Universal Relations for Transversely Isotropic Elastic Materials

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(Received 8 March 2002; Final version 13 March 2002)

Dedicated to Millard F. Beatty, respected colleague and friend.

Abstract: We adopt Hayes and Knops's approach and derive universal relations for finite deformations of a transversely isotropic elastic material. Explicit universal relations are obtained for homogeneous deformations corresponding to triaxial stretches, simple shear, and simultaneous shear and extension. Universal relations are also derived for five families of nonhomogeneous deformations.

1. INTRODUCTION

Truesdell and Noll [1, section 54] and Beatty [2] have lucidly stated the importance of universal relations for finite deformations of isotropic elastic materials; the same remarks apply to transversely isotropic elastic materials. Beatty [2, 3] has derived a class of universal relations for constrained and unconstrained isotropic elastic materials. Pucci and Saccomandi [4], have given a general approach for finding universal relations in continuum mechanics. Saccomandi and Vianello [5] have given a variational characterization of the universal relations for hemitropic materials. They have proved that the class of transversely hemitropic hyperelastic bodies is characterized by the condition that the inner product between $\mathbf{SC}-\mathbf{CS}$ and $\mathbf{W}_{\mathrm{N}}$ vanishes for all deformations of the body; $\mathbf{S}$ is the second Piola–Kirchhoff stress tensor, $\mathbf{C}$ the right Cauchy–Green tensor and $\mathbf{W}_{\mathrm{N}}$ the skew-symmetric tensor associated with the unit vector $\mathbf{N}$ which points along the axis of transverse isotropy. Here we adopt the approach employed by Hayes and Knops [6] for deriving universal relations for isotropic elastic materials and use it to deduce universal relations for transversely isotropic elastic materials. The general relations so obtained are applied to three classes, namely triaxial extension, simple shear and simultaneous shear and extension, of homogeneous deformations to get explicit universal relations for these deformations. We note that homogeneous deformations can be produced by the action of surface tractions alone in every homogeneous elastic body; Ericksen [7] proved that these are the only deformations that can be produced in every isotropic compressible elastic body. We also derive universal relations for five families of nonhomogeneous deformations which, according to Ericksen [8], can be produced in every isotropic incompressible hyperelastic body by the action of surface tractions alone.

Universal relations given here are properties of the constitutive relation for a transversely isotropic elastic material. They are valid for both static and dynamic deformations, and

 DOI: 10.1177/108128028483
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whether or not body forces are required to produce the envisaged deformations. Some of these are easily verifiable experimentally. Universal relations for nonhomogeneous deformations hold pointwise and will require local measurements.

2. UNIVERSAL RELATIONS

The constitutive relation in terms of the Cauchy stress tensor $\mathbf{T}$ and the left Cauchy–Green strain tensor $\mathbf{B}$ for a transversely isotropic elastic material with the axis of transverse isotropy along the unit vector $\mathbf{n}$ in the present configuration is (e.g. see [9])

$$\mathbf{T} = \gamma_1 \mathbf{1} + \gamma_2 \mathbf{B} + \gamma_3 \mathbf{B}^2 + \gamma_4 \mathbf{n} \otimes \mathbf{n} + \gamma_5 (\mathbf{n} \otimes \mathbf{B}\mathbf{n} + \mathbf{B}\mathbf{n} \otimes \mathbf{n}).$$

Here the response functions $\gamma_1, \gamma_2, \quad, \gamma_5$ are functions of the five principal invariants $I_1, I_2, \ldots, I_5$ defined as

$$I_1 = \mathrm{tr}(\mathbf{B}) \qquad I_2 = \mathrm{tr}(\mathbf{B}^2), \qquad I_3 = \mathrm{tr}(\mathbf{B}^3), \qquad I_4 = \mathbf{n} \cdot \mathbf{B}\mathbf{n}, \qquad I_5 \quad \mathbf{n} \cdot \mathbf{B}^2\mathbf{n}.$$

The tensor product, $\mathbf{a} \otimes \mathbf{b}$, between vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$ for every vector $\mathbf{c}$, and $\mathbf{a} \cdot \mathbf{b}$ equals the inner product between vectors $\mathbf{a}$ and $\mathbf{b}$. For an incompressible transversely isotropic elastic material, γ_1 in (2.1) is replaced by an arbitrary pressure $-p$, and $\mathbf{B}$ must satisfy $\det \mathbf{B} = 1$.

