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# An Approximate Linear Stability Analysis of Simple Shearing Deformations of a Dipolar Viscoplastic Material

XIANG-TONG ZHANG and R.C. BATRA

*Department of Mechanical and Aerospace Engineering and Engineering Mechanics, University of Missouri - Rolla, Rolla, MO 65401-0249, U.S.A.*

**Abstract**

We analyze the stability of a homogeneous solution of equations governing the simple shearing deformations of a thermally softening viscoplastic block. We also take gradients of strain and strain-rate as independent kinematic variables, and include in the theory the dipolar stress associated with the strain gradient. It is shown that the consideration of dipolar effects has a stabilizing effect on infinitesimal perturbations of the homogeneous solution.

## 1. Introduction

Shear instabilities in the form of localized narrow bands are often observed in metals deformed at high strain rates. Zener and Hollomon [1] observed 32 $\mu$m wide shear bands during the punching of a hole in a steel plate. They added that the heating caused by the plastic deformation of the material made it softer and the material became unstable when this thermal softening equalled the combined effects of strain and strain-rate hardening. Recht [2] used this criterion, i.e., the instability occurs at the peak in the stress-strain curve, to derive values of strain rate necessary to produce shear strain localization and compared these values for different materials. Recht neglected the effect of strain-rate hardening. Staker [3] used the same instability criterion and included the dependence of the flow stress upon strain rate. Assuming parabolic strain and strain-rate hardening laws, he concluded that important material parameters are the specific heat, slope of the temperature dependence of the flow stress, and parameters indicating the strain

hardening capacity of the material. The thermal conductivity, yield strength, and strain-rate sensitivity do not enter in as a first order effect.

Instead of presuming that the material becomes unstable at a stress maximum, Clifton [4] for quasistatic deformations, and Bai [5] for dynamic deformations studied the growth of infinitesimal periodic perturbations superimposed on a body deformed by a finite amount in simple shear. They used a classical, linear perturbation analysis in which the coefficients in the linear differential equations for the perturbations were taken to be constants. Molinari [6] and Fressengeas and Molinari [7] have developed a relative linear perturbation analysis that accounts, in part, for the nonsteadiness of the homogeneous solution by linearizing in the relative perturbation defined as the perturbation divided by the corresponding unperturbed quantity. Anand et. al. [8] have generalized the linear perturbation analysis of Clifton [4] and Bai [5] for one-dimensional problems to three- dimensional problems.

Wright and Walter [9] recently used the linear perturbation analysis in which the coefficients in the linear differential equations for the perturbations are taken to be functions of time. Here we generalize their results to dipolar materials and confirm the result indicated earlier by numerical experiments of Wright and Batra [10], and Batra [11], and Batra and Kim [12] that the consideration of dipolar effects tends to stabilize the infinitesimal perturbations of the homogeneous solution. We should add that the stability criteria obtained to date, including the present one, do not give an idea of the time when the severe localization of the deformation will begin.

## 2. Governing Equations and Their Homogeneous Solution

In terms of non-dimensional variables, equations governing the thermomechanical deformations of a viscoplastic block undergoing simple shearing and overall adiabatic deformations are (e.g., see Batra and Kim [12])

$$\rho \dot{v} = (s - l\sigma_{,y})_{,y} \qquad 0 < y < 1,$$

$$\dot{\theta} = k\theta_{,yy} + s\dot{\gamma} + l\sigma\ddot{d} \qquad 0 < y < 1$$

$$= v_{,y}, \qquad \ddot{d} = v_{,yy}$$

$$(s^2 + \sigma^2)^{\frac{1}{2}} = (1 - \alpha\theta)\,[1 + b\,(\dot{\gamma}^2 + l^2\dot{d}^2)^{\frac{1}{2}}]^m,$$

