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ANALYSIS OF DEFORMATION FIELDS NEAR AN INTERFACIAL CRACK TIP IN LOCKING MATERIALS

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Abstract—We study plane strain deformations near a crack tip between two isotropic and homogeneous Hookean bodies and assume that deformations obey the constraint that the volume of a material element cannot become zero. The constraint is imposed by requiring that the dilatation everywhere in the body must be greater than or equal to a constant. The region where the dilatation equals the constant is called the locking region. It is shown that when the locking region occurs in only a part of one body, the index of the singularity near the interfacial crack tip is 1/2, and the size of the locking region is determined by the values of the material parameters for the two bodies.

1. INTRODUCTION

SINCE the time Williams [1] obtained the characteristic oscillating stress singularity near the interface crack tip that implied interpenetration of the material, there have been numerous studies (e.g. see Cherapanov [2], England [3], Erdogan [4], Rice and Shih [5], Park and Earmme [6], Shih and Asaro [7], Hutchinson *et al.* [8] and Rice [9]) to understand the singular nature of deformation fields near an interface crack tip and interpret the elastic solution. Here we attempt to understand how the interfacial crack tip solution is modified by the constraint that the dilatation at every point in the body exceeds or equals a constant greater than -1. We use the terminology introduced by Prager [10] who called such materials locking materials.

2. FORMULATION OF THE PROBLEM

Our interest is to find the deformation and stress fields in the vicinity of an interfacial crack tip in a body made of two homogeneous, isotropic, and linear elastic materials. We assume that the body undergoes plane strain deformations in the x_1–x_2 plane, and that deformations of the body represent the constraint

$$\frac{\mathrm{d}v}{\mathrm{d}V} > 0, \tag{1}$$

where $\mathrm{d}v$ and $\mathrm{d}V$ equal, respectively, the volume of the same material element in the present and undeformed stress free reference configurations. In the linearized theory the constraint (1) becomes

$$\epsilon_{\alpha\alpha} \geqslant \delta > -1, \tag{2}$$

where $\epsilon_{\alpha\beta}$ is the infinitesimal strain tensor, a repeated index implies summation over the range 1, 2 of the index, and δ is a material constant. We call the region where equality holds in (2) as the "locking region". The region where inequality holds in (2) is unconstrained and deforms elastically.

In the locking region, the stresses and strains are related by

$$\epsilon_{\alpha\alpha} = \delta, \tag{3}$$

$$\sigma_{\alpha\beta} = -p\delta_{\alpha\beta} + 2\mu\epsilon_{\alpha\beta} + \lambda\epsilon_{\gamma\gamma}\delta_{\alpha\beta}, \tag{4}$$

and in the unconstrained region by

$$\sigma_{\alpha\beta} = \lambda\epsilon_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu\epsilon_{\alpha\beta}, \tag{5}$$

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where λ and μ are the Lamé constants, and $p(x_1, x_2) > 0$ is the hydrostatic pressure not determined by the deformation field. If desired, the last term on the right-hand side of eq. (4) could be absorbed in p.

In terms of the Airy stress function $F(x_1, x_2)$,

$$\sigma_{11} = F_{,22}, \sigma_{22} = F_{,11}, \sigma_{12} = -F_{,12} \tag{6}$$

and F satisfies the biharmonic equation

$$\nabla^4 F = 0. \tag{7}$$

In eq. (6), $F_{,\alpha\beta} = \partial^2 F/\partial x_\alpha \partial x_\beta$. Equations (6) and (7) hold in both regions, and imply that equilibrium equations and compatibility conditions are identically satisfied by every F. The derivation of eq. (7) for unconstrained materials is given in several textbooks (e.g. see Sokolnikoff [11]). For locking materials, eq. (7) can be derived by essentially following the same steps; the details are omitted herein. The biharmonic eq. (7) can be solved by using the method of complex variables. In terms of analytical functions $\phi_e(z)$ and $\psi_e(z)$ of the complex variable $z = x_1 + ix_2$, we have in the elastic region

$$\sigma_{11} + \sigma_{22} = 2(\phi_e' + \bar{\phi}_e'), \tag{8}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2(\bar{z}\phi_e'' + \psi_e'), \tag{9}$$

$$2\mu(u_1 + iu_2) = \kappa\phi_e - z\bar{\phi}_e' - \bar{\psi}_e, \tag{10}$$

where a superimposed bar indicates the complex conjugate of the variable, ϕ' denotes derivative of ϕ with respect to z, and $\kappa = 3 - 4\nu$, ν being the Poisson ratio. In the locking region, analogs of eqs (8)–(10) are

$$\sigma_{11} + \sigma_{22} = 2(\phi_0' + \bar{\phi}_0'), \tag{11}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2(\bar{z}\phi_0'' + \psi_0'), \tag{12}$$

$$2\mu(u_1 + iu_2) = \phi_0 - z\bar{\phi}_0' - \bar{\psi}_0 + \mu\delta z, \tag{13}$$

$$2(\lambda + \mu)\delta > \sigma_{11} + \sigma_{22}, \tag{14}$$

where the inequality (14) holds because $p > 0$.

