On Extensional Oscillations and Waves in Elastic Rods

SHANKAR KRISHNASWAMY
R. C. BATRA
Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0219, USA

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Abstract: The authors study the dispersive nature of propagating extensional waves in an infinitely long elastic rod within the framework of the linear theory of a Cosserat rod with two directors. The authors also identify certain material constants in the theory in a manner that is different from those used by others and consequently show that the resulting theory better captures the high-frequency dynamical behavior of three-dimensional rod-like bodies.

1. INTRODUCTION

Wavepropagation in linearly elastic rods has been studied extensively. Several authors have attempted to construct approximate one-dimensional theories in which the governing equations depend on one spatial variable only. Green [1] has reviewed most of the developments before 1960; many of the theories included in [1] are developed in a somewhat ad hoc manner. A systematic treatment of a theory of rods using the concept of a Cosserat (or directed) curve was initiated in the late 1960s; we mention the work of Green and Laws [2]; Cohen [3]; Green, Laws, and Naghdi [4, 5]; and Green, Naghdi, and Wenner [6, 7]. The purely kinematical aspects of directed rods were considered earlier by Ericksen and Truesdell [8]. Naghdi [9] has presented a comprehensive account of the nonlinear theory of Cosserat curves and its applicability as a model for three-dimensional rod-like bodies. In the context of directed theories of rods, few articles have addressed wave propagation in quantitative detail; some exceptions the papers by Antman and Liu [10], Cohen and Whitman [11], and being Green et al. [4]. Here, we essentially follow the developments contained in [4, 7].

We present a detailed study of wave propagation in the linear extensional theory of an infinitely long, elastic Cosserat rod. In particular, we discuss the dispersive nature of such waves and examine the limiting values of phase speeds for low and high frequencies. We recall that the classical one-dimensional theory of rods predicts only one nondispersive wave speed and does not account for thickness (or lateral) modes of deformation, let alone coupling between these and longitudinal modes that exists in a three-dimensional rod-like body. The importance of thickness modes of deformation for quasi-static contact problems has been addressed by Naghdi and Rubin [12] and, more recently, for dynamical problems

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by Nordenholz and O'Reilly [13]. These modes are also important in high-frequency dynamical problems, and this point has been emphasized for plates by Kane and Mindlin [14].

For linear extensional motions of Cosserat rods, all of the constitutive parameters entering the theory have been identified by Green et al. [6] and Green and Naghdi [15] by comparing certain exact solutions in the three-dimensional *equilibrium* theory of linear elasticity with those obtained from the rod theory. Consequently, the constitutive equations are accurate in the low-frequency regime and may not model high-frequency behavior well. We show here that this is indeed the case and propose that for high-frequency motions, the constitutive parameters be identified by comparing solutions from the two dynamical theories. This procedure forces the Cosserat theory to better mimic the high-frequency behavior of three-dimensional rod-like bodies. Thus, we extend the range of applicability of the Cosserat theory to the high-frequency regime by using values for some of the material constants that are different from those given in [6, 15].

This article is organized as follows. In Section 2, after a brief background on the nonlinear theory of a Cosserat curve, equations relevant to the linear extensional theory are presented. Section 3 discusses the dispersive behavior of extensional waves in infinite rods, examines the low- and high-frequency limits of wave speeds, and presents a procedure to identify one of the constitutive parameters. In Section 4, the frequency of pure thickness oscillation is used to obtain new values of certain other material constants. In Section 5, the group speeds associated with various branches of the frequency spectrum are studied.

2. BACKGROUND INFORMATION AND BASIC EQUATIONS

We recall that a Cosserat curve $\mathcal{C}$ is a material curve $\mathcal{L}$ embedded in a three-dimensional Euclidean space, to each point of which is attached deformable vector fields called directors. We confine attention to the case when only two directors are attached to each point of $\mathcal{L}$. Let the material points of $\mathcal{L}$ be identified by the convected coordinate ξ. The motion of $\mathcal{C}$ is specified by the following three sufficiently smooth vector functions that assign a position $\mathbf{r}$ and a pair of directors $\mathbf{d}_\alpha$ ($\alpha = 1, 2$)[1] to material points of $\mathcal{L}$ at time t:

$$\mathbf{r} = \hat{\mathbf{r}}(\xi, t), \qquad \mathbf{d}_\alpha \quad \hat{\mathbf{d}}_\alpha(\xi, t), \qquad [\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3] > 0, \tag{1}$$

where $\mathbf{d}_3 = \partial\hat{\mathbf{r}}/\partial\xi$, and $[\cdot, \cdot, \cdot]$ denotes the scalar triple product.

The velocity $\mathbf{v}$ of a material point and the velocities $\mathbf{w}_\alpha$ of the directors are defined by

$$\mathbf{v} \quad \dot{\mathbf{r}}, \qquad \mathbf{w}_\alpha = \dot{\mathbf{d}}_\alpha, \tag{2}$$

where a superposed dot indicates material time differentiation. In a fixed reference configuration of $\mathcal{C}$, we denote the position vector and directors by $\mathbf{R}$ and $\mathbf{D}_\alpha$, respectively. Thus,

$$\mathbf{R} = \hat{\mathbf{R}}(\xi), \qquad \mathbf{D}_\alpha \quad \hat{\mathbf{D}}_\alpha(\xi), \qquad [\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3] > 0, \tag{3}$$

where $\mathbf{D}_3 = \partial\hat{\mathbf{R}}/\partial\xi$.

The local field equations governing the motion of $\mathcal{C}$ are (see [9]):

$$
\begin{aligned}
&\dot{\lambda} \quad 0,\\
&\frac{\partial \mathbf{n}}{\partial \xi} + \lambda \mathbf{f} = \lambda\left(\dot{\mathbf{v}} + y^{\alpha}\dot{\mathbf{w}}_{\alpha}\right),\\
&\frac{\partial \mathbf{m}^{\alpha}}{\partial \xi} + \lambda \mathbf{l}^{\alpha} - \mathbf{k}^{\alpha} = \lambda\left(y^{\alpha}\dot{\mathbf{v}} + y^{\alpha\beta}\dot{\mathbf{w}}_{\beta}\right),\\
&\mathbf{d}_3 \times \mathbf{n} + \mathbf{d}_{\alpha} \times \mathbf{k}^{\alpha} + \frac{\partial \mathbf{d}_{\alpha}}{\partial \xi} \times \mathbf{m}^{\alpha} = \mathbf{0}.
\end{aligned}
\tag{4.4}
$$

Equations (4.1), (4.2), (4.3), and (4.4) represent the conservation of mass and balances of linear momentum, director momenta, and moment of momentum, respectively. In these equations, $\mathbf{n}$ is the contact force, $\mathbf{m}^{\alpha}$ are the contact director forces, $\mathbf{k}^{\alpha}$ are the intrinsic director forces, $\mathbf{f}$ is the assigned force, and $\mathbf{l}^{\alpha}$ are the assigned director forces, all of which depend on (ξ, t). Also, $\lambda = \rho(\mathbf{d}_3 \cdot \mathbf{d}_3)^{1/2}$, where $\rho = \rho(\xi, t)$ is the mass density per unit length of $\mathcal{L}$ in the present configuration, and the quantities y^{α} and $y^{\alpha\beta}$ are inertia coefficients that are independent of time. We note that the theory is complete only when the quantities $\mathbf{n}$, $\mathbf{m}^{\alpha}$, and $\mathbf{k}^{\alpha}$ are specified constitutively; we will discuss below the constitutive theory in the context of linear extensional deformations.

