A CRACK AT THE INTERFACE BETWEEN A KANE-MINDLIN PLATE AND A RIGID SUBSTRATE

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Abstract—The interface fracture of an elastic plate bonded to a rigid substrate is studied using Kane and Mindlin's kinematic assumptions for the quasi-three-dimensional (3-D) deformations of plates deformed in stretching. The stresses and deformations are computed for a semi-infinite plate perfectly bonded to a rigid substrate and subjected to uniform in-plane normal tractions at infinity. These agree well with those obtained from the 3-D elasticity theory when the latter are averaged over the plate thickness. The Kane–Mindlin theory predicts a boundary layer adherent to the interface between the plate and the substrate and its thickness approximately equals the plate thickness. An interface crack between the elastic plate and the rigid substrate is investigated. The crack front average stress fields consist of a plane strain (not plane stress) oscillatory singular field and an antiplane shear inverse-square-root singular field, and the three fracture modes are coupled. These agree with the existing 3-D finite element results. The effect of the plate thickness on stress intensity factors and phase angles is studied. The antiplane shear stress intensity factor approaches zero for a vanishingly thin plate, but cannot be ignored otherwise. A path-independent integral including the thickness effects is deduced and is used to establish a fracture criterion for thin plates. Finally, the interface fracture criterion is discussed within the framework of the Kane–Mindlin theory. © 1997 Published by Elsevier Science Ltd

Keywords—interface fracture, stress intensity factor, path-independent integral, fracture criterion.

1. INTRODUCTION

IT IS well known that the plane stress assumptions, i.e. the out-of-plane stresses are negligible as compared to the in-plane ones, can be used to study deformations of thin plates under in-plane loads. Equations governing the two in-plane displacements or the Airy stress function are then deduced, and the out-of-plane displacement is obtained from the in-plane stresses by using the constitutive relations. Generally speaking, the plane stress theory is valid for a plate with thickness of at least an order of magnitude smaller than a characteristic in-plane dimension. When interfaces between dissimilar materials are involved, we are particularly interested in the interface region as fracture often ensues there. The plane stress theory, however, usually predicts a jump discontinuity in the out-of-plane displacement at the interface, which violates the continuity requirements. It is expected that significant out-of-plane stresses exist at the interface region. Hence, the plane stress theory is futile in studying such problems. When a plate is subjected to dynamic loads, the plane stress theory will also fail even for homogeneous materials, when the incident wavelength is of the order of the plate thickness. It seems that a full three-dimensional (3-D) analysis is inevitable for the thin plate interface crack problems. However, a complete 3-D analytical study is complex and a quasi-3-D analysis may accurately predict quantities of significant interest. Kane and Mindlin[1] proposed a quasi-3-D theory to study vibrations of an elastic plate. In this theory, the out-of-plane displacement is independent of the in-plane deformations and the final governing equations are of order six instead of four, as in the classical elasticity. This requires three conditions on the boundary, which are consistent with the 3-D plate theory. The Kane–Mindlin (K–M) theory has been used to study static and dynamic crack problems for thin plates[2–6]. It is also expected that the behavior of interface cracks in thin plates can be addressed with the K–M theory in terms of quantities averaged over the plate thickness.

In this paper, a crack at the interface between a semi-infinite elastic plate and a rigid substrate is investigated by using the K–M theory [1]. First, the plate-substrate system without a crack and subjected to in-plane tension at infinity is studied. The stresses and deformations obtained from the K–M theory are compared with the full elasticity solution [7]. Then, an interface crack problem is investigated. The interface crack front average stress fields and their oscil-

latory behavior are determined and the effect of thickness on the stress intensity factors and the phase angles is studied. A path-independent integral including the thickness effects is deduced, which may be used to establish the fracture criterion of thin plates. Finally, the interface fracture criterion is discussed in the framework of the K–M theory.

2. PLATE EQUATIONS BASED ON THE KANE–MINDLIN ASSUMPTIONS

Consider a plate of thickness $2h$ and denote by (x_1, x_2, x_3) the rectangular Cartesian coordinate system, with $x_3 = \pm h$ describing the plate surfaces. The plate is subjected to symmetrical loads about the plane $x_3 = 0$ (the antisymmetrical loads will cause bending, which is not considered in this study). Kane and Mindlin[1] proposed the following assumptions on the displacement fields in the plate

$$u_1(x_1, x_2, x_3) = v_1(x_1, x_2), \quad u_2(x_1, x_2, x_3) = v_2(x_1, x_2), \quad u_3(x_1, x_2, x_3) = (x_3/h)w(x_1, x_2). \quad (1)$$

Here **u** denotes displacement of a point and $w(x_1, x_2)$ is the out-of-plane displacement of points on the surface $x_3 = h$.