The continuity of surface tractions and displacements across the boundary between the locking region and the unconstrained region gives

$$[\![\phi + z\bar{\phi}' + \bar{\psi}]\!] = 0, \tag{15}$$

$$\kappa\phi_e - \phi_0 - [\![z\bar{\phi}' + \bar{\psi}]\!] = \mu\delta z, \tag{16}$$

where $[\![f]\!]$ denotes the difference in the values of f on the two sides of the boundary between the two regions. Equations (15) and (16) yield the following Hilbert condition [12]

$$(\kappa + 1)\phi_e - 2\phi_0 = \mu\delta z \tag{17}$$

which we assume replaces eq. (16).

3. CRACK-TIP ANALYSIS

We assume that a crack is located at $x_2 = 0, x_1 < 0$; thus the origin of the $x_1 - x_2$ coordinate axes is at the crack tip with the crack aligned along the negative x_1-axis (e.g. see Fig. 1). In terms of the polar coordinates, the cracked body occupies the domain $0 \leqslant r < \infty$, $-\pi \leqslant \theta \leqslant \pi$. We assume that the material of the body in the upper half plane has elasticities (μ_1, κ_1) and that in the lower half (μ_2, κ_2). Here μ equals the shear modulus of the material and $\kappa = 3 - 4\nu$. We use William's [1] eigenexpansion method combined with Muskhelishvili's complex variable method [12] to get an asymptotic solution near the crack tip [5]. The solution is assumed to be expressible as the product of a function of r and a function of θ. We assume that each one of the upper and lower halves of the body can be divided into two parts, a locking region and an unconstrained region. Thus we may have four regions Ω_{11}, Ω_{12}, Ω_{21} and Ω_{22} defined by $\theta_1 \leqslant \theta \leqslant \pi$, $0 \leqslant \theta \leqslant \theta_1$, $\theta_2 \leqslant \theta \leqslant 0$, and

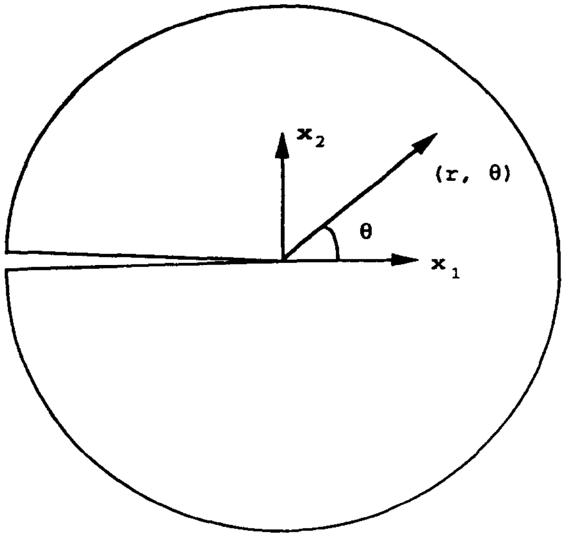

Fig. 1. A schematic sketch of the problem studied.

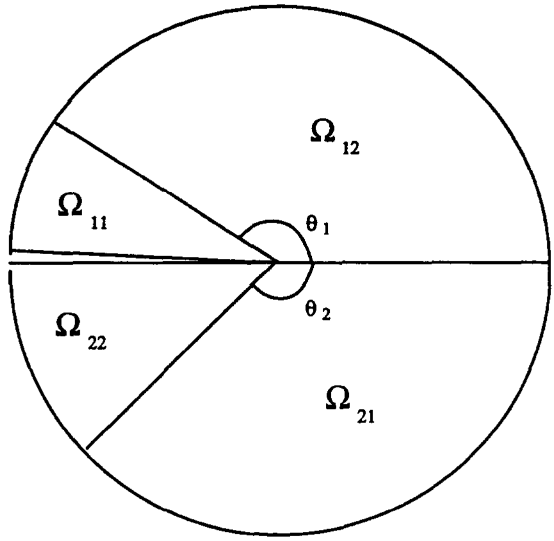

Fig. 2. A possible division of the domain into four subdomains.

$-\pi \leqslant \theta \leqslant \theta_2$ with material parameters (μ_1, κ_{11}), (μ_1, κ_{12}), (μ_2, κ_{21}) and (μ_2, κ_{22}), respectively. These are shown in Fig. 2. We note that κ equals 1 in the locking region.