We recall that the field equations (4.1) through (4.4) may be derived from the three-dimensional balance laws of continuum mechanics under the assumption that the position $\mathbf{r}^*$ of a material point is a linear function of the convected coordinates θ^{α} defining the cross-section of the rod:

$$
\mathbf{r}^*(\theta^{\alpha}, \xi, t) = \mathbf{r}(\xi, t) + \theta^{\alpha}\mathbf{d}_{\alpha}(\xi, t). \tag{5}
$$

The kinetical quantities $\mathbf{n}$, $\mathbf{m}^{\alpha}$, and $\mathbf{k}^{\alpha}$ may be identified as weighted integrals of appropriate traction vectors of the three-dimensional theory. The quantities y^{α}, $y^{\alpha\beta}$, and λ can also be identified with certain weighted integrals (see [6, 9]). We now discuss the linear theory of an elastic rod that is straight in its reference configuration. Let the rod be referred to a fixed system of Cartesian coordinates (x, y, z) with associated orthonormal basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$; let the origin be located at the center of the rod and the z-axis directed to the right so that the vector $\mathbf{e}_3$ is along the rod. We note that in the linear theory, the convected coordinate ξ may be taken to coincide with the Cartesian coordinate z, and the coordinates (x, y) are used to describe the cross-section of the rod. The directors $\mathbf{d}_1$ and $\mathbf{d}_2$ represent material fibers that in the reference configuration are parallel to $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. The displacement vector $\mathbf{u}$ and director displacement vectors $\bar{\delta}_i$ are defined through

$$
\mathbf{r} = \mathbf{R} + \mathbf{u}, \qquad \mathbf{d}_i \quad \mathbf{D}_i + \bar{\delta}_i, \tag{6}
$$

where $\mathbf{R} = z\mathbf{e}_3$ and $\mathbf{D}_i = \mathbf{e}_i$. Equations (5) and (6) imply that the three-dimensional displacement $\mathbf{u}^*(x, y, z, t)$ can be represented as

$$\mathbf{u}^*(x, y, z, t) = \mathbf{u}(z, t) + x\bar{\boldsymbol{\delta}}_1(z, t) + y\bar{\boldsymbol{\delta}}_2(z, t).$$

All vector and tensor quantities will be referred to the orthonormal basis $\{\mathbf{e}_i\}$; for example, $\bar{\boldsymbol{\delta}}_i = \bar{\delta}_{ij}\mathbf{e}_j$ and $\mathbf{u} = u_i\mathbf{e}_i$. Following [12], we define the linearized strain measures γ_{ij} and $\kappa_{\alpha i}$ by

$$\gamma_{ij} = \frac{1}{2}(\bar{\delta}_{ij} + \bar{\delta}_{ji}), \qquad \kappa_{\alpha i} = \frac{\partial \bar{\delta}_{\alpha i}}{\partial z}.$$

We choose the material curve $\mathcal{L}$ to be the line joining centroidal particles of the cross-sections of the three-dimensional rod-like body; thus, the inertia coefficients y^α vanish. In the linear theory of a rod that is straight in its reference configuration and exhibits certain material and geometrical symmetries (see [7]), the equations of motion separate into four groups, and those describing extensional motions in the absence of $\mathbf{f}$ and $\mathbf{l}^\alpha$ are[2,3]

$$\frac{\partial n}{\partial z} = \lambda \ddot{u}, \tag{9}$$

$$\frac{\partial m_1}{\partial z} - k_1 = \lambda y_{11}\ddot{\delta}_1$$

$$\frac{\partial m_2}{\partial z} - k_2 = \lambda y_{22}\ddot{\delta}_2,$$

where $\lambda = \rho_0$ is the reference mass density per unit length of $\mathcal{L}$, and[4]

$$n = \mathbf{n}\cdot\mathbf{e}_3, \qquad m_1 = \mathbf{m}_1\cdot\mathbf{e}_1, \qquad m_2 = \mathbf{m}_2\cdot\mathbf{e}_2,$$
$$k_1 = \mathbf{k}_1\cdot\mathbf{e}_1, \qquad k_2 = \mathbf{k}_2\cdot\mathbf{e}_2, \qquad \delta_1 = \bar{\delta}_{11}, \qquad \delta_2 = \bar{\delta}_{22}, \qquad u = u_3.$$

We supplement equations (9), (10), and (11) with the following constitutive equations (see [2, 3, 12])[5]:

$$n = \alpha_8\delta_1 + \alpha_9\delta_2 + \alpha_3\frac{\partial u}{\partial z},$$

$$m_1 = \alpha_{10}\frac{\partial \delta_1}{\partial z} + \alpha_{17}\frac{\partial \delta_2}{\partial z}, \qquad m_2 = \alpha_{17}\frac{\partial \delta_1}{\partial z} + \alpha_{11}\frac{\partial \delta_2}{\partial z}.$$

$$k_1 = \alpha_1\delta_1 + \alpha_7\delta_2 + \alpha_8\frac{\partial u}{\partial z}, \qquad k_2 = \alpha_7\delta_1 + \alpha_2\delta_2 + \alpha_9\frac{\partial u}{\partial z}.$$

In (12) through (14), the coefficients α_1, α_2, α_3, α_7, α_8, α_9, α_{10}, α_{11}, and α_{17} are material constants whose values will be determined later in this section and in Section 4. As will be verified subsequently, these constants satify the conditions $\alpha_3(\alpha_{10}+\alpha_{17}) \neq 0, \alpha_8 \neq 0, \alpha_2+\alpha_7 \neq 0$, and $\alpha_{10}-\alpha_{17} \neq 0$. We will study only the case when the strain energy density and the kinetic energy of the rod are invariant under transformations in which $\mathbf{d}_1$ replaces $\mathbf{d}_2$ (and vice versa) and $\mathbf{D}_1$ replaces $\mathbf{D}_2$ (and vice versa); for details, see [4]. This invariance property requires that

$$y_{11}=y_{22}, \qquad \alpha_1 \quad \alpha_2, \qquad \alpha_8=\alpha_9, \qquad \alpha_{10}=\alpha_{11}$$

