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ADDENDUM TO « A THEOREM IN THE THEORY OF INCOMPRESSIBLE NAVIER-STOKES-FOURIER FLUIDS »

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SUNTO. — Riprendendo la considerazione dei tre fluidi coi quali mi occupavo nel mio lavoro precedente, nel caso di un vaso totalmente riempito da un fluido, deduco varie limitazioni alla velocità della dissipazione dell'energia. Dimostro anche che l'energia di un corpo rigido, che è conduttore non-lineare del calore, è stazionaria e tende (nella norma L^2) esponenzialmente verso il suo valore nell'equilibrio.

In this note I obtain estimates of the rate of decay of the energy for each of the three fluids considered in [1] for the case when the vessel is completely filled with the fluid. I also show that for a non-linear heat conductor, the energy approaches exponentially, in L^2-norm, its value in the equilibrium configuration as time $t \to \infty$. To avoid repetition, I use the notations and the analysis of [1]. The equation numbers refer to the equations of [1].

5. - $\mathbf{v} = \mathbf{o}$ on $\partial \mathbf{R}$.

Assume that the vessel is completely filled with the fluid and that the fluid is homogeneous. Thus the density is uniform throughout the fluid and, because of the assumption of incompressibility, it has the same value ϱ_0 for all times. For this case, by Reynolds Transport Theorem [2, p. 15]

$$\frac{d}{dt}\int_{\chi(\mathrm{R},t)} \varrho\, \Omega\, dV \qquad \varrho_0\, \frac{d}{dt}\int_{\chi(\mathrm{R},t)} \Omega\, dV = 0\,. \tag{28}$$

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Here I have used the assumption that Ω is a function of $\mathbf{x}$ only. Equation (28) states that the potential energy of the fluid remains constant. For an inhomogeneous fluid, equation (28) need not hold. Thus for a homogeneous, incompressible Navier-Stokes-Fourier fluid, we obtain the following from the equation (21).

$$\dot{E} \leq -q_3 \int (\theta \quad \theta_0)^2 \, dV - 2 \int \mu \, d_{ij} \, d_{ij} \, dV \tag{29}$$

where

$$E(t) \equiv \int [K(\theta - \theta_0)^2 + \varrho_0 \, \mathbf{v} \cdot \mathbf{v}] \, dV$$

Using the inequalities

$$\left\{ \begin{aligned} &\int K_m (\theta \quad \theta_0)^2 \, dV \geq \int K (\theta - \theta_0)^2 \, dV \\ &\int \mu \, d_{ij} \, d_{ij} \, dV \geq \frac{c_3}{q_1 \, \varrho_0} \int \varrho_0 \, \mathbf{v} \cdot \mathbf{v} \, dV \end{aligned} \right. \tag{30}$$

I strengthen (29) to read

$$\dot{E} \leq \quad \beta E , \tag{31}$$

where

$$K_m \quad \sup_{\theta, \mathbf{x}} \frac{c(\theta, \mathbf{x})}{\theta}$$

$$\beta \quad \min \left(\frac{q_3}{K_m} , \frac{2 \, c_3}{q_1 \, \varrho_0} \right)$$

To obtain (30), I used $(12)_3$ and (13) and assumed that K_m is finite An integration of (31) results in

$$E(t) \leq E(0) \, e^{-\beta t} \tag{32}$$

This gives the rate of decay of the energy E for a Navier-Stokes-Fourier fluid. Note that β depends upon the shear viscosity, the density, the specific heat, the thermal conductivity of the fluid and the shape and the size of the vessel.

So far I have shown that $\theta \xrightarrow{L^2} \theta_0$ as $t \to \infty$. That this also implies $\varepsilon(\theta) \xrightarrow{L^2} \varepsilon(\theta_0)$ as $t \to \infty$ provided the specific heat is bounded above by C, follows from the inequality

$$|\varepsilon(\theta) - \varepsilon(\theta_0)| \leq C\,|\theta \quad \theta_0|$$

which is a version of the Mean-Value Theorem.

Sometimes the thermomechanical deformations of an incompressible Navier-Stokes-Fourier fluid are assumed to be governed by the Boussinesq equations. The Boussinesq equations account for the buoyancy force but neglect the effect of the viscous dissipation in the energy equation. Joseph [3] has studied extensively the stability of a solution of these equations and from his more general results, one can obtain an estimate of the type (32) for the present problem.

For homogeneous Reiner-Rivlin fluids, when $\partial_1 R = \partial R$ and the inequalities (24) hold, I obtain

(34) $$E_1(t) \leq E_1(0)\, e^{-(2\alpha/q_1\varrho_0)t}$$

where

$$E_1(t) \equiv \frac{\varrho_0}{2}\int \mathbf{v}\cdot\mathbf{v}\, dV\,.$$

(34) gives the rate of decay of the kinetic energy in a purely mechanical problem for homogeneous Reiner-Rivlin fluids. For homogeneous incompressible fluids of second grade, the equation (25), in view of (26) and (28), becomes

(35) $$\dot{E}_2(t) \leq \quad 2\int \mu\, tr\, \mathbf{d}^2\, dV\,,$$

$$E_2(t) \equiv \quad \frac{\varrho_0}{2}\,\mathbf{v}\cdot\mathbf{v} + \alpha_1\, tr\, \mathbf{d}^2\Big|\; dV$$

Recalling $(30)_2$ and $(26)_2$, I have

$$\dot{E}_2(t) \leq \quad \gamma\, E_2(t)$$

and hence

(36) $$E_2(t) \leq E_2(0)\, e^{-\gamma t}$$

where

$$\gamma \equiv \min \left(\frac{2\mu}{q_1 \varrho_0} , \frac{\mu}{} \right.$$

The equation (36) gives the rate of decay of the kinetic energy and the viscous dissipation for a homogeneous incompressible fluid of second grade.