Without loss of generality (e.g. see [1, 5]), we seek functions $\phi(z)$ and $\psi(z)$ of the form

$$z^{\bar{\rho}} \quad \text{and} \quad z^{\rho}; \qquad 0 < \mathrm{Re}(\rho) < 1, \tag{18}$$

in the locking and elastic regions, respectively. Here ρ is an eigenvalue to be determined. When ρ is a complex number, conditions $\epsilon_{\alpha\alpha} > \delta$ and inequality (14) cannot be satisfied because of the oscillatory nature of the singularity in the limit $r \to 0$. Thus an asymptotic solution of $r - \theta$ separable form is impossible when ρ is a complex number. Accordingly we confine ourselves to the case when ρ is a real number. We further assume that

$$\phi = Az^{\rho}, \quad \psi = Bz^{\rho} \quad \text{on } \Omega_{11}, \tag{19}$$

$$\phi = Ez^{\rho}, \quad \psi = Fz^{\rho} \quad \text{on } \Omega_{12}, \tag{20}$$

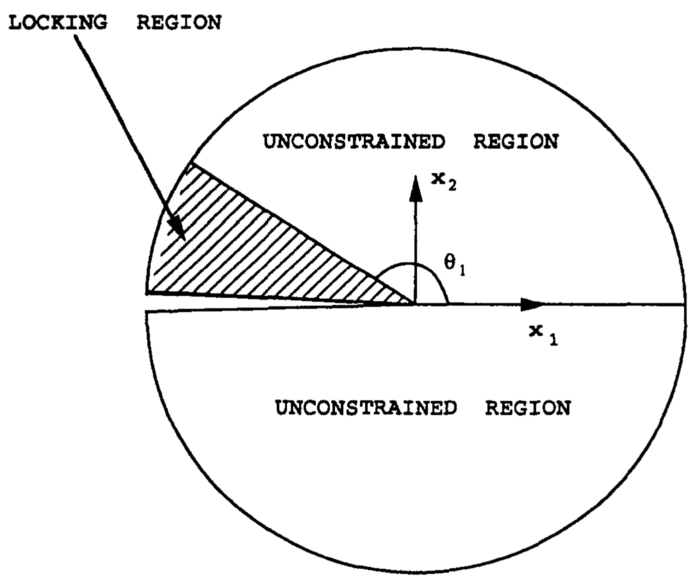

Fig. 3. Locking region in one body only.

$$\phi = Gz^{\rho}, \quad \psi = Hz^{\rho} \quad \text{on } \Omega_{21}, \tag{21}$$

$$\phi = Mz^{\rho}, \quad \psi = Nz^{\rho} \quad \text{on } \Omega_{22}, \tag{22}$$

where A, B, E, F, G, H, M and N are eight undetermined complex constants.

The requirement that surfaces $\theta = \pm\pi$ of the crack are traction free, and continuity conditions across the bounding surfaces $\theta = \theta_1$, 0 and θ_2 give the following eight homogeneous equations for the determination of A, B, E, F, G, H, M and N.

$$\bar{A}e^{-2i\rho\pi} + A\rho + B = 0, \tag{23}$$

$$\bar{M}e^{2i\rho\pi} + M\rho + N = 0, \tag{24}$$

$$A(\kappa_{11} + 1) - E(\kappa_{12} + 1) = 0, \tag{25}$$

$$\bar{A}e^{-2i\rho\theta_1} + A\rho e^{-2i\theta_1} + B - (\bar{E}e^{-2i\rho\theta_1} + E\rho e^{-2i\theta_1} + F) = 0, \tag{26}$$

$$E + \rho\bar{E} + \bar{F} - (G + \rho\bar{G} + \bar{H}) = 0, \tag{27}$$

$$\mu_1(\kappa_{21}G - \rho\bar{G} - \bar{H}) - \mu_2(\kappa_{12}E - \rho\bar{E} - \bar{F}) = 0, \tag{28}$$

$$M(\kappa_{22} + 1) - G(\kappa_{21} + 1) = 0, \tag{29}$$

$$\bar{M}e^{-2i\rho\theta_2} + M\rho e^{-2i\theta_2} + N - (\bar{G}e^{-2i\rho\theta_2} + G\rho e^{-2i\theta_2} + H) = 0. \tag{30}$$

These equations have a nontrivial solution for A, B, E, F, G, H, M and N only for certain values of ρ.