Substitution from (12) through (15) into (9) through (11) results in the following three second-order partial differential equations for u, δ_1, and δ_2:

$$\begin{aligned}
\alpha_8(\delta_{1,z}+\delta_{2,z})+\alpha_3 u_{,zz} &= \lambda \ddot{u}, \\
\alpha_{10}\delta_{1,zz}+\alpha_{17}\delta_{2,zz}-\alpha_2\delta_1-\alpha_7\delta_2-\alpha_8 u_{,z} &= \lambda y_{11}\ddot{\delta}_1, \\
\alpha_{17}\delta_{1,zz}+\alpha_{10}\delta_{2,zz}-\alpha_7\delta_1-\alpha_2\delta_2-\alpha_8 u_{,z} &= \lambda y_{11}\ddot{\delta}_2,
\end{aligned} \tag{18}$$

where a comma followed by z indicates partial differentiation with respect to z. If we further consider only those motions of the rod for which $\delta_1=\delta_2=\delta$, for example, as we might expect a uniform circular rod to deform under certain circumstances, then (17) and (18) collapse into a single equation

$$(\alpha_{10}+\alpha_{17})\delta_{,zz}-(\alpha_2+\alpha_7)\delta-\alpha_8 u_{,z}=\lambda y_{11}\ddot{\delta}, \tag{19}$$

and (16) becomes

$$2\alpha_8\delta_{,z}+\alpha_3 u_{,zz}=\lambda \ddot{u}.$$

This special case of the present Cosserat theory is similar to Mindlin and Herrmann's [18] theory for uniform straight circular rods.

3. WAVES IN AN INFINITE ROD

We express the equations of motion (16) through (18) in terms of director displacement potentials Φ and Ψ, which satisfy

$$\delta_1=\Phi_{,z}+\Psi_{,z}, \qquad \delta_2=\Phi_{,z}-\Psi_{,z}$$

For motions that are harmonic in time, we may express Φ, Ψ, and u as

$$\Phi(z,t) \quad \varphi(z)e^{-i\omega t}, \qquad \Psi(z,t)=\psi(z)e^{-i\omega t} \qquad u(z,t) \quad \tilde{u}(z)e^{-i\omega t}$$

where ω is the frequency of oscillation, and $i=\sqrt{-1}$.

Substitution of (21) and (22) into (16), (17), and (18) results in

$$(\alpha_{10}-\alpha_{17})\frac{d^2\psi}{dz^2}+(\alpha_7-\alpha_2+\lambda y\omega^2)\psi=0,$$

$$(\alpha_{10}+\alpha_{17})\frac{d^2\varphi}{dz^2}+(\alpha_2+\alpha_7-\lambda y\omega^2)\varphi-\alpha_8\tilde{u}=0,$$

$$2\alpha_8\frac{d^2\varphi}{dz^2}+\alpha_3\frac{d^2\tilde{u}}{dz^2}+\lambda\omega^2\tilde{u}=0,$$

where $y = y_{11}$. Henceforth, y will equal y_{11} rather than the y-coordinate of a material point, as used in (7).

By eliminating the variable $\tilde{u}$ from (24) and (25), we obtain the following fourth-order ordinary differential equation for φ:

$$\frac{d^4\varphi}{dz^4}+(\zeta_2^2+\zeta_3^2)\frac{d^2\varphi}{dz^2}+\zeta_2^2\zeta_3^2\varphi=0,$$

where

$$\zeta_2^2+\zeta_3^2=\frac{\{2\alpha_8^2-\alpha_3(\alpha_2+\alpha_7-\lambda y\omega^2)+\lambda\omega^2(\alpha_{10}+\alpha_{17})\}}{\alpha_3(\alpha_{10}+\alpha_{17})}$$

$$\zeta_2^2\zeta_3^2=\frac{\lambda\omega^2(\lambda y\omega^2-\alpha_2-\alpha_7)}{\alpha_3(\alpha_{10}+\alpha_{17})}$$

Following [14], we express (26) as two distinct second-order ordinary differential equations

$$\frac{d^2\varphi_2}{dz^2}+\zeta_2^2\varphi_2=0, \qquad \frac{d^2\varphi_3}{dz^2}+\zeta_3^2\varphi_3=0,$$

where $\varphi=\varphi_2+\varphi_3$. Also, we rewrite equation (23) in the form

$$\frac{d^2\psi}{dz^2}+\zeta_1^2\psi=0$$

where

$$\zeta_1^2=\frac{(\alpha_7-\alpha_2+\lambda y\omega^2)}{(\alpha_{10}-\alpha_{17})}.$$

Thus, the extensional equations (23) through (25) may be equivalently expressed elegantly in terms of three potential functions φ_2, φ_3, and ψ, which are governed by $(29)_{1,2}$ and (30).

Finally, we note that the displacement u and director displacements δ_1 and δ_2 are given by

$$\delta_1 = \frac{d}{dz}(\varphi_2 + \varphi_3 + \psi)e^{-i\omega t} \qquad \delta_2 = \frac{d}{dz}(\varphi_2 + \varphi_3 - \psi)e^{-i\omega t} \qquad u \quad (\sigma_2\varphi_2 + \sigma_3\varphi_3)e^{-i\omega t}, \tag{32}$$

where

$$\sigma_\beta \quad \left\{(\lambda y\omega^2 - \alpha_2 \quad \alpha_7) - (\alpha_{10} + \alpha_{17})\zeta_\beta^2\right\}/\alpha_8, \quad \beta = 2, 3,$$

and $(32)_3$ is obtained from $(29)_{1,2}$ and (24).

Equations $(29)_{1,2}$ and (30) facilitate the calculation of the three types of waves that can propagate along the rod. We assume that

$$\psi(z) = Ae^{ikz} \qquad \varphi_2(z) = Be^{ikz} \qquad \varphi_3(z) = Ce^{ikz}$$

where k is the wave number and A, B, and C are constants. From (34) and (22), we see that the potentials Ψ and Φ have the familiar propagating wave expressions

$$\Psi(z, t) \quad Ae^{i(kz-\omega t)} \qquad \Phi \quad (B + C)e^{i(kz-\omega t)}$$

Substituting (34) into $(29)_{1,2}$ and (30), we obtain the following three values for k:

$$k_i \quad \zeta_i^2$$

From (27), (28), (31), and (36), the wave (or phase) speeds c_1, c_2, and c_3 are computed to be

$$c_1^2 = \frac{p}{\lambda} + \frac{q}{\lambda k^2} - \frac{1}{\lambda}\left\{\left(p + \frac{q}{k^2}\right)^2 - f - \frac{g}{k^2}\right\}^{1/2}$$

$$c_2^2 \quad \frac{p}{\lambda} + \frac{q}{\lambda k^2} + \frac{1}{\lambda}\left\{\left(p + \frac{q}{k^2}\right)^2 - f - \frac{g}{k^2}\right\}^{1/2}$$

$$c_3^2 \quad \frac{(\alpha_{10} - \alpha_{17})}{\lambda y} + \frac{(\alpha_2 - \alpha_7)}{\lambda y k^2}$$

where

$$p = \frac{\alpha_{10} + \alpha_{17}}{2y} + \frac{\alpha_3}{2}, \qquad q \quad \frac{\alpha_2 + \alpha_7}{2y} \tag{40}$$

$$f = \frac{\alpha_3(\alpha_{10} + \alpha_{17})}{y} \qquad g = \frac{\alpha_3(\alpha_2 + \alpha_7) - 2\alpha_8^2}{y} \tag{41}$$