Introduce the following stress and strain resultants

$$\{N_{\alpha\beta}(x_1, x_2), N_{33}(x_1, x_2)\} = \frac{1}{2h} \int_{-h}^{h} \{\sigma_{\alpha\beta}(x_1, x_2, x_3), \sigma_{33}(x_1, x_2, x_3)\} dx_3$$

$$\{\gamma_{\alpha\beta}(x_1, x_2), \gamma_{33}(x_1, x_2)\} = \frac{1}{2h} \int_{-h}^{h} \{\varepsilon_{\alpha\beta}(x_1, x_2, x_3), \varepsilon_{33}(x_1, x_2, x_3)\} dx_3$$

$$\{R_\alpha(x_1, x_2), \Gamma_\alpha(x_1, x_2)\} = \frac{1}{2h} \int_{-h}^{h} x_3 \{\sigma_{\alpha 3}(x_1, x_2, x_3), 2\varepsilon_{\alpha 3}(x_1, x_2, x_3)\} dx_3 \quad (2)$$

where σ_{ij} and ϵ_{ij} are the components of the stress and infinitesimal strain tensors, respectively, indices i and j take values 1, 2, 3 and Greek indices, α and β, have the range 1 and 2. The plate equations in the framework of the K–M theory are

$$N_{\alpha\beta,\beta} = 0, \, R_{\alpha,\alpha} - N_{33} = 0$$

$$(\gamma_{\alpha\beta}, \gamma_{33}) = \frac{1+\nu}{E}(N_{\alpha\beta}, N_{33}) - \frac{\nu}{E} N_{kk}(\delta_{\alpha\beta}, 1)$$

$$\Gamma_\alpha = \frac{1}{\mu} R_\alpha$$

$$\gamma_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}), \, \gamma_{33} = \frac{w}{h}, \, \Gamma_\alpha = \frac{h}{3} w_{,\alpha} \quad (3)$$

where E is Young's modulus, ν is Poisson's ratio, μ is the shear modulus, a comma followed by an index i implies partial differentiation with respect to x_i, $\delta_{\alpha\beta}$ is the Kronecker delta, and a repeated index implies summation over the range of the index. The equations governing the Airy stress function Φ and the out-of-plane displacement w are[5]

$$\nabla^2 \left\{ \frac{(1-\nu^2)}{E\nu} \nabla^2 \Phi - \frac{w}{h} \right\} = 0,$$

$$\frac{h}{6(1+\nu)} \nabla^2 w - \frac{w}{h} - \frac{\nu}{E} \nabla^2 \Phi = 0 \quad (4)$$

where ∇^2 is the two-dimensional (2-D) Laplace operator. In terms of the non-dimensional variable Ψ[5]

$$\Psi = \frac{(1 - \nu^2)}{E\nu} \nabla^2 \Phi - \frac{w}{h} \tag{5}$$

eqs (4) and (5) can be written as

$$\nabla^2 \Psi = 0$$

$$\frac{1 - \nu^2}{E\nu} \nabla^2 \Phi - \frac{w}{h} - \Psi = 0$$

$$\delta^2 \nabla^2 \left(\frac{w}{h} \right) - \frac{w}{h} - \nu^2 \Psi = 0 \tag{6}$$

where

$$\delta = \sqrt{\frac{1 - \nu}{6}} h. \tag{7}$$

The stress resultants and in-plane displacements are related to Φ, Ψ and w as follows

$$N_{\alpha\beta} = \nabla^2 \Phi \delta_{\alpha\beta} - \Phi_{,\alpha\beta}$$

$$N_{33} = E \left(\frac{w}{h} + \nu^2 \Psi \right) / (1 - \nu^2)$$

$$R_\alpha = \frac{\mu h}{3} w_{,\alpha} \tag{8}$$

and

$$v_{1,1} = -\frac{(1 + \nu)}{E} \Phi_{,11} + \nu\Psi$$

$$v_{2,2} = -\frac{(1 + \nu)}{E} \Phi_{,22} + \nu\Psi$$

$$v_{1,2} + v_{2,1} = -\frac{2(1 + \nu)}{E} \Phi_{,12}. \tag{9}$$

Equations (6), (8) and (9) imply that the K–M theory includes some 3-D effects. It allows for three conditions on the boundary to be specified, which is consistent with the 3-D elasticity theory. These features make it possible to determine 3-D stresses and deformations, though some results should be interpreted as average over the plate thickness. It can also be seen from eq. (6) that the K–M theory exhibits a boundary layer for small values of the parameter δ. Since the plane stress equations can be recovered by letting $\delta \to 0$ in eqs (6)–(9), therefore, the plane stress theory prevails outside the boundary layer.

