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CONSERVATION LAWS IN LINEAR PIEZOELECTRICITY

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Abstract—We use Noether's theorem on variational principles invariant under a group of infinitesimal transformations to obtain a class of conservation laws for linear piezoelectric materials and linear elastic dielectrics.

1. INTRODUCTION

CONSERVATION laws and related path-independent integrals are important in studying forces acting on a defect [1]. An important tool in deriving the conservation laws is Noether's theorem [2] on variational principles invariant under a group of infinitesimal transformations. Knowles and Sternberg [3] and Fletcher [4] have used this theorem to derive conservation laws in elastostatics and linear elastodynamics respectively. Li [5] used Noether's theorem in conjunction with the variational principle of complementary energy to obtain another set of conservation laws for linear elastostatics which he called the dual conservation laws.

The previously derived conservation laws for piezoelectricity and the more general theory of dielectrics [6–9] are for static deformations of the body. Here we use Noether's theorem in conjunction with four variational formulations of quasistatic piezoelectricity [10] to obtain a class of conservation laws. The quasistatic piezoelectricity theory is defined as one in which inertia forces associated with mechanical deformation are considered but the time-derivative of the magnetic induction in Faraday's law of induction is neglected. These conservation laws represent extensions of Fletcher's and Li's results to quasistatic piezoelectricity. We note that Knowles and Sternberg [3] and Fletcher [4] prove the completeness of conservation laws, but we do not.

2. NOETHER'S THEOREM

We state a version of Noether's theorem appropriate for our work. Consider the following functional or action integral

$$\pi(\mathbf{y}) = \int_V \Sigma\left(\mathbf{x}, \mathbf{y}, \frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right) \mathrm{d}V \tag{1}$$

where y_α, $\alpha = 1, 2, \ldots, N$, are functions of x_i, $i = 1, 2, \ldots, M$, and the integration is over the M-dimensional domain V. The Euler–Lagrange equations associated with the functional π are

$$\frac{\partial \Sigma}{\partial y_\alpha} - \frac{\partial}{\partial x_i}\left(\frac{\partial \Sigma}{\partial y_{\alpha,i}}\right) = 0; \quad y_{\alpha,i} \equiv \frac{\partial y_\alpha}{\partial x_i}. \tag{2}$$

Here and below a repeated index implies summation over the range of the index. Also consider an infinitesimal parametric transformation

$$\begin{aligned} x_i' &= x_i + f_{iK}(\mathbf{x}, \mathbf{y})\epsilon_K + o(\epsilon), \\ y_\alpha' &= y_\alpha + g_{\alpha K}(\mathbf{x}, \mathbf{y})\epsilon_K + o(\epsilon), \end{aligned} \tag{3}$$

where ϵ_K, $K = 1, 2, \ldots, P$ are the infinitesimal parameters of the transformation, and

$$\epsilon = (\epsilon_K, \epsilon_K)^{1/2}, \quad \lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0. \tag{4}$$

If under the transformation (3) the functional (1) is invariant, or more specifically,

$$\int_{V'} \Sigma\left(\mathbf{x}', \mathbf{y}', \frac{\partial \mathbf{y}'}{\partial \mathbf{x}'}\right) \mathrm{d}V' - \int_V \Sigma\left(\mathbf{x}, \mathbf{y}, \frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right) \mathrm{d}V = o(\epsilon), \tag{5}$$

and the Euler–Lagrange eq. (2) is satisfied, then Noether's theorem [2] states that

$$\frac{\partial}{\partial x_i}\left[\left(\sum \delta_{ij} - y_{\alpha,j}\frac{\partial \Sigma}{\partial y_{\alpha,i}}\right) f_{jK} + \frac{\partial \Sigma}{\partial y_{\alpha,i}} g_{\alpha K}\right] = 0. \tag{6}$$

Equation (6) is in a zero divergence form and is called a conservation law in physical problems. These can be transformed into path-independent integrals by using the divergence theorem.