All three wave speeds depend on the wave number; hence, the waves are dispersive. We now investigate the short and long wavelength limits of the three wave speeds. In the former case, the wave number $k \to \infty$ and the three limiting speeds are

$$\lim_{k\to\infty} c_1^2 = \frac{p}{\lambda} - \frac{1}{\lambda}(p^2 - f)^{1/2} = \frac{\alpha_{10}+\alpha_{17}}{\lambda y} \qquad \lim_{k\to\infty} c_2^2 = \frac{p}{\lambda} + \frac{1}{\lambda}(p^2 \quad f)^{1/2} \quad \frac{\alpha_3}{\lambda}$$

$$\lim_{k\to\infty} c_3^2 \quad \frac{\alpha_{10}-\alpha_{17}}{\lambda y} \tag{43}$$

whereas in the latter, $k \to 0$, two of the limiting speeds are ∞, and the only finite limiting speed is given by

$$\lim_{k\to 0} c^2 = \frac{g}{\quad} = \frac{1}{\lambda}\left[\alpha_3 - \frac{2\alpha_8^2}{\alpha_2+\alpha_7}\right] \tag{44}$$

In the special case of the theory when $\delta_1 = \delta_2 = \delta$, equations (19) and (20) may be expressed in a form involving potentials; we do not record these but just write the frequency equation in the form

$$\left[\frac{\alpha_{10}+\alpha_{17}}{\lambda y} + \frac{\alpha_2+\alpha_7}{\lambda y k^2} - c^2\right]\left[\frac{\alpha_3}{\lambda} - c^2\right] \quad \frac{2\alpha_8^2}{\lambda^2 y k^2} \quad 0. \tag{45}$$

It is clear that the special case yields a two-mode theory, whereas the general case gives a three-mode theory. The limiting values of the two phase speeds in the special case are given by $(42)_{1,2}$ for high frequencies. For low frequencies, one of the two limits is infinity, and the second is given by (44).

We now discuss the specification of the material constants that enter the constitutive equations (12) through (14). The constants α_1, α_2, α_3, α_7, α_8, and α_9—identified by Green et al. [6] by comparing exact equilibrium solutions from three-dimensional linear elasticity with corresponding solutions in the Cosserat theory, are given by

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{EA(1 \quad \nu)}{(1+\nu)(1-2\nu)}, \qquad \alpha_7 = \alpha_8 = \alpha_9 \quad \frac{EA\nu}{(1+\nu)(1-2\nu)},$$

where E is Young's modulus, ν is Poisson's ratio, and A is the cross-sectional area of the rod. The coefficients α_{10}, α_{11}, and α_{17} have been obtained using a similar procedure (involving an orthotropic rod) by Green and Naghdi [15]. Their results, when specialized to isotropic rods for which $y_{11} = y_{22} = y$, yield

$$\alpha_{10} \quad \alpha_{11} = \frac{EI}{2(1+\nu)}, \qquad \alpha_{17} \quad 0,$$

where $I(=Ay)$ is the area moment of inertia of the cross-section. We note that for $-1 < \nu < 0.5$, the values of the α's given in (46) and (47) satisfy the conditions noted following (14). If the three-dimensional rod has a uniform reference density ρ_0^*, then $\lambda = \rho_0 = \rho_0^* A$, and in view of $(47)_2$, the limiting phase speeds in $(42)_1$ and (43) become equal so that the high-frequency limits are

$$\lim_{k\to\infty} c_1^2 = \lim_{k\to\infty} c_3^2 = \frac{E}{2(1+\nu)\rho_0^*} = c_S^2, \quad \lim_{k\to\infty} c_2^2 = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho_0^*} = c_D^2,$$

where c_S and c_D are the speeds of propagation of shear and dilatational waves, respectively, in an unbounded linear elastic body. The low-frequency limit (44) reduces to

$$\lim_{k\to 0} c_1^2 = \frac{E}{\rho_0^*} = c_0^2,$$

where c_0 is the speed of propagation of longitudinal waves in the classical one-dimensional rod theory. The classical theory is a one-mode theory with nondispersive waves; it is a reasonable model only for low-frequency waves in a slender three-dimensional rod-like body. The Cosserat theory captures some limited but important three-dimensional effects; it predicts dispersion and accounts for coupling between longitudinal and thickness modes of wave propagation while retaining the simplicity of being a one-dimensional theory.

To ascertain the range of validity of the Cosserat theory, we now compare results with the corresponding results from the three-dimensional theory. We first recall that in the latter, the Pochhammer-Chree frequency equation associated with torsionless axisymmetric waves in a traction-free infinite circular cylinder is (see, e.g., Graff [19])

$$\frac{2\alpha}{a}(\beta^2+k^2)J_1(\alpha a)J_1(\beta a) - (\beta^2 - k^2)^2 J_0(\alpha a)J_1(\beta a) - 4k^2\alpha\beta J_1(\alpha a)J_0(\beta a) = 0,$$

where a is the radius of the cylinder, $\alpha^2 = \omega^2/c_D^2 - k^2$, $\beta^2 = \omega^2/c_S^2 - k^2$, and J_n is the Bessel function of order n. In the low-frequency limit, the phase speed corresponding to the first branch approaches c_0. For high frequencies, the phase speed of the first branch approaches the speed c_R of Rayleigh waves in a traction-free elastic half-space, indicating that high-frequency waves are confined to the lateral surface of the cylinder. This speed is given by the real root η of the cubic equation

$$\eta^3 - 8\eta^2 + (24 - 16\chi)\eta - 16(1-\chi) = 0,$$

where $\chi = (1-2\nu)/(1-\nu)$, and $\eta = c^2/c_S^2$. An approximate expression for c_R is given by the following (see [19, p. 326]):

$$c_R \simeq \frac{(0.87+1.12\nu)}{(1+\nu)} c_S = \frac{(0.87+1.12\nu)}{(1+\nu)} \left[\frac{E}{2(1+\nu)\rho_0^*}\right]^{1/2}$$