$$sv_{,yy} = v_{,y}\sigma/l.$$

These equations for dipolar materials reduce to those for nonpolar materials when $l$ is set equal to zero. Equations (2.1) and (2.2) express, respectively, the balance of linear momentum and the balance of internal energy. Equations $(2.3)_1$ and $(2.3)_2$ define the plastic strain rate $\dot{\gamma}$ and its gradient $\dot{d}$ in terms of the spatial derivatives of the velocity $v$ of a material particle in the direction of shearing. Equation (2.4) is the kinetic relation between the shear stress $s$, the dipolar stress $\sigma$, the temperature rise $\theta$, the coefficient of thermal softening $\alpha$, and the material characteristic length $l$. The parameter $b$ equals the characteristic time for the material, and the parameter $m$ signifies the strain-rate sensitivity of the material. Elastic deformations have been neglected, and all of the plastic working given by $(s\dot{\gamma} + l\sigma\dot{d})$ has been assumed to be converted into heat. In equation (2.1) through (2.5), $\rho$ is the constant mass density, $k$ is the constant thermal conductivity, a superimposed dot indicates the material time derivative, and a comma followed by $y$ signifies partial differentiation with respect to $y$.

We presume that overall deformations of the block are adiabatic, the lower surface is at rest, and the upper surface is assigned a constant velocity of 1.0. Thus

$$\theta_{,y}(0,t) = 0,\ \theta_{,y}(1,t) = 0,\ v(0,t) = 0,\ v(1,t) = 1, \sigma(0,t) = 0,\ \ \sigma(1,t) = 0.$$

The boundary conditions $(2.6)_5$ and $(2.6)_6$ can be motivated as follows. Assuming that a material defect or inhomogeneity is located near the center $y = 0$ of the block, sharp gradients of the deformation occur only in a small neighborhood of $y = 0$. Therefore, $v_{,yy}$ equals essentially zero at $y = \pm 1$. A numerical solution of the problem on the domain $-1 \le y \le 1$ with boundary conditions $\sigma(\pm 1, t) = 0$ revealed that $\sigma(0,t) = 0$. Here we study the problem on the domain $0 \le y \le 1$ and thus take $(2.6)_5$ and $(2.6)_6$ as the boundary conditions for $\sigma$.

When the initial conditions are uniform, i.e.,

$$v(y,0) = y,\ \theta(y,0) = 0$$

equations (2.1)-(2.6) have the following simple homogeneous solution

$$v = y,\ s = \frac{\Gamma}{a} e^{-\Gamma t},\ \theta = \frac{1}{a}(1 - e^{-\Gamma t}),\ \sigma = o,$$

where $\Gamma = \alpha (1 + b)^m$. Note that for the homogeneous solution, $d = v_{,yy} = 0$, and the dipolar effects make no contribution. The solution (2.8) is the same as that given by Wright and Walter [9], who analyzed shear bands in simple (or nonpolar) materials.

## 3. Linear Perturbation Analysis

We now assume that the initial temperature is nonuniform, and is given by

$$\theta(y, 0) = \varepsilon \psi_0(y), \text{ where } \int_0^1 \psi_0(y)\, dy = 1$$

That is, the initial temperature is nearly equal to the reference temperature, and its average defines a small positive number $\varepsilon$. We assume that the solution of (2.1) through (2.6) is of the form

$$s = s_H + \varepsilon \tilde{s} + \quad , \quad \theta = \theta_H + \varepsilon \tilde{\theta} + \ldots,$$

$$v = v_H + \varepsilon \tilde{v} + \quad ., \quad \sigma = \sigma_H + \varepsilon \tilde{\sigma} +$$

where $s_H$, $\theta_H$, $v_H$ and $\sigma_H$ are given by (2.8) where the subscript $H$ is not indicated, and $\varepsilon$ is a small parameter defined by the initial temperature distribution (3.1). We recall that numerical experiments (e.g., see Batra and Kim [12]) indicate that $s$ and $\sigma$ are uniform in $y$ to a high degree of approximation unless the nominal strain rate is very large. Accordingly, we neglect the inertia term in the equations of first variation, written below without tildes.