Values of θ_1 and θ_2 are determined by

$$\begin{aligned} &\phi'_e + \bar{\phi}'_e > 0 \text{ in the elastic domain,} \\ &\phi'_0 + \bar{\phi}'_0 < 0 \text{ in the locking domain.} \end{aligned} \tag{31}$$

For the special case when $\mu_1 = \mu_2$, we have

$$M(\kappa_{22} + 1) = G(\kappa_{21} + 1) = A(\kappa_{11} + 1) = E(\kappa_{12} + 1), \tag{32}$$

and eq. (31) takes the form

$$(\sigma_{11} + \sigma_{22}) \sim \operatorname{Re}\phi' \sim \operatorname{Re}(Ae^{i(\rho-1)\theta}) \quad \text{for } -\pi \leqslant \theta \leqslant \pi, \tag{33}$$

which simplifies the analysis. Therefore, we first study this case.

3.1. *The case with* $\mu_1 = \mu_2$

3.1.1. *The locking region in one material only.* Without any loss of generality we assume that only the upper-half of the domain with $\nu = \nu_1$ is divided into locking and unconstrained regions, and the former is given by $0 \leqslant \theta_1 \leqslant \theta \leqslant \pi$. We set

$$\mu_1 = \mu_2 = \mu, \quad \kappa_{12} = \kappa_1, \quad \kappa_{21} = \kappa_{22} = \kappa_2. \tag{34}$$

Recall that $\kappa_{11} = 1$. Equations (29) and (30) yield

$$M = G, \quad N = H, \tag{35}$$

$$(\kappa_1 + 1)E = (\kappa_2 + 1)M = 2A. \tag{36}$$

Substitution from eqs (35) and (36) into eqs (23)–(28) yields the following complex eigenequations for $A \neq 0$.

$$A\rho\left(\frac{\kappa_1 - 1}{\kappa_1 + 1}\right)(e^{-2i\theta_1} - 1) + \bar{A}\left[\frac{\kappa_1 - 1}{\kappa_1 + 1}e^{-2i\rho\theta_1} + \frac{2(\kappa_2 - \kappa_1)}{(\kappa_1 + 1)(\kappa_2 + 1)} + \frac{2e^{2i\rho\pi}}{(\kappa_2 + 1)} - e^{-2i\rho\pi}\right] = 0. \tag{37}$$

Let $A = R + iI$. Then

$$A = (R^2 + I^2)^{1/2}e^{i\omega}, \quad \operatorname{tg}\omega = I/R \tag{38}$$

and A changes its sign when ω increases by $\pm\pi$. The conditions (31) and (33) require that

$$(\sigma_{11}+\sigma_{22}) \sim \operatorname{Re}\phi' \sim \cos[(\rho-1)\theta+\omega] > 0 \quad \text{when } -\pi \leqslant \theta < \theta_1, \tag{39}$$

$$\sim \cos[(\rho-1)\theta+\omega] < 0 \quad \text{when } \theta_1 < \theta \leqslant \pi. \tag{40}$$

Therefore,

$$(\rho-1)\theta_1+\omega = -\frac{\pi}{2} \pm 2n\pi, \quad n = 1, 2, \ldots, \tag{41}$$

and

$$\frac{I}{R} = \frac{\cos(\rho-1)\theta_1}{\sin(\rho-1)\theta_1}, \tag{42}$$

or

$$|(\rho-1)(\theta-\theta_1)| \leqslant \pi, \quad -\pi \leqslant \theta \leqslant \pi, \tag{43}$$

and

$$\bar{A} = -A\, e^{2i(\rho-1)\theta_1}. \tag{44}$$

Substituting from (44) into (37) and the requirement that $A \neq 0$ gives the following complex equation for the determination of ρ and θ_1.

$$\gamma_1[e^{2i\rho\theta_1} + \rho(1-e^{2i\theta_1}) - 1] + \gamma_2(e^{2i\rho\pi}-1)e^{2i\rho\theta_1} = 2i\sin 2\rho\pi\, e^{2i\rho\theta_1}, \tag{45}$$

where the material parameter γ_α ($\alpha = 1, 2$) is defined by

$$\gamma_\alpha = \frac{\kappa_\alpha - 1}{\kappa_\alpha + 1} = \frac{1-2\nu_\alpha}{2-2\nu_\alpha}, \quad 0 \leqslant \gamma_\alpha \leqslant 1/2. \tag{46}$$

We note that $\theta_1 = 0$ implies that $\gamma_2 = 0$, i.e. material 2 is incompressible; and $\theta_1 = \pi$ gives that $\gamma_1 = \gamma_2$ and $\rho = 1/2$, i.e. the body is homogeneous. We will study later the problem for the homogeneous body.