3. AN APPLICATION OF THE KANE–MINDLIN THEORY

As an application of the K–M theory, we consider a semi-infinite plate perfectly bonded to a rigid substrate along the x_1-axis as shown in Fig. 1(a). The plate is simply-supported at $x_1 \to \infty$ and loaded uniformly by a tensile traction $\sigma_0 = N_0$ at $x_2 \to \infty$. The solution of the plane stress problem is

$$\sigma_{22} = \sigma_0, \sigma_{11} = \nu\sigma_0, \sigma_{12} = 0 \tag{10}$$

$$u_1 = 0, \quad u_2 = \frac{1 - \nu^2}{E} \sigma_0 x_2, \quad u_3 = -\frac{\nu(1 + \nu)}{E} \sigma_0 x_3. \tag{11}$$
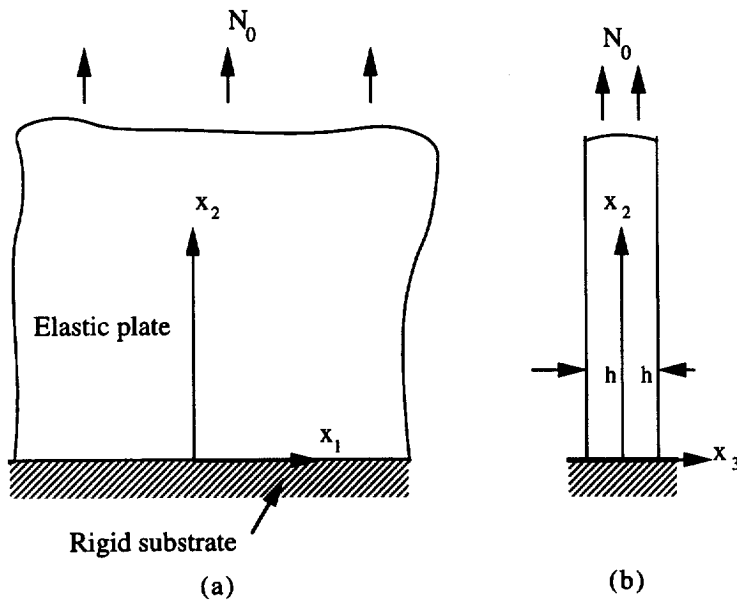
Fig. 1. A semi-infinite plate on a rigid substrate subjected to remote tension.

It is clear from eq. (11)$_3$ that the out-of-plane displacement does not vanish everywhere at the interface $x_2 = 0$, between the plate and the rigid substrate, except at the point $x_3 = 0$. This violates the requirement that $u_3 = 0$ at $x_2 = 0$. Hence, the plane stress theory fails to give a correct solution near the interface.

Now we solve the problem in the framework of the K–M theory by assuming that the solution is independent of x_1. The governing eq. (4) reduces to

$$\frac{1 - v^2}{Ev}\varphi'''' - \left(\frac{w}{h}\right)'' = 0$$

$$\frac{h^2}{6(1 + v)}\left(\frac{w}{h}\right)'' - \frac{w}{h} - \frac{v}{E}(N_0 + \phi'') = 0 \tag{12}$$

and the Airy stress function has the form

$$\Phi = \frac{1}{2}N_0 x_1^2 + \varphi(x_2). \tag{13}$$

In eq. (12), a prime denotes differentiation with respect to x_2.

The solution of eqs (12) and (13) satisfying the traction conditions at $x_2 \rightarrow \infty$ and zero displacements at $x_2 = 0$ is

$$N_{11}/(vN_0) = 1 + \frac{v}{1 - v}e^{-x_2/\delta}, \quad N_{22}/N_0 = 1, \quad N_{12} = 0,$$

$$N_{33}/N_0 = \frac{v}{1 - v}e^{-x_2/\delta}, \quad R_1 = 0$$

$$R_2/(hN_0) = -\frac{v}{\sqrt{6(1 - v)}}e^{-x_2/\delta}$$

$$v_1 = 0, \quad v_2 = \frac{1 - v^2}{E}N_0\left[x_2 + \delta\left(\frac{v}{1 - v}\right)^2(e^{-x_2/\delta} - 1)\right]$$
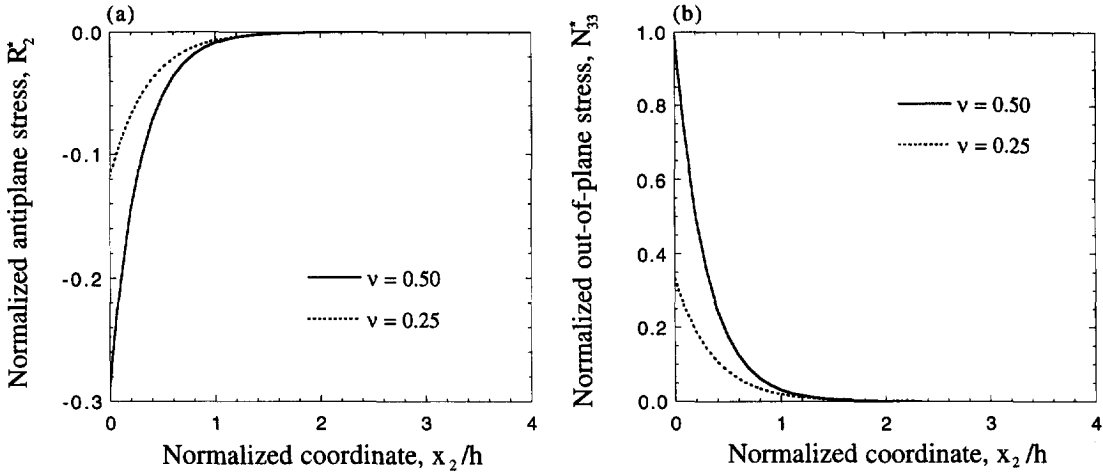
Fig. 2. Normalized (a) antiplane shear R_2^* and (b) out-of-plane normal stress N_{33}^* vs x_2/h.