3. GOVERNING EQUATIONS AND ENERGY DENSITY FUNCTIONS FOR QUASISTATIC PIEZOELECTRICITY

Throughout this paper we use rectangular Cartesian coordinates. For a piezoelectric material, the internal energy density U is a function of the infinitesimal strain tensor ϵ_{ij} and the electric displacement D_i. The stress tensor σ_{ij} appropriate for small deformations and the electric field E_i are related to U by

$$\sigma_{ij} = \frac{\partial U}{\partial \epsilon_{ij}}, \quad E_i = \frac{\partial U}{\partial D_i}. \tag{7}$$

Whereas in linear elasticity we have two energy density functions, viz, the strain energy density and the complementary energy density, in linear piezoelectricity, the following three additional energy density functions can be introduced through Legendre transforms [10].

$$H(\epsilon, \mathbf{E}) = U - E_i D_i, \tag{8}$$

$$M(\boldsymbol{\sigma}, \mathbf{D}) = U - \sigma_{ij}\epsilon_{ij}, \tag{9}$$

$$G(\boldsymbol{\sigma}, \mathbf{E}) = U - E_i D_i - \sigma_{ij}\epsilon_{ij}. \tag{10}$$

The corresponding constitutive relations are given by

$$\sigma_{ij} = \frac{\partial H}{\partial \epsilon_{ij}}, \qquad D_i = -\frac{\partial H}{\partial E_i}, \tag{11}$$

$$\epsilon_{ij} = -\frac{\partial M}{\partial \sigma_{ij}}, \quad E_i = \frac{\partial M}{\partial D_i}, \tag{12}$$

$$\epsilon_{ij} = -\frac{\partial G}{\partial \sigma_{ij}}, \quad D_i = -\frac{\partial G}{\partial E_i}. \tag{13}$$

Equations governing the infinitesimal deformations of a piezoelectric body are

$$\sigma_{ji,j} = \rho \ddot{u}_i, \quad D_{i,i} = 0, \tag{14}$$

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad E_i = -\phi_{,i}, \tag{15}$$

where $\mathbf{u}$ is the mechanical displacement and ϕ the electric potential. Equation $(14)_1$ expresses the balance of linear momentum and eq. $(14)_2$ is the Gauss equation. A theory in which one considers the inertia forces associated with mechanical deformations but neglects time-derivatives of the magnetic induction in Faraday's law of induction is usually referred to as quasistatic piezoelectricity.

4. VARIATIONAL PRINCIPLES FOR QUASISTATIC PIEZOELECTRICITY

Corresponding to each energy density function, there is a variational principle for quasistatic piezoelectricity [10]. We list below the functional and the corresponding stationary conditions

$$\pi_1(\mathbf{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}, \phi, \mathbf{E}, \mathbf{D}) = \int_{t_0}^{t} \mathrm{d}\tau \int_V \Sigma_1 \,\mathrm{d}V, \tag{16}$$

$$\Sigma_1 = H(\boldsymbol{\epsilon}, \mathbf{E}) - \sigma_{ij}(\epsilon_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i})) + D_i(E_i + \phi_{,i}) - \tfrac{1}{2}\rho \dot{u}_i \dot{u}_i, \tag{17}$$

$$\begin{aligned} &\sigma_{ji,j} = \rho \ddot{u}_i, \qquad D_{i,i} = 0, \\ &\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad E_i = -\phi_{,i}, \\ &\sigma_{ij} = \frac{\partial H}{\partial \epsilon_{ij}}, \qquad D_i = -\frac{\partial H}{\partial E_i}; \end{aligned} \tag{18}$$

$$\pi_2(\mathbf{u}, \boldsymbol{\sigma}, \phi, \mathbf{D}) = \int_{t_0}^{t} \mathrm{d}\tau \int_V \Sigma_2 \,\mathrm{d}V, \tag{19}$$

$$\Sigma_2 = M(\boldsymbol{\sigma}, \mathbf{D}) - \sigma_{ji,j}u_i - D_{i,i}\phi - \tfrac{1}{2}\rho \dot{u}_i \dot{u}_i, \tag{20}$$

$$\begin{aligned} &\sigma_{ji,j} = \rho \ddot{u}_i, \quad D_{i,i} = 0, \\ &\tfrac{1}{2}(u_{i,j} + u_{j,i}) = -\frac{\partial M}{\partial \sigma_{ij}}, \quad \phi_{,i} = -\frac{\partial M}{\partial D_i}; \end{aligned} \tag{21}$$