We now examine solutions of eq. (45). For the special case of $\cos(\rho-1)\theta_1 = 0$ and thus $I = 0$, $(1-\rho)\theta_1 = \pi/2$, eq. (45) reduces to

$$\gamma_1[1 + e^{-2i\theta_1} + \rho(1-e^{-2i\theta_1})] + \gamma_2(e^{2i\rho\pi}-1) = e^{2i\rho\pi} - e^{-2i\rho\pi}, \tag{47}$$

and we get $\rho = 1/2$ when $\theta_1 = \pi$. However, when $\sin(\rho-1)\theta_1 = 0$ and thus $R = 0$, $\theta_1 = 0$, eq. (45) becomes

$$\gamma_2(e^{2i\rho\pi}-1) = e^{2i\rho\pi} - e^{-2i\rho\pi}, \tag{48}$$

and we obtain $\gamma_2 = 0$, $\rho = 1/2$. We note that eq. (45) relates (ρ, θ_1) to material parameters γ_1 and γ_2. We rewrite it as

$$\gamma_1 X + \gamma_2 Y = (e^{-2i\rho\pi}+1)Y = 2i\sin 2\rho\pi\, e^{2i\rho\theta_1}, \tag{49}$$

where

$$X = e^{2i\rho\theta_1} + \rho(1-e^{2i\theta_1}) - 1; \quad Y = (e^{2i\rho\pi}-1)\,e^{2i\rho\theta_1}. \tag{50}$$

Thus, when $\operatorname{Im}(XY^*) \neq 0$, the unique solution of γ_1 and γ_2 is

$$\gamma_1 = -4\sin 2\rho\pi \sin^2\rho\pi/\operatorname{Im}(XY^*), \tag{51}$$

$$\gamma_2 = (1+\cos 2\rho\pi) + \sin 2\rho\pi\, \operatorname{Re}(XY^*)/\operatorname{Im}(XY^*), \tag{52}$$

where

$$XY^* = (e^{-2i\rho\pi}-1)[1 + (\rho-1)\,e^{-2i\rho\theta_1} - \rho\, e^{2i(1-\rho)\theta_1}], \tag{53}$$

$$\begin{aligned}\operatorname{Re}(XY^*) = {} & (\cos 2\rho\pi - 1) + (1-\rho)[\cos 2\rho\theta_1 - \cos 2\rho(\pi+\theta_1)] \\ & + \rho[\cos 2(1-\rho)\theta_1 - \cos 2(\theta_1 - \rho(\pi+\theta_1))],\end{aligned} \tag{54}$$

and

$$\mathrm{Im}(XY^*) = -\sin 2\rho\pi + (\rho - 1)[\sin 2\rho\theta_1 - \sin 2\rho(\pi + \theta_1)] + \rho[\sin 2(1-\rho)\theta_1 - \sin 2(\theta_1 - \rho(\pi + \theta_1))]. \quad (55)$$

On the other hand, if $\mathrm{Im}(XY^*) = 0$, the necessary and sufficient condition for the existence of a solution is that

$$\rho = 1/2. \quad (56)$$

For $\rho = 1/2$, we have

$$X = e^{i\theta_1} - \tfrac{1}{2}e^{2i\theta_1} - \tfrac{1}{2}, \quad Y = -2\,e^{i\theta_1}, \quad (57)$$

and the complex eq. (49) reduces to the following real relation

$$2\gamma_2 = (1 - \cos\theta_1)\gamma_1, \quad (58)$$

or

$$\gamma_2 = \gamma_1 \sin^2\theta_1/2 = \gamma_1 \cos^2[(\pi - \theta_1)/2], \quad (59)$$

which determines θ_1. When $\gamma_2 = \gamma_1$, $\pi - \theta_1 = 0$, and when $\gamma_2 = 0$ but $\gamma_1 \neq 0$, $\theta_1 = 0$.

In summary, when $\mu_1 = \mu_2$, there always exists a solution with the singularity index $\rho = 1/2$ with the locking region located in the domain 1 with $v = v_1 \leqslant v_2$, and the angle $(\pi - \theta_1)$ subtended by the locking region is given by eqs (58) and (59). The value of θ_1 is determined by the values of Poisson's ratio for the two materials.

3.1.2. *The domain is divided into four regions.* On the interface between the unconstrained and locking regions $\sigma_{\alpha\alpha} = 0$ as $r \to 0$, and the sign of $\sigma_{\alpha\alpha}$ is determined by

$$\mathrm{Re}\,\phi \sim \cos[(\rho - 1)\theta + \omega], \quad -\pi \leqslant \theta \leqslant \pi. \quad (60)$$

Thus the angle $\Delta\theta$ between two adjacent interboundaries must satisfy

$$|(\rho - 1)\,\Delta\theta| = \pi, \quad (61)$$

which implies that for a $r - \theta$ separable solution to exist, the domain can be divided into at most four regions. As before we assume that $\mu_1 = \mu_2$.