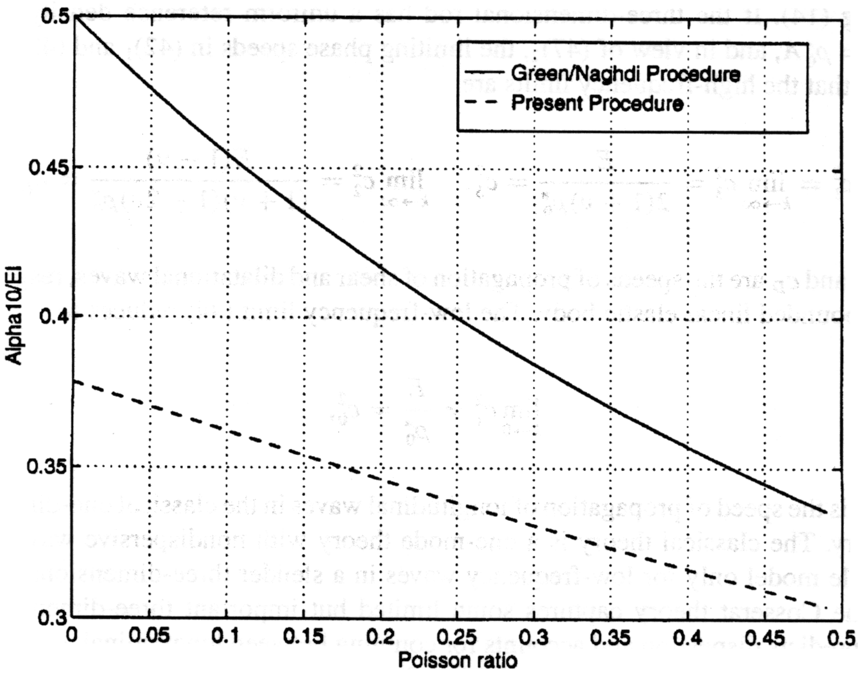

Fig. Variation of α_{10}/EI with Poisson's ratio.

To improve the high-frequency prediction made by the Cosserat theory, we choose the value of α_{10} (while retaining $\alpha_{17} = 0$), so that the phase speed in $(42)_1$ equals c_R—that is,

$$\alpha_{10} = \lambda y c_R^2 = \rho_0^* I c_R^2 \simeq \frac{(0.615 + 0.792\nu)^2}{(1 + \nu)^3} EI \tag{53}$$

For dynamic problems involving a Cosserat rod, it is preferable to assign α_{10} the value (53) rather than (47);[6] these two values of α_{10}/EI as a function of the Poisson ratio ν are plotted in Figure 1. The value (53) of α_{10} improves the high-frequency behavior of the first and third branches but is indifferent to the second since the latter has a limiting phase speed given by $(42)_2$, which is independent of α_{10}.

It is now appropriate to compare our procedure with that advocated by Rubin [20] and Naghdi and Rubin [21], although not in the context of elastic rods. Rubin's idea, when adapted to rods, suggests that the match of the first high-frequency limit be achieved by assigning a different value to the inertia coefficient y rather than by modifying the material constant α_{10}. This has the advantage of maintaining a match of static solutions as well. However, once a value of y is thus fixed, there is no more freedom in the theory to match the wave speeds of higher branches. The method proposed herein improves the high-frequency limiting speeds of all three branches, but a match of static solutions is lost.

4. PURE THICKNESS OSCILLATIONS

For a circular rod, we observe that the limiting phase speed of the second branch is c_S from the Pochhammer-Chree equation but is c_D in the special case of the Cosserat theory (see (45) and the paragraph following (45)). A similar situation arises in the theory of [18], who comment that "this appears to be a defect in the present theory, but perhaps judgement should be withheld until solutions for finite bars of very short length are studied in detail" (p. 190). The Mindlin-Herrmann theory has been discussed by Graff [19], who has remarked that the aforementioned defect cannot be corrected within the framework of their theory. We now present a procedure within the framework of the Cosserat theory that partially rectifies this defect. The procedure is motivated by a paper by Deresiewicz and Mindlin [22], who discuss the determination of Timoshenko's shear coefficient in the flexural vibrations of beams. While we cannot make the phase speed of the second branch approach c_S in the high-frequency limit, we demonstrate that it can be improved from c_D to a value closer to c_S by an appropriate choice of the coefficient α_3.

We recall that Green et al. [7] and Green and Naghdi [15] have determined the constitutive constants entering the Cosserat theory of rods on the basis of exact solutions in the three-dimensional static theory of elasticity (i.e., at zero frequency).[7] Hence, these values are acceptable for dynamical problems in which the frequencies involved are small. However, as the frequencies become large, the frequency spectrum changes significantly at a frequency $\bar{\omega}$ when the infinite rod undergoes pure thickness oscillations, that is, oscillations in the absence of longitudinal displacement of particles on the curve $\mathcal{L}$. It is therefore desirable that the value $\bar{\omega}$, as obtained from the Cosserat theory, match that predicted by the three-dimensional theory. In the latter, a traction-free infinite circular cylinder undergoes pure radial oscillations at this frequency. With this objective in mind, we calculate the frequency of pure thickness oscillations in the two theories.

Consider the special case of the Cosserat theory governed by equations (19) and (20). We set the longitudinal displacement u to be zero and observe that (20) then forces δ to be uniform in z. Consequently, δ depends on time only, and (19) implies that it is governed by the simple harmonic equation

$$\ddot{\delta} + \frac{\alpha_2 + \alpha_7}{\lambda y}\delta \quad 0.$$

Hence, the circular frequency of pure thickness oscillations is given by

$$\bar{\omega}^2 \quad \frac{\alpha_2 + \alpha_7}{\lambda y}.$$

In the three-dimensional theory, the equation governing pure radial oscillations of an infinitely long circular cylinder of radius a is obtained by assuming that the displacement $\mathbf{u}$ is of the form

$$\mathbf{u} = U(r)e^{i\bar{\omega}t}\mathbf{e}_r.$$

where r is the radial polar coordinate and $\mathbf{e}_r$ is the associated unit vector. Substituting (56) into the equations of motion yields the one nontrivial equation

$$r^2 U'' + rU' + \left(\frac{\bar{\omega}^2}{c_D^2} - 1\right) U \quad 0,$$

where a prime denotes differentiation with respect to r. For a traction-free lateral surface, the only nontrivial boundary condition to be satisfied is the vanishing of the radial stress

$$\tau_{rr}|_{r=a} \quad \left(\rho_0^* c_D^2 U' + \frac{E\nu}{(1+\nu)(1-2\nu)} \frac{U}{r}\right)\Bigg|_{r=a} = 0. \tag{58}$$

The solution of (57), which is bounded at $r \quad 0$, is given by

$$U(r) \quad A J_0(\beta r), \qquad \beta^2 \quad \frac{\bar{\omega}^2}{c_D^2}$$

The boundary condition (58) requires that

$$\chi J_1(\beta a) = \beta a J_0(\beta a), \tag{60}$$

whose roots yield frequencies of pure radial oscillations. By matching the frequency generated from the second nonzero root with the expression (55), we determine $\alpha_2 + \alpha_7$. The choice of the second root, rather than the first one, will be justified shortly. The variation of $\alpha_2 + \alpha_7$, determined with ν, is plotted in Figure 2. In the same figure, we also plot $\alpha_2 + \alpha_7$, as determined by $(46)_{1,2}$. We note that for $\nu = 0.29$, the value of $\alpha_2 + \alpha_7$ from $(46)_{1,2}$ equals 1.846 EA but is 9.589 EA from the procedure just described.

