}{2}+\rho_1\epsilon+\rho_2\epsilon^2+\cdots, \tag{38}$$

where

$$\rho_1 = \left.\frac{d\rho}{d\theta_o}\right|_{\theta_o = \pi}. \tag{39}$$

Noting that

$$(\sin 2\rho\theta_o + \rho \sin 2\theta_o) \sim -2\pi\rho_1\epsilon \tag{40}$$

and substituting from (38) into (36) and the result into (37), and letting $\epsilon \to 0$, we find that

$$H_{11} \sim \epsilon^2, \quad H_{12} \sim \epsilon, \quad H_{21} \sim \epsilon, \quad H_{22} \sim \epsilon^2. \tag{41}$$

Thus, $\rho_1 = 0$ (as $\epsilon \to 0$) implying thereby that the expression $(38)_2$ should be replaced by

$$\rho = \tfrac{1}{2} + \rho_2\epsilon^2 + \cdots. \tag{42}$$

An exercise similar to that used to find ρ_1 gives $\rho_2 = 0$. Thus, we assume that

$$\rho = \tfrac{1}{2} + \rho_3\epsilon^3, \tag{43}$$

where $6\rho_3 = d^3\rho/d\theta_o^3|_{\theta_o=\pi}$. The dominant terms in eqs (36) are

$$H_{11} = -3(\kappa + 1)\epsilon^2/8, \tag{44.1}$$

$$H_{12} = -6(\kappa + 1)\epsilon^3/8 + 12a\mu(2m\pi + \tfrac{1}{2})\epsilon^3, \tag{44.2}$$

$$H_{21} = -3(\kappa + 1)\epsilon^3/8 + 4a\mu(2m\pi + \tfrac{1}{2})\epsilon^3, \tag{44.3}$$

$$H_{22} = -6(\kappa + 1)\epsilon^4/8 + 12a\mu(2m\pi + \tfrac{1}{2})\epsilon^4, \tag{44.4}$$

where $\kappa = 3$ (defined at $\epsilon = 0$, that is, at $\nu = 0$). Equation (37) has two possible roots. However, one of these, $m = -1/(4\pi)$, is unacceptable since it leads to $(\epsilon_{rr}\epsilon_{\theta\theta} - \epsilon_{r\theta}^2) < 0$ at $\theta = \theta_o$. Thus, we consider the other root corresponding to

$$-6(\kappa + 1)/8 + 12a\mu(2m\pi + \tfrac{1}{2}) = 0,$$

that is

$$m = 0 \tag{45}$$

and obtain

$$(\rho - \tfrac{1}{2}) = O[(\theta_o - \pi)^4] \tag{46}$$

at the neighborhood of $\theta_o = \pi$ (that is, $\nu = 0$). We can now find the value of θ_o from (25.2). Noting that

$$f_2 \sim O[(\theta_o - \pi)^2] f_3$$

we obtain

$$(M/R) = \tfrac{1}{2} + O((\theta_o - \pi)^3)\ldots. \tag{47}$$

Substitution from (46) and (47) into (25.2) yields, up to the ϵ^4 terms

$$(\theta_o - \pi)^2 = 16\nu + o(\nu). \tag{48}$$

Thus $\epsilon^2 = 0$ when $\nu = 0$ as expected. It may be proved that the condition (24.2) is satisfied. Since $p_3(\theta - \pi)^3$ is a small term of higher-order, we set

$$p(\theta) \approx \lambda + p_2(\theta - \pi)^2 + p_4(\theta - \pi)^4$$

whose minimum value is

$$\lambda - p_2^2/4p_4.$$

Noting that

$$p_4 \approx 9\rho^2(\lambda + 2\mu)f_3^2/4f_2^2 \sim O[(\theta_o - \pi)^{-4}]; \quad (\theta_o - \pi)^2 \sim \nu$$

we obtain

$$\lambda - p_2^2/4p_4 > 0$$

for a sufficiently small value of v. Therefore $p(\theta) > 0$ over domain 2.

4. CONCLUSIONS

We have studied plane-strain deformations near a crack tip in a linear-elastic isotropic and homogeneous body under the constraint that the dilatation must be non-negative. When dominant terms are kept in the constraint equation, we obtain a nonlinear equation. The region wherein the nonlinear constraint equation holds has been identified as the locking region, a term borrowed from Prager [10]; the remaining region is unconstrained. Equations that delimit the sizes of these regions have been derived. It is shown that the nonlinear problem so formulated has a series-type nontrivial solution in the neighborhood of the classical solution. The order of singularity of the present problem is nearly the same as that of the classical solution for sufficiently small values of Poisson's ratio. It is possible that the nonlinear problem has other solutions that are not r, θ separable. Such solutions may exhibit different types of singularity.

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APPENDIX

In rectangular Cartesian coordinates and for plane-strain deformations in the x–y plane,

$$dv/dV = 1 + u_x + v_y + u_x v_y - u_y v_x \tag{A1}$$

where $u_x = \partial u/\partial x$ etc. and u, v denote the components of the displacement along x and y axes, respectively. Therefore, $dv/dV > 0$ is equivalent to

$$u_x + v_y + u_x v_y - u_y v_x > -1. \tag{A2}$$

Since

$$u_y v_x = \epsilon_{xy}^2 - (u_y - v_x)^2/4, \tag{A3}$$

it follows that

$$(u_x v_y - u_y v_x) \geqslant (\epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}^2), \tag{A4}$$

where ϵ_{xx}, ϵ_{yy} and ϵ_{xy} are components of the infinitesimal strain tensor. Hence condition

$$\epsilon_{xx} + \epsilon_{yy} + (\epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}^2) > -1 \tag{A5}$$

is sufficient for the inequality (A2) to hold.

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