$$0 = (s - l\sigma_{,y})_{,y}, \tag{3.3}$$

$$\theta_{,t} = k\theta_{,yy} + \frac{1 + \overline{m}}{\overline{m}} s + \frac{\Gamma}{\overline{m}} \theta, \tag{3.4}$$

$$v_{,y} = \frac{\alpha}{\overline{m}\Gamma}(e^{\Gamma t}) s + \frac{\alpha}{\overline{m}}(e^{\Gamma t})\theta, \tag{3.5}$$

$$v_{,yy} = \frac{\alpha}{\Gamma} e^{\Gamma t} \sigma / l,$$

where $\overline{m} = mb/(1+b)$. Note that these equations lose validity for large times. The pertinent boundary and initial conditions are

$$v(0,t) = v(1,t) = 0, \quad \theta_{,y}(0,t) = \theta_{,y}(1,t) = 0,$$

$$\sigma(0,t) = \sigma(1,t) = 0, \quad \theta(y,0) = \psi_0(y)$$

Integration of (3.5) with respect to $y$ from 0 to 1 and the use of the boundary conditions $(3.7)_1$ and $(3.7)_2$ gives

$$\int_0^1 s\,dy = -\Gamma \int_0^1 \theta\,dy.$$

Integrating (3.3) twice and using the boundary conditions $(3.7)_5$ and $(3.7)_6$ we obtain

$$s - l\sigma_{,y} = f(t), \text{ and } \int_0^1 s\,dy = f(t).$$

Now integrate (3.4) and use (3.8) to obtain

$$\frac{d}{dt}\int_0^1 \theta\,dy = \left[-\frac{(1+\overline{m})}{\overline{m}}\Gamma + \frac{\Gamma}{\overline{m}}\right] \int_0^1 \theta\,dy = -\Gamma \int_0^1 \theta\,dy$$

Hence

$$\int_0^1 \theta\,dy = e^{-\Gamma t}, \text{ and } f(t) = -\Gamma e^{-\Gamma t}$$

where the last relation follows from $(3.11)_1$, (3.8), and (3.9). Differentiating (3.5) with respect to $y$, and substituting for $v_{,yy}$ into (3.6), we obtain

$$\sigma = \frac{l}{\overline{m}}(s_{,y} + \Gamma \theta_{,y}). \tag{3.12}$$

Numerical results (e.g., see Batra and Kim [12] indicate that for small time $t$, $|s_{,y}| << \Gamma |\theta_{,y}|$. Since we are interested in only small times, therefore, we approximate (3.12) by

$$\sigma = \frac{l\Gamma}{\bar{m}}\theta_{,y} \tag{3.13}$$

Substitution for $f(t)$ from (3.11) and for $\sigma$ from (3.13) into $(3.9)_1$ and the resulting expression for $s$ into (3.4), yields

$$\theta_{,t} = \bar{k}\,\theta_{,yy} + \frac{\Gamma}{\bar{m}}\theta - \Gamma\frac{1+\bar{m}}{\bar{m}}e^{-\Gamma t},$$

where

$$\bar{k} = k + \frac{1+\bar{m}}{\bar{m}^2}l^2\Gamma \tag{3.15}$$

With the decomposition

$$\theta = e^{(\Gamma t/\bar{m})}(\psi(y,t) + \varphi(t)),$$

equation (3.14) may be split into the following two simpler equations

$$\psi_{,t} = k\psi_{,yy}, \quad \varphi_{,t} = -\Gamma\frac{1+\bar{m}}{\bar{m}}e^{-\Gamma(1+1/\bar{m})t}$$

The appropriate choices for initial and boundary conditions are

$$\psi_{,y}(0,t) = \psi_{,y}(1,t) = 0, \quad \psi(y,0) = \psi_0(y), \varphi(0) = 0 \tag{3.17}$$

The solution (3.16) and (3.17) is

$$\varphi(t) = e^{-\Gamma[(1+\bar{m})/\bar{m}]t} - 1,$$

$$\psi(y,t) = \sum_{n=0}^{\infty}\alpha_n e^{\alpha_n t}\cos n\pi y, \quad \alpha_n = -n^2\pi^2\bar{k},$$

and $\alpha_n$ and can chosen to satisfy the initial conditions. Substitution for $\theta$ into (3.13), and then using $(3.11)_2$ and $(3.9)_1$, we obtain the following solution of (3.3)-(3.7) which is exact to first order in $\varepsilon$.