$$w = \frac{\nu(1 + \nu)}{E} N_0 h (-1 + e^{-x_2/\delta}).$$ (14)

It can be seen from eqs (10), (11) and (14) and recalling eqs (1) and (2) that the plane stress solution can be recovered by letting $x_2/h \rightarrow \infty$. The out-of-plane stresses R_2 and N_{33} decay exponentially with x_2/h. The difference between the out-of-plane displacement and the corresponding plane stress one also decays exponentially with x_2/h. Though the in-plane stress N_{22} is not influenced by the plate thickness, the other in-plane stress N_{11} and the in-plane displacement v_2 are different from their values in the plane stress case. It is apparent that the solution (14) exhibits a boundary layer effect. The boundary layer adheres to the interface between the plate and the rigid substrate and its thickness is of the order of the plate thickness. Significant out-of-plane stresses exist within the boundary layer, while the plane stress solution prevails out of the boundary layer. The plate thickness effect is most severe for an incompressible material ($\nu = 0.5$) and is absent in a material with zero Poisson's ratio.

Figure 2 shows variations, in the x_2-direction, of the normalized out-of-plane stresses $R_2^* = R_2/(hN_0)$ and $N_{33}^* = N_{33}/N_0$ for two values of Poisson's ratio. It is clear that the stresses change dramatically near the interface. The stresses become essentially zero (the plane stress assumptions) at a distance equal to $2h$ from the interface. Figure 3 shows the variation of the out-of-plane displacement, w, normalized by its value in the plane stress case. It is seen that the displacement, w, converges to the plane stress one at a distance equal to the plate thickness from the interface. Results from Figs 2 and 3 suggest that the boundary layer extends in the x_2-direction for a distance equal to the plate thickness, which agrees with the common understanding of the plate thickness effects.

The semi-infinite plate problem discussed above can also be regarded as a plane strain problem in the x_2–x_3 plane, i.e. a semi-infinite strip with width $2h$ perfectly bonded to a rigid substrate at $x_2 = 0$ and subjected to a uniform tensile traction $\sigma_0 = N_0$ at $x_2 \rightarrow \infty$, see Fig. 1(b). As the solution of the plane strain strip problem is also the exact solution of the 3-D elasticity problem, we can compare it with the solution obtained by using the K–M theory. Benthem [7] has solved the plane strain strip problem. By using his numerical data, the average stresses N_{22} and R_2 at $x_2 = 0$, i.e. the interface between the plate and the rigid substrate, are evaluated as $1.002N_0$ and $-0.09N_0h$ for $\nu = 0.24$; the corresponding values obtained by using the K–M theory are N_0 and $-0.11N_0h$, respectively. Thus, the K–M theory gives acceptable results for engineering applications. Note that the plane stress theory assumes $R_2 = 0$.
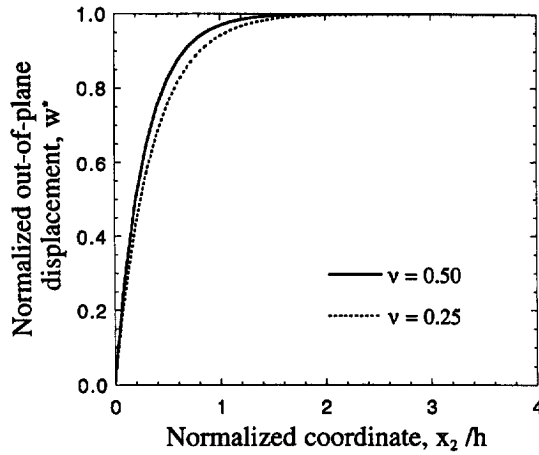
Fig. 3. Normalized out-of-plane displacement w^* vs x_2/h.

4. AN INTERFACE CRACK BETWEEN A SEMI-INFINITE PLATE AND A RIGID SUBSTRATE

Consider a through crack of length $2a$ at the interface between a semi-infinite plate and a rigid substrate as shown in Fig. 4. The crack surface is described by $x_2 = 0$, $|x_1| \leq a$ and $|x_3| \leq h$. The plate is subjected to a uniform tensile traction N_0 at $x_2 \to \infty$. By using the super-position method, stresses and displacements in the plate may be computed by adding the solution obtained in the last section to the solution of the following crack problem

$$N_{12} = 0, \quad N_{22} = -N_0, \quad R_2 = R_2^0 = \frac{hN_0\nu}{\sqrt{6(1-\nu)}}, \quad |x_1| \leq a, \quad x_2 = 0 \tag{15}$$