Equations (23)–(30) give the following eigenequation for A.

$$A\rho[c\,e^{-2i\theta_2} - a\,e^{-2i\theta_1} + a - c] + \bar{A}[c\,e^{-2i\rho\theta_2} - a\,e^{-2i\rho\theta_1} + (b + a - 1) - (c + b)\,e^{2i\rho\pi} + e^{-2i\rho\pi}] = 0, \quad (62)$$

where

$$a = (\kappa_{12} - \kappa_{11})/(\kappa_{12} + 1), \quad b = (1 + \kappa_{11})/(\kappa_{21} + 1), \quad c = [(1 + \kappa_{11})/(\kappa_{22} + 1) - b]. \quad (63)$$

Substitution from eqs (32), (34) and (36) into eq. (62) yields the following complex eigenequation to determine ρ and θ_1.

$$\rho\,e^{2i(1-\rho)\theta_1}[c\,e^{-2i\theta_2} - a\,e^{-2i\theta_1} + a - c] = c\,e^{-2i\rho\theta_2} - a\,e^{-2i\rho\theta_1} + (b + a - 1) - (c + b)\,e^{2i\rho\pi} + e^{-2i\rho\pi}. \quad (64)$$

The conditions (31) require that

$$(\sigma_{11} + \sigma_{22}) \sim \mathrm{Re}(\phi') \sim \cos[(\rho - 1)\theta + \omega] > 0 \text{ inside the elastic domain, and}$$
$$\sim \cos[(\rho - 1)\theta + \omega] < 0 \text{ inside the locking domain.} \quad (65)$$

Thus,

$$(1 - \rho)(\theta_1 - \theta_2) = \pi \quad (66)$$

and the admissible solution must satisfy

$$-\pi \leqslant \theta_2 < \theta_1 \leqslant \pi \quad \text{and} \quad \rho \leqslant 1/2. \quad (67)$$

Using eq. (66), eq. (64) becomes

$$\rho\, e^{2i(1-\rho)\theta_1}(a-c) = (1-\rho)c\, e^{-2i\rho\theta_2} + (\rho-1)a\, e^{-2i\rho\theta_1} + (b+a-1) - (c+b)\, e^{2i\rho\pi} + e^{-2i\rho\pi}. \tag{68}$$

For the special case of $\cos(\rho-1)\theta_1 = 0$ (thus $I = 0$, $(1-\rho)\theta_1 = \pi/2$ and $\theta_2 = -\theta_1$), eq. (68) reduces to

$$c(\rho-1)\, e^{2i\theta_1} - a(\rho-1)\, e^{-2i\theta_1} + \rho(a-c) + (b+a-1) - (c+b)\, e^{2i\rho\pi} + e^{-2i\rho\pi} = 0. \tag{69}$$

The other special case $\sin(\rho-1)\theta_1 = 0$ is excluded by eq. (66) and (67). We now distinguish between the following two cases.

(I) $\kappa_{11} = \kappa_{22} = 1, \quad \kappa_{12} = \kappa_1, \quad \kappa_{21} = \kappa_2.$

For this case,

$$a = 1 - 2/(\kappa_1+1), \quad b = 2/(\kappa_2+1), \quad c = 1-b, \tag{70}$$

and eq. (68) reduces to

$$a[\rho(e^{2i(1-\rho)\theta_1} - e^{-2i\rho\theta_1}) + e^{-2i\rho\theta_1} - 1] + c[\rho(e^{-2i\rho\theta_2} - e^{2i(1-\rho)\theta_1}) - e^{-2i\rho\theta_2} + 1] = (e^{-2i\rho\pi} - e^{2i\rho\pi}) = -2i\sin 2\rho\pi. \tag{71}$$

Two conditions for the linear dependence of its real and imaginary parts are

$$\rho[\cos 2(1-\rho)\theta_1 - \cos 2\rho\theta_1] + \cos 2\rho\theta_1 - 1 = 0,$$
$$\rho[\cos 2(1-\rho)\theta_1 - \cos 2\rho\theta_2] + \cos 2\rho\theta_2 - 1 = 0, \tag{72}$$

which imply that $\cos 2\rho\theta_1 = \cos 2\rho\theta_2$. Since $\rho \leqslant 1/2$, we must have $\theta_1 = -\theta_2$ which gives the following contradiction

$$(1-\rho)\cos 2\rho\theta_1 = (1+\rho). \tag{73}$$

Therefore, eq. (71) cannot be reduced to a real equation.