or

$$\pi_3(\mathbf{u}, \boldsymbol{\sigma}, \phi, \mathbf{E}, \mathbf{D}) = \int_{t_0}^{t} \mathrm{d}\tau \int_V \Sigma_3 \,\mathrm{d}V, \tag{22}$$

$$\Sigma_3 = G(\boldsymbol{\sigma}, \mathbf{E}) - \sigma_{ji,j}u_i + D_i(E_i + \phi_{,i}) - \tfrac{1}{2}\rho \dot{u}_i \dot{u}_i, \tag{23}$$

$$\sigma_{ji,j} = \rho \ddot{u}_i, \quad D_{i,i} = 0, \quad E_i = -\phi_{,i}, \tag{24}$$

$$\tfrac{1}{2}(u_{i,j} + u_{j,i}) = -\frac{\partial G}{\partial \sigma_{ij}}, \quad D_i = -\frac{\partial G}{\partial E_i};$$

$$\pi_4(\mathbf{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}, \phi, \mathbf{D}) = \int_{t_0}^{t} \mathrm{d}\tau \int_V \Sigma_4 \,\mathrm{d}V, \tag{25}$$

$$\Sigma_4 = U(\boldsymbol{\epsilon}, \mathbf{D}) - \sigma_{ij}(\epsilon_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i})) - D_{i,i}\phi - \tfrac{1}{2}\rho \dot{u}_i \dot{u}_i, \tag{26}$$

$$\begin{aligned} &\sigma_{ji,j} = \rho \ddot{u}_i, \quad D_{i,i} = 0, \\ &\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad \sigma_{ij} = \frac{\partial U}{\partial \epsilon_{ij}}, \quad \phi_{,i} = -\frac{\partial U}{\partial D_i}. \end{aligned} \tag{27}$$

These mixed variational principles are generalizations of the Hellinger–Reissner and the Hu–Washizu principles in linear elasticity. Each principle gives a different but equivalent set of equations as stationary conditions. The details of deriving the stationary conditions have been omitted and can be found in the cited references, e.g. see [3, 4]. Functions Σ_1, Σ_2, Σ_3, and Σ_4 depend implicity upon time t through the dependence of their arguments on t.

5. INVARIANCE UNDER TRANSLATIONS

For a homogeneous piezoelectric body, energy density functions U, H, M, and G do not depend explicitly upon the Cartesian coordinates $\mathbf{x}$, and the functionals π_1, π_2, π_3, and π_4 are invariant under the translational transformation

$$x_i' = x_i + \epsilon, \quad t' = t, \tag{28}$$

since

$$u_i' = u_i, \quad \epsilon_{ij}' = \epsilon_{ij}, \quad \sigma_{ij}' = \sigma_{ij},$$

$$\phi' = \phi, \quad E_i' = E_i, \quad D_i' = D_i, \tag{29}$$

under the transformation (28). The application of Noether's theorem (6)–(17), (20), (23) and (26) with $\mathbf{x} = (x_1, x_2, x_3, t)$, $\mathbf{y} = (\mathbf{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}, \phi, \mathbf{E}, \mathbf{D})$ for (17), $\mathbf{y} = (\mathbf{u}, \boldsymbol{\sigma}, \phi, \mathbf{D})$ for (20), $\mathbf{y} = (\mathbf{u}, \boldsymbol{\sigma}, \phi, \mathbf{E}, \mathbf{D})$ for (23) and $\mathbf{y} = (\mathbf{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}, \phi, \mathbf{D})$ for (26) gives the following set of conservation laws

$$\frac{\partial f_i}{\partial t} + \frac{\partial g_{ik}^{(\alpha)}}{\partial x_k} = 0, \quad \alpha = 1, 2, 3, 4, \tag{30}$$