$$v_1 = 0, \quad v_2 = 0, \quad w = 0, \quad |x_1| > a, \quad x_2 = 0 \tag{16}$$

$$N_{\alpha\beta} \longrightarrow 0, \quad N_{33} \longrightarrow 0, R_\alpha \longrightarrow 0, \quad x_\alpha x_\alpha \longrightarrow \infty. \tag{17}$$
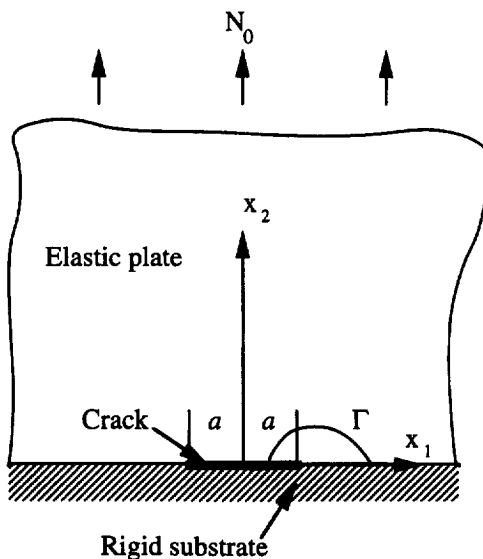

Fig. 4. A crack at the interface between an elastic plate and a rigid substrate.

4.1. *Integral equations*

By using the Fourier transform method, the crack problem is reduced to the following system of singular integral equations

$$-\beta f_2(x_1) + \frac{1}{\pi}\int_{-a}^{a}\frac{f_1(t)}{t-x_1}\,\mathrm{d}t + \frac{1}{\pi}\int_{-a}^{a}k_{1j}(x_1,t)f_j(t)\mathrm{d}t = 0, \ |x_1| \le a$$

$$\beta f_1(x_1) + \frac{1}{\pi}\int_{-a}^{a}\frac{f_2(t)}{t-x_1}\,\mathrm{d}t + \frac{1}{\pi}\int_{-a}^{a}k_{2j}(x_1,t)f_j(t)\mathrm{d}t = -2K_0\frac{N_0}{E}, \ |x_1| \le a$$

$$\frac{1}{\pi}\int_{-a}^{a}\frac{f_3(t)}{t-x_1}\,\mathrm{d}t + \frac{1}{\pi}\int_{-a}^{a}k_{3j}(x_1,t)f_j(t)\mathrm{d}t = \frac{6\nu(1+\nu)}{\sqrt{6(1-\nu)}}\frac{N_0}{E}, \ |x_1| \le a \tag{18}$$

where

$$f_1(x_1) = v_{1,1}|_{x_2=0}, \ f_2(x_1) = v_{2,1}|_{x_2=0}, \ f_3(x_1) = w_{,1}|_{x_2=0} \tag{19}$$

$$\beta = (2\nu - 1)/2(1-\nu) \tag{20}$$

is the Dundurs' parameter for plane strain deformations,

$$K_0 = (1+\nu)(3-4\nu)/4(1-\nu) \tag{21}$$

and the Fredholm kernels $k_{ij}(x,t)$ $(i, j = 1,2,3)$ are given in Appendix A. We note that the three dislocation density functions $f_i(x_1)$ on the crack face are coupled. The boundary conditions (16) require that $f_i(x_1)$ satisfy

$$\int_{-a}^{a}f_i(t)\mathrm{d}t = 0, \ i = 1, 2, 3. \tag{22}$$

By introducing a complex displacement dislocation density $f(x_1)$,

$$f(x_1) = \frac{N_0}{E}[f_1(x_1) + if_2(x_1)] \tag{23}$$

the system of integral eqs (18) may be simplified as follows

$$\beta f(r) + \frac{1}{\pi i}\int_{-1}^{1}\frac{f(s)}{r-s}\,\mathrm{d}s + \int_{-1}^{1}\{K_{11}(r,s)f(s) + K_{12}(r,s)\overline{f(s)}$$

$$+K_{13}(r,s)f_3(s)\}\mathrm{d}s = -2K_0, \ |r| \le 1$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f_3(s)}{r-s}\,\mathrm{d}s + \int_{-1}^{1}\{K_{31}(r,s)f(s) + K_{32}(r,s)\overline{f(s)}$$

$$+K_{33}(r,s)f_3(s)\}\mathrm{d}s = \frac{6\nu(1+\nu)}{\sqrt{6(1-\nu)}}, \ |r| \le 1 \tag{24}$$

where $\overline{f(r)} = \frac{N_0}{E}[f_1(r) - if_2(r)]$ is the complex conjugate of $f(r)$, $f_3(r)$ is normalized by N_0/E,

$$x_1 = ar, t = as \tag{25}$$

and the Fredholm kernels $K_{ij}(r,s)$ $(i = 1,3, j = 1,2,3)$ are given in Appendix A.