$$s = \frac{\Gamma}{\alpha}(1-\alpha\varepsilon)e^{-\Gamma t} + \varepsilon\frac{l^2\Gamma}{\bar{m}}e^{\Gamma t/\bar{m}}\psi_{,yy} + O(\varepsilon^2),$$

$$\sigma = \varepsilon\frac{l\Gamma}{\bar{m}}e^{\Gamma t/\bar{m}}\psi_{,y} + O(\varepsilon^2), \tag{3.19}$$

$$v_{,y} = 1 + \varepsilon\frac{\alpha}{\bar{m}}e^{(1+1/\bar{m})\Gamma t}(\psi-1) + \frac{\varepsilon\alpha l^2}{\bar{m}^2}e^{(1+1/\bar{m})} + O(\varepsilon^2).$$

$$\theta = \frac{1}{\alpha}[1-(1-\alpha\varepsilon)e^{-\Gamma t}] + \varepsilon e^{\Gamma t/\bar{m}}(\psi-1) + O(\varepsilon^2).$$

Thus to first order in $\varepsilon$, the consideration of dipolar effects does not influence the evolution of $\theta$, but does affect the evolution of $s$, $\sigma$, and $v_{,y}$. The solution variables $s, \sigma, v_{,y}$ and $\theta$ will grow because of the positive exponential term.

The $n$th Fourier component of $s, \sigma, v_{,y}$ and $\theta$ in (3.19) will decay initially if

$$\left(1+\frac{1}{\bar{m}}\right)\Gamma < \left(k + \frac{1+\bar{m}}{\bar{m}^2}l^2\Gamma\right)n^2\pi^2, \tag{3.20}$$

where $\bar{m} = mb/(1+b)$ and $\Gamma = \alpha(1+b)^m$ For metals, $\bar{m} << 1$ and $(1+1/\bar{m}) \approx 1/\bar{m}$.

Thus (3.20) becomes

$$\frac{\Gamma}{\bar{m}} < \left(k + \frac{l^2}{\bar{m}^2}\Gamma\right)n^2\pi^2. \tag{3.21}$$

Since the material characteristic length $l$ appears on the right-hand side of (3.21), it is clear that the consideration of dipolar effects has a stabilizing effect on the perturbations. For $l = 0$, the result (3.21) reduces to that obtained by Wright and Walter [9] for nonpolar materials, as it should. In dimensional terms, (3.21) becomes

$$\kappa_0 \bar{\alpha} H^2 (1 + \bar{b}\dot{\gamma}_0)^{m+1} < (\overline{kbm} + \frac{l^2}{} 1 + \bar{b}\dot{\gamma}_0)^{m+2-} \alpha \kappa_0) n^2 \pi^2 \tag{3.22}$$

where the dimensional quantities are indicated by a superimposed bar, $\dot{\gamma}_0$ is the average strain-rate, $2H$ equals the height of the block (or the gauge length) and $\kappa_0$ is the yield stress in a quasistatic reference simple shear test. For nonpolar materials (i.e., $l = 0$), one can conclude from equation (3.22) that increase in strength $\kappa_0$, thermal softening coefficient $\bar{\alpha}$, slab height $H$, or the nominal strain-rate $\dot{\gamma}_0$ tend to be destabilizing; and an increase in thermal conductivity $\bar{k}$ tends to be stabilizing. For dipolar materials, increase in the thermal conductivity and the material characteristic length are stabilizing and the increase in the slab height is destabilizing. Other parameters appears on both sides on eqn. (3.22).

With $n = 1$ the inequality (3.22) gives a criterion for absolute stability which may be used to determine the threshold value of $\dot{\gamma}_0$ below which all Fourier components decay.

## 4. Conclusions

We have analyzed the stability of the homogeneous solution of equations governing the simple shearing deformations of a thermally softening viscoplastic block. The approximations, suggested by the prior numerical solutions of such problems, are that for small times, the flux of linear momentum is essentially uniform in the spatial variable, and $| s_{xy} | << \Gamma | \theta_{xy} |$ where $\Gamma$ is defined in terms of the material variables.it is found that the increase in the thermal conductivity and the material characteristic length has a stabilizing effect, and the increase in the slab height has a destabilizing effect on small perturbations superimposed on the homogeneous solution. The specific heat does not appear in the stability criterion (3.22)

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