where

$$f_i = \rho \dot{u}_j u_{j,i},$$

$$g_{ik}^{(1)} = \Sigma_1 \delta_{ik} - u_{j,i}\sigma_{jk} - \phi_{,i} D_k,$$

$$g_{ik}^{(2)} = \Sigma_2 \delta_{ik} + \sigma_{kj,i} u_j + D_{k,i}\phi,$$

$$g_{ik}^{(3)} = \Sigma_3 \delta_{ik} + \sigma_{kj,i} u_j - \phi_{,i} D_k,$$

$$g_{ik}^{(4)} = \Sigma_4 \delta_{ik} - u_{j,i}\sigma_{jk} + D_{k,i}\phi, \tag{31}$$

and δ_{ij} is the Kronecker delta. These conservation laws can be verified by direct differentiation, and reduce to Fletcher's [4] results when quantities associated with the electric field are neglected, and the Li's [5] results when dynamic terms are also ignored.

6. INVARIANCE UNDER ROTATIONS

For a homogeneous and isotropic material, the energy density functions are isotropic functions. For example, the energy density function H satisfies

$$H(\mathbf{Q}\boldsymbol{\epsilon}\mathbf{Q}^T, \mathbf{QE}) = H(\boldsymbol{\epsilon}, \mathbf{E}) \tag{32}$$

for any orthogonal tensor $\mathbf{Q}(\epsilon)$ satisfying $\mathbf{Q}(0) = \mathbf{1}$. The functions Σ_1, Σ_2, Σ_3, and Σ_4 are invariant under the transformations

$$x_i' = Q_{ij} x_j, \quad t' = t,$$

$$u_i' = Q_{ij} u_j, \quad \epsilon_{ij}' = Q_{im} Q_{jn} \epsilon_{mn}, \quad \sigma_{ij}' = Q_{im} Q_{jn} \sigma_{mn},$$

$$\phi' = \phi, \quad E_i' = Q_{ij} E_j, \quad D_i' = Q_{ij} D_j. \tag{33}$$

Noether's theorem (6) when applied to (17), (20), (23), and (26), with $\mathbf{x}$ and $\mathbf{y}$ in (6) identified as in the previous section, gives the following set of conservation laws.

$$e_{ijk}\frac{\partial h_{jk}}{\partial t} + \frac{\partial q_{ik}^{(\alpha)}}{\partial x_k} = 0, \quad \alpha = 1, 2, 3, 4, \tag{34}$$

where

$$h_{jk} = \rho e_{ijk}(x_j \dot{u}_m u_{m,k} + \dot{u}_j u_k),$$

$$q_{ik}^{(1)} = e_{imj}(x_m \Sigma_1 \delta_{jk} - x_m u_{l,j}\sigma_{lk} - x_m \phi_{,j} D_k + \sigma_{jk} u_m),$$

$$q_{ik}^{(2)} = e_{imj}(x_m \Sigma_2 \delta_{jk} + x_m \sigma_{kl,j} u_l + x_m D_{k,j}\phi + \sigma_{lj} u_l \delta_{mk} + \sigma_{kj} u_m - \delta_{jk}\phi D_m),$$

$$q_{ik}^{(3)} = e_{imj}(x_m \Sigma_3 \delta_{jk} + x_m \sigma_{kl,j} u_l - x_m \phi_{,j} D_k + \sigma_{lj} u_l \delta_{mk} + \sigma_{kj} u_m),$$

$$q_{ik}^{(4)} = e_{imj}(x_m \Sigma_4 \delta_{jk} - x_m u_{l,j}\sigma_{lk} + x_m D_{k,j}\phi + \sigma_{jk} u_m - \delta_{jk} D_m \phi), \tag{35}$$

and e_{ijk} is the permutation symbol. These conservation laws also reduce to Fletcher's [4] results for linear elastodynamics and to Li's [5] results for linear elastostatics.