According to the singular integral equation method[8,9], eq. (24) has solutions of the form

$$f(r) = \frac{(1-r)^{-i\varepsilon}(1+r)^{i\varepsilon}}{\sqrt{1-r^2}}F(r), \ |r| \le 1$$

$$f_3(r) = \frac{F_3(r)}{\sqrt{1-r^2}}, \ |r| \le 1 \tag{26}$$

where $F(r) = F_1(r) + iF_2(r)$ and $F_3(r)$ are continuous bounded functions on the interval $[-1,1]$, and

$$\varepsilon = \frac{1}{2\pi} \ln \frac{1 - \beta}{1 + \beta} \tag{27}$$

is the so-called oscillatory index in the plane strain case.

4.2. Crack front fields

It can be shown that the crack front average stress fields consist of a plane strain (not plane stress) oscillatory singular field and an antiplane shear square-root singular field. The stresses at the crack extended line $(x_1 > a, x_2 = 0)$ are given by

$$N_{22} + iN_{12} = K(x_1 - a)^{i\varepsilon}/\sqrt{2\pi(x_1 - a)} \tag{28a}$$

$$R_2 = hK_{III}/\sqrt{2\pi(x_1 - a)} \tag{28b}$$

where the stress intensity factors $K = K_1 + iK_2$ and K_{III} are

$$K = K_1 + iK_2 = -\frac{N_0\sqrt{\pi a}[F_2(1) + iF_1(1)]}{2K_0(2a)^{i\varepsilon}\cosh(\pi\varepsilon)} \tag{29a}$$

$$K_{III} = -\frac{N_0\sqrt{\pi a}F_3(1)}{6(1 + \nu)}. \tag{29b}$$

Equations (24), (26), (28) and (29) reveal that all three fracture modes are coupled together and the oscillatory behavior of the crack tip fields is determined by the Dundurs' parameter for plane strain deformations. Nakamura[10], by using the finite element method, arrived at similar conclusions.

For incompressible materials $(\nu = 0.5)$, the oscillatory index $\varepsilon = 0$ and eqs (28a) and (29a) reduce to

$$N_{22} + iN_{12} = K/\sqrt{2\pi(x_1 - a)} \tag{30}$$

$$K = K_1 + iK_2 = K_I + iK_{II} = -N_0\sqrt{\pi a}[F_2(1) + iF_1(1)]/2K_0. \tag{31}$$

The crack front average energy release rate is evaluated as

$$G = \frac{(1 - \nu^2)K\bar{K}}{2E\cosh^2(\pi\varepsilon)} + \frac{3K_{III}^2}{4\mu} \tag{32}$$

and two normalized phase angles are introduced to represent the relative strengths of the mode mixity

$$\psi = \frac{2}{\pi} \tan^{-1}\left\{\frac{\text{Im}(K\hat{r}^{i\varepsilon})}{\text{Re}(K\hat{r}^{i\varepsilon})}\right\}, \quad \phi = \frac{2}{\pi} \sin^{-1}\left\{\frac{K_{III}}{\sqrt{\frac{4}{3}\mu G}}\right\} \tag{33}$$

in which $\hat{r}$ is a reference length. Rice[11] has discussed how to choose $\hat{r}$ for plane strain interface fracture.

4.3. A path-independent integral and a fracture criterion

Path-independent integrals play an important role in fracture mechanics[12]. Rice[12] proposed a path-independent integral, or the well-known J-integral and related it to the energy release rate associated with the quasi-static crack growth. Rice's J-integral is a special case of the Eshelby's energy–momentum tensor[13]. We now extend the J-integral to include the plate thickness effects in the framework of the K–M theory. Like the J-integral, the proposed path-

independent integral is also a special case of one of the conservation laws in general three-dimensional elastostatics[14].

Consider the line integral

$$J_{TH} = \int_C \left[W dx_2 - N_{\alpha\beta} n_\beta v_{\alpha,1} dl - \frac{1}{h} R_\alpha n_\alpha w_{,1} dl \right] \tag{34}$$

where C is a simple closed curve, n_α is the unit outward normal vector of C, dl is an infinitesimal length on C, $W (\gamma_{\alpha\beta}, \gamma_{33}, \gamma_\alpha)$ is given by

$$W(\gamma_{\alpha\beta}, \gamma_{33}, \gamma_\alpha) = \mu\left(\gamma_{\alpha\beta}\gamma_{\alpha\beta} + \gamma_{33}^2\right) + \frac{1}{2}\lambda\left(\gamma_{\alpha\alpha} + \gamma_{33}\right)^2 + \frac{h^2}{6}\mu\gamma_\alpha\gamma_\alpha \tag{35}$$

$$\gamma_\alpha = \frac{1}{h}w_{,\alpha} = \frac{3}{h^2}\Gamma_\alpha \tag{36}$$

and λ is the Lamé constant.

The stress resultants $N_{\alpha\beta}$, N_{33} and R_α are related to W by

$$N_{\alpha\beta} = \frac{\partial W}{\partial\gamma_{\alpha\beta}}, \quad N_{33} = \frac{\partial W}{\partial\gamma_{33}}, \quad R_\alpha = \frac{\partial W}{\partial\gamma_\alpha} \tag{37}$$

which are equivalent to the stress–strain relations in eq. (3). In fact, W is the integral of the strain energy density over the plate thickness divided by $2h$ provided that the displacements are given by eq. (1).