We recall that the low-frequency limit (44) reduces to the Pochhammer-Chree prediction when α_2 and α_7 are assigned the values $(46)_{1,2}$. However, the new prescription for $\alpha_2 + \alpha_7$ will affect this low-frequency limit; we now remedy this problem. Besides $\alpha_2 + \alpha_7$, the phase speed in (44) depends on α_3 and α_8. We now demand that (44) still reduce to $c_0^2 = E/\rho_0^*$ with the new value of $\alpha_2 + \alpha_7$ by an appropriate choice of either α_3 or α_8. There is a compelling reason to modify α_3 because if we retain α_3 as in $(46)_1$, the high-frequency limiting speed $(42)_2$ of the second branch in the special case of the Cosserat theory reduces to c_D (see $(48)_2$), while the Pochhammer-Chree equation predicts a limiting speed c_S. This defect was mentioned above. So, we retain α_8 as in $(46)_2$, use the new value of $\alpha_2 + \alpha_7$, and choose α_3 so that the expression in (44) reduces to $c_0^2 = E/\rho_0^*$. Fortunately, this choice of α_3 helps remedy the defect, as we shall see in the next paragraph. Figure 3 depicts the variation of α_3 thus obtained with ν and also that of α_3 computed from $(46)_1$. We note that for $\nu = 0.29$, $(46)_1$ and the present procedure respectively yield $\alpha_3 = 1.060$ EA and $\alpha_3 = 1.310$ EA.

Our prescription of α_3 first entailed a modification of $\alpha_2 + \alpha_7$; the value of the latter is influenced by the roots of (60). Since α_3 affects the phase speed of the second mode of the special case of the Cosserat theory (see $(42)_2$), we match $\alpha_2 + \alpha_7$ from the second

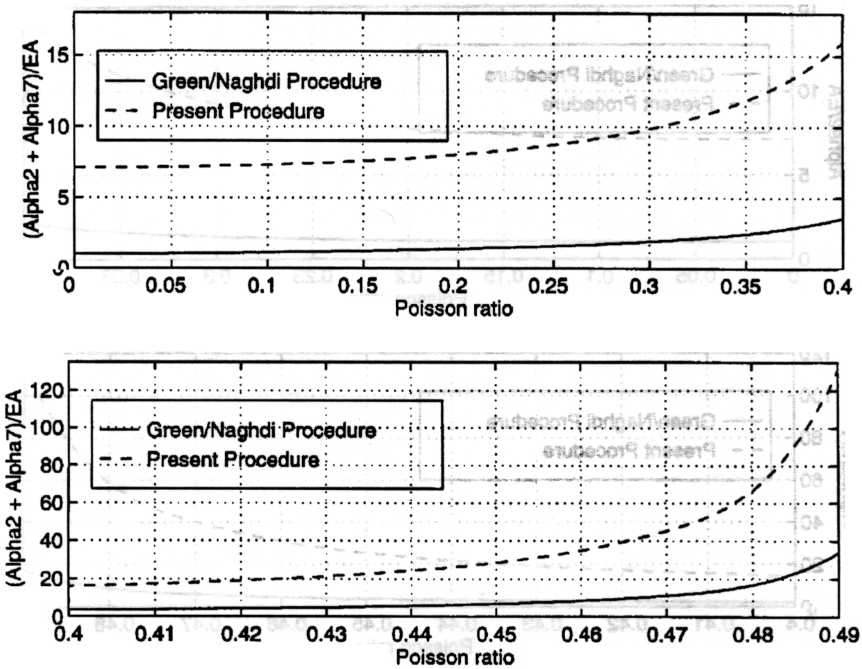

Fig. 2. Variation of $(\alpha_2+\alpha_7)/EA$ with Poisson's ratio ν (a) for $0 \le \nu \le 0.4$, and (b) for $0.4 \le \nu \le 0.49$.

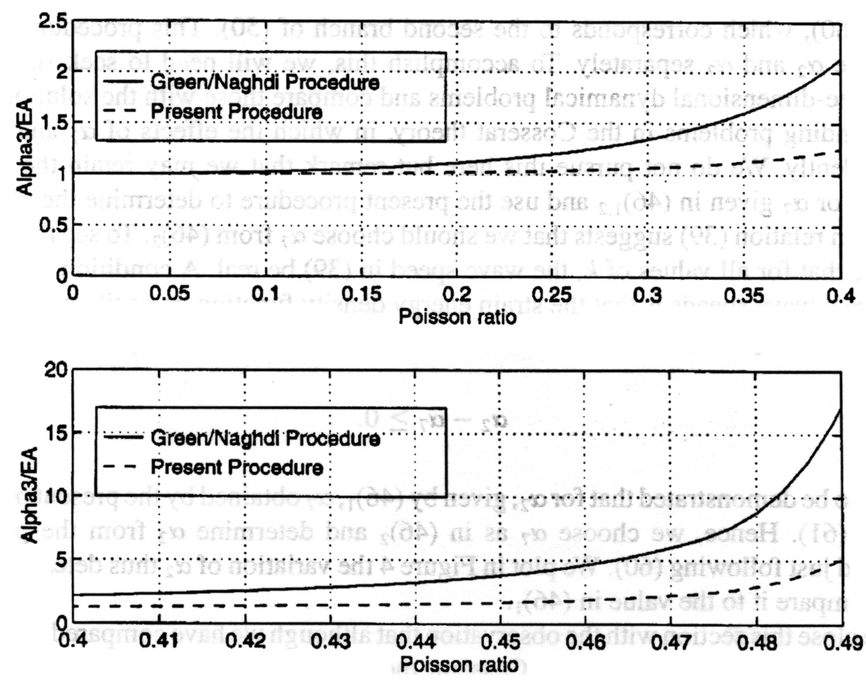

Fig. 3. Variation of α_3/EA with Poisson's ratio ν (a) for $0 \le \nu \le 0.4$, and (b) for $0.4 \le \nu \le 0.49$.

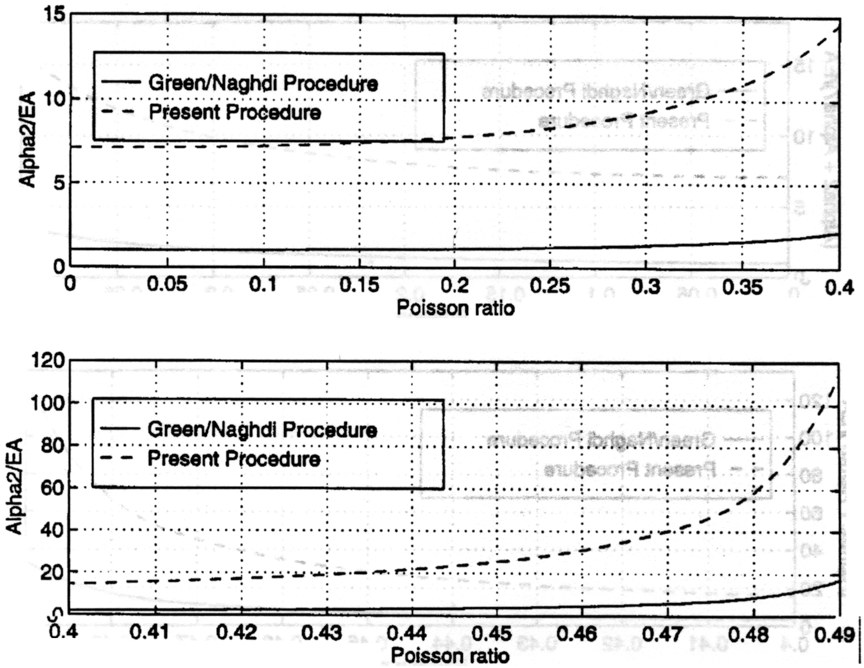

Fig. 4. Variation of α_2/EA with Poisson's ratio ν (a) for $0 \leq \nu \leq 0.4$, and (b) for $0.4 \leq \nu \leq 0.49$.