(II) $\kappa_{12} = \kappa_{21} = 1, \quad \kappa_{11} = \kappa_1, \quad \kappa_{22} = \kappa_2.$

We now have

$$a = (1-\kappa_1)/2 = 1-b, \quad b = (1+\kappa_1)/2, \quad c = (1+\kappa_1)/(\kappa_2+1) - b, \tag{74}$$

and eq. (68) becomes

$$a[\rho(e^{2i(1-\rho)\theta_1} - e^{-2i\rho\theta_1}) + e^{-2i\rho\theta_1} - e^{2i\rho\pi}] + c[\rho(e^{-2i\rho\theta_2} - e^{2i(1-\rho)\theta_1}) - e^{-2i\rho\theta_2} + e^{2i\rho\pi}] = (e^{-2i\rho\pi} - e^{2i\rho\pi}) = -2i\sin 2\rho\pi. \tag{75}$$

Two conditions for linear dependence of its real and imaginary parts are

$$\rho[\cos 2(1-\rho)\theta_1 - \cos 2\rho\theta_1] + \cos 2\rho\theta_1 - \cos 2\rho\pi = 0, \tag{76}$$

$$\rho[\cos 2(1-\rho)\theta_1 - \cos 2\rho\theta_2] + \cos 2\rho\theta_2 - \cos 2\rho\pi = 0, \tag{77}$$

which imply that $\theta_1 = -\theta_2$; however, as discussed above this is impossible.

We are unable to study thoroughly the four region case in its complete generality. However, some concrete results can be obtained for the special case of a homogeneous material, and the following two cases.

(III) $\kappa_{11} = \kappa_{22} = 1, \quad \kappa_{12} = \kappa_{21} = \kappa.$

For these values of κ_{11}, κ_{22}, κ_{12} and κ_{21} we have

$$a = c = (\kappa-1)/(\kappa+1) \tag{78}$$

and eq. (71) reduces to

$$a(1-\rho)(e^{-2i\rho\theta_1}-e^{-2i\rho\theta_2})=-2i\sin 2\rho\pi \tag{79}$$

which implies that $\cos 2\rho\theta_1=\cos 2\rho\theta_2$. However, due to $\rho\leqslant 1/2$ and $0\leqslant\theta_1\leqslant\pi$, we must have $\theta_1=-\theta_2$ and $2(1-\rho)\theta_1=\pi$. Thus,

$$a(1-\rho)\sin 2\rho\theta_1=\sin 2\rho\pi \tag{80}$$

and it turns out that the unique solution is $\rho=1/2$, $\theta_1=\pi$.

(IV) $\kappa_{12}=\kappa_{21}=1$, $\kappa_{11}=\kappa_{22}=\kappa$.

We have

$$a=c=(1-\kappa)/2. \tag{81}$$

Thus, eq. (75) becomes

$$a(1-\rho)(e^{-2i\rho\theta_1}-e^{-2i\rho\theta_2})=-2i\sin 2\rho\pi, \tag{82}$$

which implies that $\theta_1=-\theta_2$, $2(1-\rho)\theta_1=\pi$, and

$$a(1-\rho)\sin 2\rho\theta_1=\sin 2\rho\pi. \tag{83}$$

Again, the unique solution is $\rho=1/2$ and $\theta_1=\pi$.

3.1.2.1. *Homogeneous body*. When $\mu_1=\mu_2$ and $\kappa_1=\kappa_2$, eq. (37) becomes

$$A\rho(e^{-2i\theta_1}-1)+\bar{A}[e^{-2i\rho\theta_1}-\cos 2\rho\pi-i\sin 2\rho\pi/\alpha]=0, \tag{84}$$

and the complex equation for the determination of ρ and θ_1 is

$$\rho(e^{2i(1-\rho)\theta_1}-e^{-2i\rho\theta_1})+e^{-2i\rho\theta_1}-\cos 2\rho\pi-i\sin 2\rho\pi/\alpha=0, \tag{85}$$

where $\alpha=(1-\kappa)/(3+\kappa)$. Thus, two real equations for ρ and θ_1 obtained from eq. (85) are

$$(\rho-1)\cos 2\rho\theta_1-\rho\cos 2(1-\rho)\theta_1=-\cos 2\rho\pi, \tag{86}$$

$$(\rho-1)\sin 2\rho\theta_1+\rho\sin 2(1-\rho)\theta_1=\sin 2\rho\pi/\alpha. \tag{87}$$

Squaring and adding eqs (86) and (87), we obtain

$$4\rho(\rho-1)\sin^2\theta_1=(1-\alpha^2)\sin^2 2\rho\pi/\alpha^2. \tag{88}$$

Since $|\alpha|\leqslant 1/3$, therefore, we conclude that the unique solution of (88) is $\rho=1/2$ and $\theta_1=\pi$.

3.2. *The solution for the single locking domain* ($\mu_1\neq\mu_2$)

We now consider the general case of $\mu_1\neq\mu_2$. Motivated by the results of previous sections, we confine ourselves to the solution of a single locking domain located near the crack-surface, see Fig. 3.