By using the Gauss–Green theorem and eqs (3) and (35)–(37), it can be proved that

$$J_{TH} = 0 \tag{38}$$

for any closed curve C enclosing no field singularities. If we assume that the displacements in the plate are given by eq. (1) and note that the plate surfaces are traction-free, it can be shown that eq. (38) is a special case of one of the generalized three-dimensional conservation laws[14].

If J_{TH} is evaluated along a curve Γ beginning on the interface and ending on the crack face as shown in Fig. 4, J_{TH} will be independent of the selection of Γ due to its conservative property (38) and the perfect bonding interface conditions. It can be shown that J_{TH} on Γ is the average energy release rate associated with the quasi-static interface crack growth, i.e.

$$J_{TH} = \frac{(1 - \nu^2)K\bar{K}}{2E\cosh^2(\pi\varepsilon)} + \frac{3K_{III}^2}{4\mu}. \tag{39}$$

Hence, a fracture criterion may be established for the interface crack growth in terms of J_{TH}-integral

$$J_{TH} = J_{TH}^c(\psi, \varphi) = G_c(\psi, \varphi) \tag{40}$$

where $J_{TH}^c(\psi, \varphi)$ or $G_c(\psi, \varphi)$ represents the interface fracture resistance which depends on the phase angles ψ and φ and the length scale $\hat{r}$. Detailed properties of interface fracture resistance have been discussed by Cao and Evans[15] in two-dimensional cases and by Wang et al.[16] for combined plane strain and antiplane strain conditions. It is noted that for plane stress interface fracture, φ equals zero in eq. (40). Hence, G_c is a curve in the K-plane. In the K–M theory, φ is not zero and G_c has to be considered to be a surface in the K-K_{III} space as in 3-D fracture[16]. Since the interface fracture resistance is strongly affected by the phase angles[15], the interface fracture resistance of thin plates cannot be thoroughly studied under the plane stress assumptions.

4.4. Numerical results

In computations of stress intensity factors (SIFs), we only consider the case of incompressible materials because the effect of the plate thickness is most predominant for this case, and the oscillatory index equals zero which facilitates numerical computations. Figure 5 shows the normalized SIFs vs the non-dimensional plate-thickness h/a. The SIFs are normalized by $N_0\sqrt{(\pi a)}$.
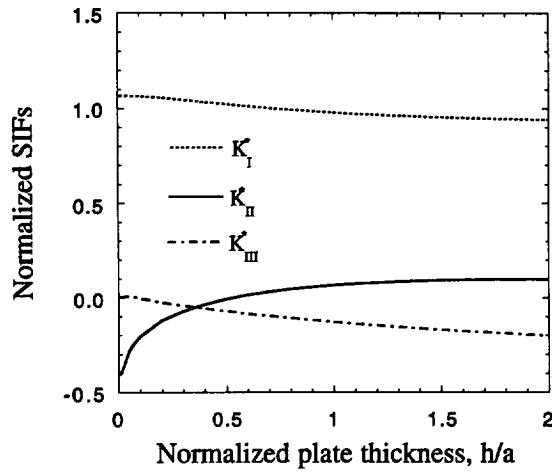
Fig. 5. Effect of plate thickness on the stress intensity factors.

It can be seen that K_I decreases with an increase in h/a. K_I is slightly higher than unity for $h/a < 0.7$. The absolute value of K_{II} decreases with an increase in h/a for $h/a < 0.5$ and becomes less than 0.1 for $h/a > 0.25$. K_{II} changes sign from negative to positive at about $h/a = 0.5$. It is also seen from Fig. 5 that K_{III} is negative for $h/a > 0.1$, becomes positive when $h/a < 0.1$ and approaches zero for a very thin plate ($h/a \to 0$). The absolute value of K_{III} is very small when $h/a < 0.1$ and increases monotonically with increasing h/a for $h/a > 0.1$. It seems peculiar that K_{II} and K_{III} change signs at about $h/a = 0.5$ and 0.1, respectively. This may be due to the interaction between in-plane and antiplane shearing. If the antiplane shear traction R_2^0 in the boundary condition (15) is neglected, we will have a negative K_{II} and positive K_{III} regardless of the plate thickness. When R_2^0 is applied, a negative K_{III} will be generated. This negative K_{III}, in turn, affects the final pattern of K_{II} and K_{III} as depicted in Fig. 5. It is clear from Fig. 5 that all three stress intensity factors are not zero. Figure 6 depicts the average energy release rate J_{TH} normalized by $[(1 - \nu^2)/E] N_0 \pi a$ and the normalized phase angles ψ and φ. Their values in the plane stress case are 0.64, -0.14 and zero, respectively, if we choose $\hat{r}$ in eq. (33) as $a/50$. Note that the plane stress field is oscillatory as the oscillatory index is not zero. It is clear from Fig. 6 that the plate thickness has a significant effect on the phase angles. The energy release rate is lower than its value in the plane stress case except for a vanishingly thin plate for which the two values are nearly equal. As the phase angles strongly affect the interface fracture resistance[15], an interface fracture toughness based on the plane stress theory is not appropriate for thin plates. It should be noted that, in general, the plate solution obtained here neither approaches the plane stress one as $h/a \to 0$ nor the plane strain one as $h/a \to \infty$. The crack tip field structure is now different from that for the plane stress case and this is not influenced by the plate
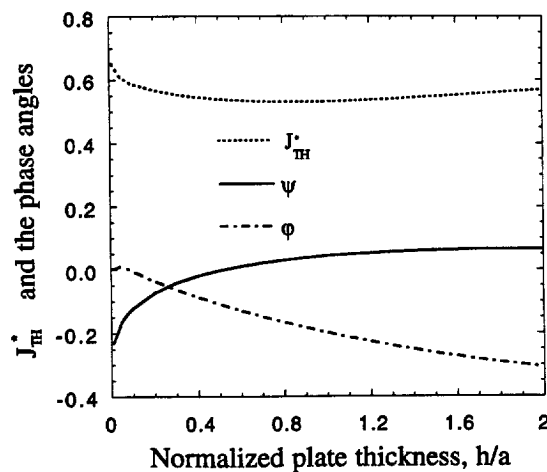