root of (60), which corresponds to the second branch of (50). This procedure does not determine α_2 and α_7 separately. To accomplish this, we will need to seek solutions of other three-dimensional dynamical problems and compare those with the solutions of the corresponding problems in the Cosserat theory, in which the effects of α_2 and α_7 arise independently. We do not pursue this here but remark that we may retain the value of either α_2 or α_7 given in $(46)_{1,2}$ and use the present procedure to determine the other. The dispersion relation (39) suggests that we should choose α_7 from $(46)_2$. To see this, we first stipulate that for all values of k, the wave speed in (39) be real. A condition sufficient to ensure real wave speeds is that the strain energy density function be positive definite (see [4]), and this implies that[8]

$$\alpha_2 - \alpha_7 \geq 0. \tag{61}$$

It can also be demonstrated that for α_2, given by $(46)_1$, α_7 obtained by the present procedure violates (61). Hence, we choose α_7 as in $(46)_2$ and determine α_2 from the procedure described just following (60). We plot in Figure 4 the variation of α_2 thus determined with ν and compare it to the value in $(46)_1$.

We close this section with the observation that although we have compared the limiting wave speeds of all branches of the Cosserat theory with those of the three-dimensional theory, we have not addressed their modes of deformation. We note that equations $(32)_{1,2,3}$ may be used to calculate these modes.

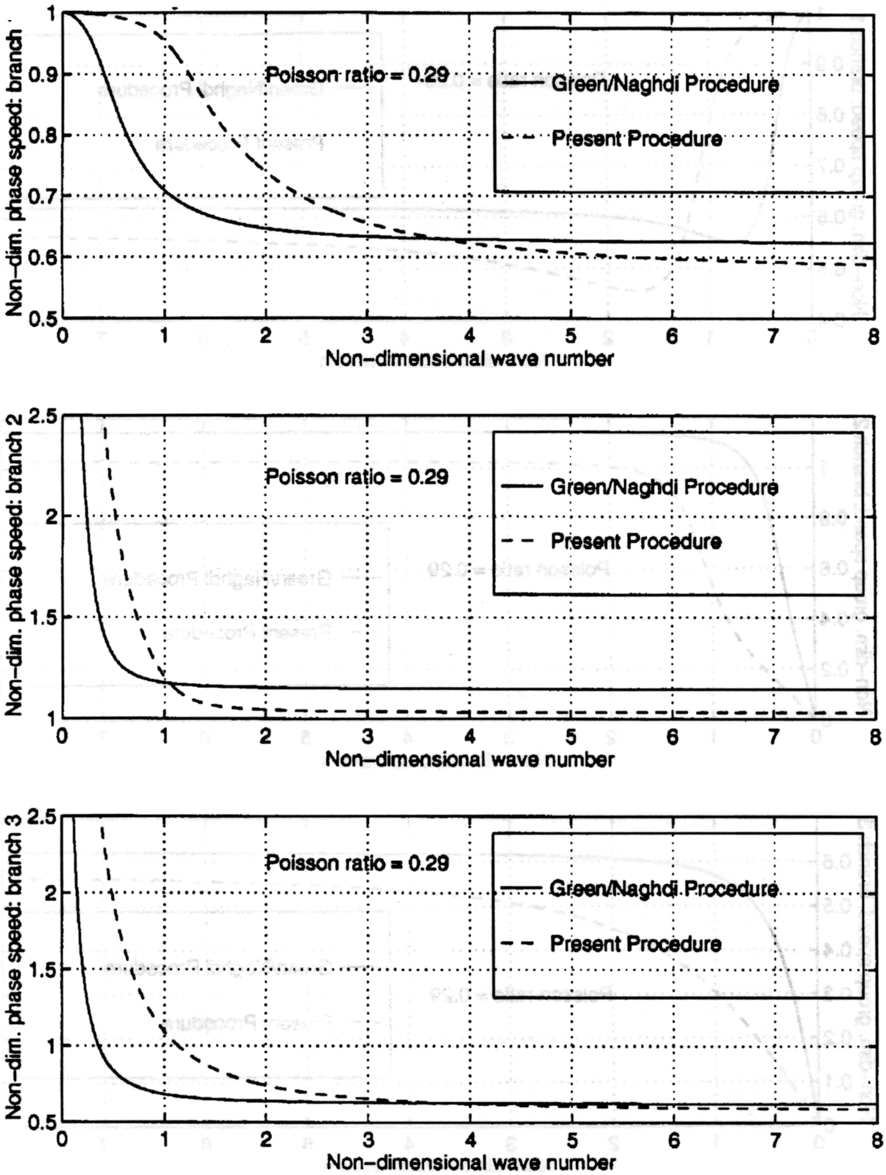

Fig. 5. Dispersion curves for the three branches.

5. DISPERSION: PHASE SPEEDS AND GROUP SPEEDS

For Poisson's ratio to equal 0.29, Figure 5 depicts the variation of the nondimensional wave speed $\bar{c} = c/c_0$ with the nondimensional wave number $\bar{k} = ka/2\pi$ for a circular rod of radius a for each of the branches given by (37), (38), and (39), computed with the new values of α_{10}, α_2, and α_3 and also with their values given by (46) and (47).