Thus, we assume that $0\leqslant\theta_1\leqslant\pi$ and set

$$\kappa_{11}=1,\quad \kappa_{12}=\kappa_1,\quad \kappa_{22}=\kappa_{21}=\kappa_2. \tag{89}$$

Then, from eqs (23) and (24) we obtain

$$G=M,\quad H=N \tag{90}$$

and the following six conditions to determine six complex constants A, B, E, F, M, N.

$$\bar{A}e^{-2i\rho\pi}+A\rho+B=0, \tag{91}$$

$$\bar{M}e^{2i\rho\pi}+M\rho+N=0, \tag{92}$$

$$2A=E(\kappa_1+1), \tag{93}$$

$$\bar{A}e^{-2i\rho\theta_1}+A\rho e^{-2i\theta_1}+B=\bar{E}e^{-2i\rho\theta_1}+E\rho e^{-2i\theta_1}+F, \tag{94}$$

$$E+\rho\bar{E}+\bar{F}=M+\rho\bar{M}+\bar{N}, \tag{95}$$

$$\mu_1[\kappa_2 M-\rho\bar{M}-\bar{N}]=\mu_2[\kappa_1 E-\rho\bar{E}-\bar{F}]. \tag{96}$$

These equations give

$$M[\mu_2(1-e^{-2i\rho\pi})+\mu_1(\kappa_2+e^{-2i\rho\pi})]=\mu_2(\kappa_1+1)E. \tag{97}$$

Thus, if $\rho \neq 1/2$, M is proportional to E with a non-real factor and eq. (33) no longer holds. This complicates the analysis of the problem. On the other hand, the solution with $\rho = 1/2$ is of some importance; therefore, we seek the solution with $\rho = 1/2$.

Substitution of $\rho = 1/2$ into eq. (97) gives

$$M[2\mu_2+\mu_1(\kappa_2-1)]=\mu_2(\kappa_1+1)E=2\mu_2 A. \tag{98}$$

Since $[2\mu_2+\mu_1(\kappa_2-1)]>0$, $\mu_2(\kappa_1+1)>0$, we still have

$$(\sigma_{11}+\sigma_{22})\sim \mathrm{Re}[\phi']\sim \mathrm{Re}[A\,e^{i(\rho-1)\theta}] \quad \text{for } -\pi\leqslant\theta\leqslant\pi, \tag{99}$$

and thus

$$\bar{A}=-A\,e^{2i(\rho-1)\theta_1}. \tag{100}$$

Substituting from eq. (98) into eqs (91)–(96), we obtain the following eigenequation for A.

$$A(\kappa_1-1)(e^{-2i\theta_1}-1)/[2(1+\kappa_1)]+\bar{A}[(\kappa_1-1)\,e^{-i\theta_1}/(1+\kappa_1)$$
$$+(\kappa_1+3)/(\kappa_1+1)-4\mu_2/[2\mu_2+\mu_1(\kappa_2-1)]]=0, \tag{101}$$

which gives the simple real relationship

$$(\kappa_1-1)\sin^2(\theta_1/2)/(\kappa_1+1)=\mu_1(\kappa_2-1)/[2\mu_2+\mu_1(\kappa_2-1)]. \tag{102}$$

Thus there exists a solution with the domain divided into three regions provided that

$$\mu_1(\kappa_2-1)\leqslant\mu_2(\kappa_1-1), \tag{103}$$

and the corresponding locking region is located in region 1 and its angle $(\pi-\theta_1)$ is determined by eq. (102).

When $\mu_1=\mu_2$, eq. (102) reduces to eqs (58) and (59). On the other hand, if $\kappa_1=\kappa_2=\kappa$, condition (103) becomes $\mu_1\leqslant\mu_2$, and eq. (102) reduces to

$$\mathrm{ctg}^2(\theta_1/2)=2(\mu_2-\mu_1)/[\mu_1(\kappa+1)]. \tag{104}$$

Thus for $\kappa_1=\kappa_2=\kappa$, $\mu_1\leqslant\mu_2$, there always exists a solution with the singularity index $\rho = 1/2$, and the corresponding locking region is located in the domain 1 with $\mu=\mu_1$ and its angle $(\pi-\theta_1)$ is determined by values of material constants μ_1 and μ_2.

4. CONCLUSIONS

We have studied plane strain deformations near a crack tip with the crack in the interface between two isotropic and homogeneous linear elastic bodies under the constraint that the dilatation must be greater than or equal to a certain constant. The region in which the dilatation equals the constant is identified as the locking zone, and in the remaining region the deformation is unconstrained. After having formulated the problem for the general case, we first examined the case when the shear moduli of the two materials are equal to each other but their Poisson's ratios are not. It is shown that there can be at most four regions of which two are locking. For several special combinations of the values of material parameters, including the case of a homogeneous body, it is found that the singularity index equals 1/2. For the case of dissimilar materials and the locking region in one material only, the singularity index is found to equal 1/2 and the size of the locking zone is determined by the values of material parameters.

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