7. INVARIANCE UNDER CHANGES OF SCALE

For homogeneous linear piezoelectric body, the energy density functions are quadratic functions of their arguments. The functions Σ_1, Σ_2, Σ_3, and Σ_4 are invariant under the scale change

$$x_1' = e^{\epsilon} x_i, \quad t' = e^{\epsilon} t, \quad u_i' = e^{-\epsilon} u_i, \quad \phi' = e^{-\epsilon}\phi, \tag{36}$$

$$\epsilon_{ij}' = e^{-2\epsilon}\epsilon_{ij}, \quad E_i' = e^{-2\epsilon} E_i,$$

$$\sigma_{ij}' = e^{-2\epsilon}\sigma_{ij}, \quad D_i' = e^{-2\epsilon} D_i. \tag{37}$$

By using Noether's theorem (6) we obtain

$$\frac{d\tilde{f}^{(\alpha)}}{\partial t} + \frac{\partial \tilde{g}_k^{(\alpha)}}{\partial x_k} = 0, \quad \alpha = 1, 2, 3, 4, \tag{38}$$

where

$$\tilde{f}^{(\alpha)} = \rho \dot{u}_j (u_j + x_m u_{j,m} + t\dot{u}_j) + t\Sigma_\alpha,$$

$$\tilde{g}_k^{(1)} = x_k \Sigma_1 - \sigma_{jk}(u_j + x_m u_{j,m} + t\dot{u}_j) - D_k(\phi + x_m \phi_{,m} + t\dot{\phi})$$

$$\tilde{g}_k^{(2)} = x_k \Sigma_2 + u_j(2\sigma_{jk} + x_m \sigma_{jk,m} + t\dot{\sigma}_{jk}) + \phi(2D_k + x_m D_{k,m} + t\dot{D}_k),$$

$$\tilde{g}_k^{(3)} = x_k \Sigma_3 + u_j(2\sigma_{jk} + x_m \sigma_{jk,m} + t\dot{\sigma}_{jk}) - D_k(\phi + x_m \phi_{,m} + t\dot{\phi}),$$

$$\tilde{g}_k^{(4)} = x_k \Sigma_4 - \sigma_{jk}(u_j + x_m u_{j,m} + t\dot{u}_j) + \phi(2D_k + x_m D_{k,m} + t\dot{D}_k). \tag{39}$$

These conservation laws can be verified by carrying out the indicated differentiations, and they also reduce to Fletcher's [4] results when quantities associated with the electric field are neglected. However, these cannot be reduced to Li's [5] results for linear elastostatics because the scale change for the static case [9] is different from that given in (32).

For static problems the functionals are invariant under the following scale change

$$x_i' = e^{-2\epsilon} x_i, \qquad u_i' = e^{\epsilon} u_i, \quad \phi' = e^{\epsilon}\phi \tag{40}$$

which results in

$$\epsilon_{ij}' = e^{3\epsilon}\epsilon_{ij}, \quad E_i' = e^{3\epsilon} E_i, \quad \sigma_{ij}' = e^{3\epsilon}\sigma_{ij}, \quad D_i' = e^{3\epsilon} D_i. \tag{41}$$

The application of Noether's theorem (6) with $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (\mathbf{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}, \phi, \mathbf{E}, \mathbf{D})$ for (17), $\mathbf{y} = (\mathbf{u}, \boldsymbol{\sigma}, \phi, \mathbf{D})$ for (20), $\mathbf{y} = (\mathbf{u}, \boldsymbol{\sigma}, \phi, \mathbf{E}, \mathbf{D})$ for (23) and $\mathbf{y} = (\mathbf{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}, \phi, \mathbf{D})$ for (26) results in the following conservation laws

$$\frac{\partial}{\partial x_i}(x_i\Sigma_1 - x_j u_{k,j}\sigma_{ki} - x_j D_i\phi_{,j} - \tfrac{1}{2}\sigma_{ji}u_j - \tfrac{1}{2}D_i\phi) = 0,$$

$$\frac{\partial}{\partial x_i}(x_i\Sigma_2 + x_j\sigma_{ik,j}u_k + x_j D_{i,j}\phi + \tfrac{3}{2}\sigma_{ji}u_j + \tfrac{3}{2}D_i\phi) = 0,$$

$$\frac{\partial}{\partial x_i}(x_i\Sigma_3 + x_j u_{ik,j}u_k - x_j D_i\phi_{,j} + \tfrac{3}{2}\sigma_{ji}u_j - \tfrac{1}{2}D_i\phi) = 0,$$

$$\frac{\partial}{\partial x_i}(x_i\Sigma_4 - x_j u_{k,j}\sigma_{ki} + x_j D_{i,j}\phi - \tfrac{1}{2}\sigma_{ji}u_j + \tfrac{3}{2}D_i\phi) = 0 \tag{42}$$

which reduce to Li's [5] results when quantities associated with the electric field are neglected.