The two dispersion curves do not differ significantly for $\nu = 0.29$. However, when the new values of α_{10}, α_2, and α_3 differ noticeably from those given by (46) and (47), the two

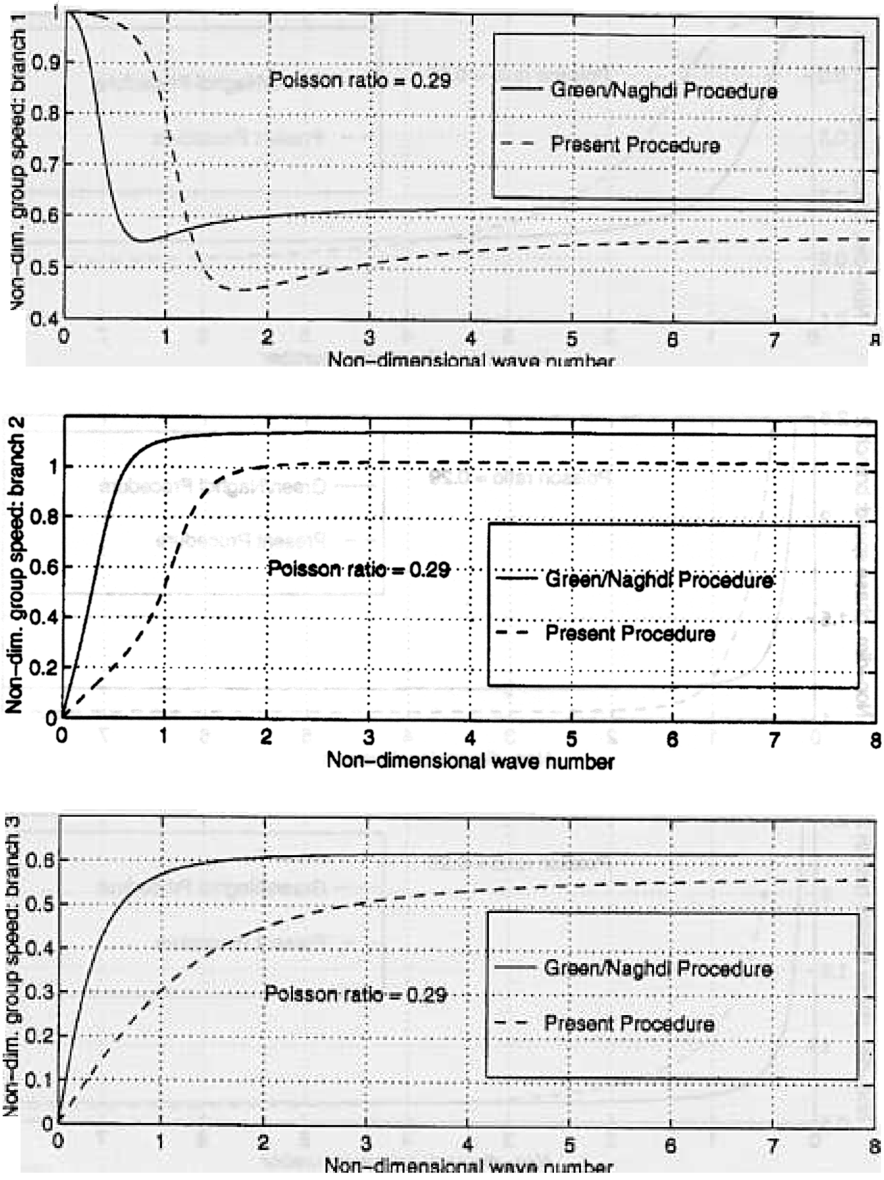

Fig. 6. Variation of nondimensional group speed with nondimensional wave number for three branches.

dispersion curves might be further apart. From a theoretical viewpoint, the present values are desirable as they approach the correct Pochhammer-Chree limits for both low and high frequencies. Bancroft [24] and Davies [25] have discussed the exact dispersion curves. Our results compare well with the analytical results for the two higher branches. For the first mode, at intermediate frequencies, the present results are slightly worse possibly due to the present choice of α_2. We recall that we could only fix the sum $\alpha_2 + \alpha_7$, we then chose to retain α_7 in $(46)_2$ to determine α_2. It may be more desirable to obtain α_7 from the exact solution of another dynamical problem.

In dispersive media, the group speed c_g represents the speed of propagation of a wave packet consisting of waves whose wavelengths are close to a certain fixed value. Also, it is the speed at which energy is transmitted in such media. Recalling that

$$c_g = c + k\frac{dc}{dk},$$

we compute the group speeds for each of the three branches of the spectrum to be

$$c_{g1} = c_1 + \frac{(\alpha_7 - \alpha_2)}{\lambda y c_1 k^2},$$

$$c_{g2} = c_2 - \frac{q}{\lambda c_2 k^2} - \frac{1}{2\lambda c_2 k^2}\frac{\left\{g - 2q\left(p + \frac{q}{k^2}\right)\right\}}{\left\{\left(p + \frac{q}{k^2}\right)^2 - f - \frac{g}{k^2}\right\}^{1/2}}, \tag{64}$$

$$c_{g3} = c_3 - \frac{q}{\lambda c_3 k^2} + \frac{1}{2\lambda c_3 k^2}\frac{\left\{g - 2q\left(p + \frac{q}{k^2}\right)\right\}}{\left\{\left(p + \frac{q}{k^2}\right)^2 - f - \frac{g}{k^2}\right\}^{1/2}}$$

We plot the variation of the nondimensional group speed $\bar{c}_g = c_g/c_0$ with the nondimensional wave number $\bar{k} = ka/2\pi$ in Figure 6 for the three branches (63) through (65); Davies [25] plots the group speeds of the first two branches calculated from the Pochhammer-Chree equation. Our result for the first branch agrees well with the exact result. However, for a small range of low frequencies for the second branch, the Cosserat theory appears to smear out the variation depicted in [25]. We suspect that it is also the case for the third branch, although the exact c_g curve is not plotted in [25].

For noncircular geometries, exact results are intractable, while the Cosserat theory is capable of generating explicit expressions for wave speeds as well as group speeds. We recognize that the latter theory represents an approximation of the three-dimensional behavior of rod-like bodies but does provide a powerful theoretical tool for the analysis of wave propagation in such bodies.

NOTES

1. Unless stated otherwise, Greek subscripts and superscripts range from 1 to 2, and Latin ones range from 1 to 3.
2. The vanishing of $\mathbf{f}$ and $\mathbf{l}^\alpha$ corresponds to the absence of both body forces and tractions on the lateral surface in the three-dimensional rod-like body.

3. For a discussion of invariance requirements under superposed rigid motions of a Cosserat rod undergoing infinitesimal deformations, the reader is referred to Green, Naghdi, and Wenner [7] and O'Reilly and Turcotte [16].
4. When Cartesian coordinates are used, no distinction need be made between superscripts and subscripts, and repeated subscripts imply summation over the range of indices.
5. A recent paper by Green and Naghdi [17] contains several illuminating remarks on constitutive coefficients for elastic rods and shells.
6. The situation here is similar to a corresponding situation in the shear deformation theory of plate bending, in which Reissner's shear correction factor is 5/6, whereas Mindlin's is $\pi^2/12$.
7. In a recent paper, Rubin [23] presents a different procedure to identify a material constant in the context of the linear flexural theory of Cosserat rods. Once again, this procedure is based on a comparison with static three-dimensional solutions.
8. In the notation of Green, Laws, and Naghdi [4], we only need consider an isothermal deformation in which $\gamma_{11} = -\gamma_{22}$, $\gamma_{ij} = 0$ $(i \neq j)$ and $\kappa_{\alpha i} = 0$ to obtain this result.

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