It seems worth mentioning that the analysis for the thermo-mechanical problem given in detail for Navier-Stokes-Fourier fluids can easily be carried over to heat conducting Reiner-Rivlin fluids and heat conducting incompressible fluids of second grade. Also the requirement $(12)_4$ can be relaxed to the weaker condition

$$\theta_0 \int \frac{k}{\theta^2} \theta_{,i} \theta_{,i} \, dV \geq \text{const.} \int \theta_{,i} \theta_{,i} \, dV \tag{37}$$

without affecting the analysis.

6. - Non-Linear Heat Conductor.

For a stationary, rigid, inhomogeneous and anisotropic nonlinear heat conductor, the only relevant balance law is the energy equation

$$\varepsilon \qquad q_{i,i} \tag{38}$$

In (38), for the sake of simplicity, I have taken the source term to be zero. Assume that for the heat conductor

$$\left\{ \begin{aligned} \varepsilon (\mathbf{X}, t) &= \varepsilon (\theta, \theta_{,i}, \mathbf{X}) , \\ \mathbf{q} (\mathbf{X}, t) &= \mathbf{q} (\theta, \theta_{,i}, \mathbf{X}) , \\ \eta (\mathbf{X}, t) &= \eta (\theta, \theta_{,i}, \mathbf{X}) , \end{aligned} \right. \tag{39}$$

where η is the specific entropy. Substitution of $(39)_{1,2}$ into (38) gives an equation for the determination of the temperature θ. If we require that every solution of (38) and $(39)_{1,2}$ satisfy the Clausius-Duhem inequality

$$\dot{\eta} + \left(\frac{q_i}{\theta} \right)_{,i} \geq 0 ,$$

we obtain [4]

$$
(40) \qquad \left\{
\begin{array}{ll}
\varepsilon\,(\mathbf{X}\,,t) & \varepsilon\,(\theta\,,\mathbf{X})\,; \\
\eta\,(\mathbf{X}\,,t) & \eta\,(\theta\,,\mathbf{X}) \\
\dfrac{\partial\,\varepsilon}{\partial\,\theta} = \theta\,\dfrac{\partial\,\eta}{\partial\,\theta}\,, & \\
\mathbf{q}\,(\mathbf{X}\,,t) = \mathbf{q}\,(\theta\,,\theta_{,i}\,,\mathbf{X})\,, & \\
q_i\,\theta_{,i} \leq 0\,. &
\end{array}
\right.
$$

In what follows I shall take $(40)_{1,2,4}$ as the constitutive relations for ε, η and $\mathbf{q}$ and assume that $\mathbf{q}$ satisfies the inequality

$$
(41) \qquad \int \frac{q_i\,\theta_{,i}}{\theta^2}\,d\,V \geq k_1 \int \theta_{,i}\,\theta_{,i}\,d\,V
$$

where k_1 is a positive constant. If $\mathbf{q}$ is given by $(2)_3$, then (41) is equivalent to (37).

Proceeding as I did in section 3 for the thermomechanical problem, I now obtain instead of (21) the following

$$
(42) \qquad \dot{H} \leq \quad \delta\,H
$$

where

$$
H\,(t) \quad \int K\,(\theta\,(\cdot\,,t) - \theta_0)^2\,d\,V
$$

and

$$
\delta \qquad \frac{k_1}{K_m} \quad or \quad \frac{1}{K_m\,q_2}\,\min \quad k_1\,,\inf_{\theta,\mathbf{X}}\,\frac{b}{\theta}\,\Big)
$$

according as the alternative (a) or (b) in $(12)_6$ holds. Here $K_m = \sup\limits_{\theta,\mathbf{X}} \dfrac{c\,(\theta,\mathbf{X})}{\theta}$ and I assume K_m is finite.

Note that one can obtain (42) from (29) by setting $\mathbf{v} \equiv \mathbf{o}$.

An integration of (42) gives

$$
(43) \qquad H\,(t) \leq H\,(0)\,e^{-\delta t}
$$

It follows from (33), $(12)_4$ and (43) that

$$
\int |\varepsilon\,(\theta) \quad \varepsilon\,(\theta_0)\,|^2\,d\,V \leq \frac{C^2}{c_4}\,H\,(0)\,e^{-\delta t}.
$$

Hence for a non-linear heat conductor, a weak solution of (38) and $(40)_{1,4}$, under the boundary conditions $(10)_{3,4}$ exhibits the behaviour

$$\left\{\begin{array}{l} \theta \xrightarrow{L^2} \theta_0 \text{ exponentially as } t \to \infty \\ \varepsilon(\theta) \to \varepsilon(\theta_0) \text{ exponentially as } t \to \infty \end{array}\right. \tag{44}$$

provided $(12)_4$, (41), $(12)_6$, and

$$c(\theta, \mathbf{X}) \leq C \quad \text{and} \quad \frac{c(\theta, \mathbf{X})}{\theta} \leq K_m,$$

hold. I obtained a similar result for a linear heat conductor in [5]. Note that δ in (43) depends upon the shape of the body and on the bounds of b, the specific heat and the thermal conductivity.

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