Fig. 6. Energy release rate and phase angles.

thickness, except that $K_{III} \to 0$ when $h/a \to 0$. The K–M theory was developed for a thin or a moderately thick plate and is not appropriate for a plate with a large thickness. Hence, we should not expect the plate theory solution to approach the plane strain one for large values of h/a. Finally, the present results agree qualitatively with the 3-D finite element calculations for a thin plate interface crack problem[10].

5. CONCLUDING REMARKS

Interface cracking between a semi-infinite elastic plate and a rigid substrate is studied by using the Kane and Mindlin's kinematic assumptions. A simple solution for the plate-substrate system without cracks under uniform remote tension is first obtained and results are compared with the full 3-D solution showing that the K–M theory predicts acceptable values of stresses averaged over the plate thickness. An interface crack between an elastic plate and a rigid substrate is investigated. The crack front average stress fields consist of a plane strain (not plane stress) oscillatory singular field and an antiplane shear inverse square-root singular field and the three fracture modes are always coupled. The plate thickness influences strongly stress intensity factors and phase angles. For a very thin plate, the energy release rate and the out-of-plane stress intensity factor approach their values for the plane stress case, but the in-plane stress intensity factors and the associated phase angles are different from those for the plane stress theory. The interface fracture resistance is specified by a surface in the K–K_{III} space instead of a curve in the K-plane as for the plane stress fracture. As the phase angles strongly influence the interface fracture resistance, the interface fracture toughness of thin plates cannot be studied by using the plane stress theory. However, the K–M theory offers a reasonable basis for studying interface crack problems in thin plates.

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APPENDIX A

The Fredholm kernels $k_{ij}(x, t)$ $(i, j = 1,2,3)$ in the integral eqs (18) are given by

$$k_{ij}(x, t) = \int_0^\infty f_{ij}(\xi)\sin[(x - t)\xi]d\xi, \quad (i,j) = (1, 1), (2, 2), (3, 3), (2, 3), (3, 2) \qquad (A1)$$

and

$$k_{ij}(x, t) = \int_0^\infty f_{ij}(\xi)\cos[(x - t)\xi]d\xi, \quad (i,j) = (1, 2), (2, 1), (1, 3), (3, 1), \qquad (A2)$$

where

$$f_{11}(\xi) = f_{12}(\xi) = -f_{21}(\xi) = f_{22}(\xi) = 1 + (3 - 4v)/[(1 - v)\Delta(\xi)],$$

$$f_{13}(\xi) = f_{23}(\xi) = -4K_0(\omega^2/h)(\alpha - \xi)/\Delta(\xi),$$

$$f_{31}(\xi) = f_{32}(\xi) = (2v/h)(\alpha - \xi)/\Delta(\xi),$$

$$f_{33}(\xi) = -2v(1 + v)\omega^2(\alpha - \xi)^2/\Delta(\xi) - (\alpha - \xi)/\xi \qquad (A3)$$

with

$$\Delta(\xi) = v - 3 + 2v(1 - v)\omega^2\xi(\alpha - \xi)$$

$$\alpha(\xi) = \sqrt{\xi^2 + \delta^{-2}}$$

$$\omega^2 = \frac{v}{1 - v^2}\delta^2. \qquad (A4)$$

The Fredholm kernels $K_{ij}(x,t)$ $(i = 1,3, j = 1,2,3)$ in eq. (24) are given by

$$K_{11}(x, t) = -a(k_{12} + ik_{22}),$$

$$K_{13}(x, t) = a(k_{23} - ik_{13}),$$

$$K_{31}(x, t) = a(k_{31} - ik_{32})/2,$$

$$K_{32}(x, t) = a(k_{31} + ik_{32})/2,$$

$$K_{33}(x, t) = ak_{33}. \qquad (A5)$$