8. ELASTIC DIELECTRICS WITH POLARIZATION GRADIENT

We now study deformations of a homogeneous linear elastic dielectric material for which the energy density W of deformation and polarization depends upon the infinitesimal strain tensor ϵ_{ij}, the polarization vector P_i and its gradients $P_{i,j}$. Mindlin [11] generalized Toupin's [12] theory of piezoelectricity by incorporating the dependence of the energy density of deformation and polarization upon the gradients $P_{i,j}$ of the polarization vector P_i and studied linear elastic dielectrics. Governing equations for such a material in the absence of external body forces are [11]

$$\sigma_{ij,i} = \rho\ddot{u}_j,$$
$$\bar{E}_j + E_{ij,i} - \phi_{,j} = 0,$$
$$-e_0\phi_{,ii} + P_{i,i} = 0, \tag{43}$$

where

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} = \sigma_{ji}, \quad \bar{E}_i = -\frac{\partial W}{\partial P_i}, \quad E_{ij} \equiv \frac{\partial W}{P_{j,i}}, \tag{44}$$

and e_0 is the permittivity of a vacuum. For W, Mindlin [11] proposed the following expression,

$$W = b^0_{ij}P_{j,i} + \tfrac{1}{2}a_{ij}P_iP_j + \tfrac{1}{2}b_{ijkl}P_{j,i}P_{l,k} + \tfrac{1}{2}C_{ijkl}\epsilon_{ij}\epsilon_{kl} + d_{ijkl}P_{j,i}\epsilon_{kl} + f_{ijk}\epsilon_{ij}P_k + g_{ijk}P_iP_{k,j}. \tag{45}$$

We note that eqs (43) are Euler–Lagrange equations for the functional

$$\pi(\epsilon, \mathbf{P}, \phi) = \int_0^t \mathrm{d}\tau \int \Sigma_5\,\mathrm{d}V, \tag{46}$$

where

$$\Sigma_5(\epsilon, \boldsymbol{P}, \phi) = W(\epsilon, \mathbf{P}, \phi) - \tfrac{1}{2}e_0\phi_{,i}\phi_{,i} + \phi_{,i}P_i - \tfrac{1}{2}\rho\dot{u}_i\dot{u}_i. \tag{47}$$

The invariance under translations (28), rotations (33) and the application of Noether's theorem (6) with $\mathbf{x} = (x_1, x_2, x_3, t)$ and $\mathbf{y} = (\mathbf{u}, \mathbf{P}, \phi)$ give

$$\frac{\partial}{\partial t}(\rho\dot{u}_j u_{j,i}) + \frac{\partial}{\partial x_k}(\Sigma_5\,\delta_{ik} - u_{j,i}\sigma_{jk} - \phi_{,i}(P_k - e_0\phi_{,k}) - P_{j,i}E_{kj}) = 0, \tag{48}$$

$$\frac{\partial}{\partial t}e_{ijk}\rho(x_j\dot{u}_m u_{m,k} + \dot{u}_j u_k) + \frac{\partial}{\partial x_k}e_{imj}(x_m\Sigma_5\,\delta_{jk} - x_m u_{l,j}\sigma_{lk} - x_m\phi_{,j}(P_k - e_0\phi_{,k})$$
$$- x_m P_{l,j}E_{kl} + \sigma_{jk}u_m + E_{kj}P_m) = 0. \tag{49}$$

These conservation laws can be verified by performing the indicated differentiation and they reduce to conservation laws for static problems once the time dependent terms are omitted. For static problems, there is no need to perform the integration with respect to time in eq. (46).

9. CONCLUSIONS

We have derived a class of conservation laws for linear piezoelectric materials by using Noether's theorem in conjunction with the invariance of various functionals under translations, rotations, and changes of scale. Our results generalize conservation laws for linear elastodynamics derived by Fletcher to linear piezoelectricity and Mindlin's elastic dielectrics.

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