Equation (2.1) implies that if $\mathbf{n}$ is an eigenvector (or proper or principal vector) of $\mathbf{B}$, then it is also an eigenvector of $\mathbf{T}$. Also, the other two eigenvectors of $\mathbf{B}$ which are perpendicular to $\mathbf{n}$ are also eigenvectors of $\mathbf{T}$. If any two eigenvalues of $\mathbf{B}$ are equal, then the corresponding eigenvalues of $\mathbf{T}$ are also equal only if the eigenvector of $\mathbf{B}$ corresponding to the distinct eigenvalue is parallel to $\mathbf{n}$. However, if $\mathbf{n}$ is not an eigenvector of $\mathbf{B}$, then an eigenvector of $\mathbf{B}$ is not an eigenvector of $\mathbf{T}$. For

$$\gamma_2 + I_1 \gamma_3 > 0, \qquad \gamma_3 \leq 0, \qquad \gamma_4 = \gamma_5 = 0,$$

it follows from Batra's [10] theorem that eigenvectors of $\mathbf{T}$ and $\mathbf{B}$ coincide and if two eigenvalues of $\mathbf{T}$ are equal, then the corresponding two eigenvalues of $\mathbf{B}$ are also equal. Note that the response functions $\gamma_1, \gamma_2, \ldots, \gamma_5$ while satisfying (2.2) may still depend upon $\mathbf{n}$ through their dependence on the principal invariants I_4 and I_5.

We first consider homogeneous deformations of a homogeneous body so that the deformation gradient $\mathbf{F}$, the left Cauchy–Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and the Cauchy stress tensor $\mathbf{T}$ are constants within the body. Hence equations of equilibrium with zero body force are identically satisfied. Without any loss of generality, we use rectangular Cartesian coordinates with orthonormal basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

Following Hayes and Knops [6], we note that if, for a given deformation, there is a relationship of the form

$$a_{ij} T_{ij} = 0$$

where $\mathbf{a}$ is a symmetric second-order tensor independent of the response functions $\gamma_1, \gamma_2, \ldots, \gamma_5$, and a repeated index implies summation over the range of the index, then equation (2.3) will be a universal relation. Hayes and Knops used it to derive universal relations for an unconstrained isotropic elastic body. Here we use it to find universal relations for an unconstrained and also for an incompressible transversely isotropic elastic body.

Let λ_1^2, λ_2^2 and λ_3^2 be eigenvalues of $\mathbf{B}$ with the corresponding orthonormal eigenvectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$. Thus $\mathbf{B}$ and $\mathbf{B}^2$ have the representations

$$B_{ij} \qquad \lambda_1^2 p_i p_j + \lambda_2^2 q_i q_j + \lambda_3^2 r_i r_j, \tag{2.4$_1$}$$
$$(B^2)_{ij} \qquad \lambda_1^4 p_i p_j + \lambda_2^4 q_i q_j + \lambda_3^4 r_i r_j. \tag{2.4$_2$}$$

We assume that the deformation gradient $\mathbf{F}$ and the left Cauchy–Green tensor $\mathbf{B}$ are given. Substitution from (2.4) into (2.1) and the result into (2.3) gives

$$\begin{aligned} &\gamma_1 a_{ii} + \gamma_2 a_{ij} \left(\lambda_1^2 p_i p_j + \lambda_2^2 q_i q_j + \lambda_3^2 r_i r_j\right) + \gamma_3 a_{ij} \left(\lambda_1^4 p_i p_j + \lambda_2^4 q_i q_j + \lambda_3^4 r_i r_j\right. \\ + \;& \gamma_4 a_{ij}\, n_i n_j + \gamma_5 \big[a_{ij} \left(\lambda_1^2 n_i p_j p_k n_k + \lambda_2^2 n_i q_j q_k n_k\right. \\ + \;& \lambda_3^2 n_i r_j r_k n_k + \lambda_1^2 n_j p_i p_k n_k + \lambda_2^2 n_j q_i q_k n_k + \lambda_3^2 n_j r_i r_k n_k)\big] = 0. \qquad (\end{aligned}$$

In order that equation (2.5) hold for all transversely isotropic elastic materials, it must be satisfied for all choices of $\gamma_1, \gamma_2, \ldots, \gamma_5$. Thus

$$\begin{aligned} a_{ii} &= 0, \\ a_{ij} \left(\lambda_1^2 p_i p_j + \lambda_2^2 q_i q_j + \lambda_3^2 r_i r_j\right) &= 0, \\ a_{ij} \left(\lambda_1^4 p_i p_j + \lambda_2^4 q_i q_j + \lambda_3^4 r_i r_j\right) &= 0, \\ a_{ij}\, n_i n_j &= 0, \\ a_{ij} \left[\lambda_1^2 (n_i p_j + n_j p_i) p_k n_k + \lambda_2^2 (n_i q_j + n_j q_i) q_k n_k + \lambda_3^2 (n_i r_j + n_j r_i) r_k n_k\right] &= 0. \end{aligned} \tag{2.6}$$

In terms of the eigenvectors of $\mathbf{B}$, let $\mathbf{a}$ have the representation

$$a_{ij} = \alpha p_i p_j + \beta q_i q_j \quad \gamma r_i r_j + \delta(p_i q_j + p_j q_i) + \epsilon(q_i r_j + q_j r_i) + \mu(r_i p_j + r_j p_i), \tag{2.7}$$

where constants $\alpha, \beta, \gamma, \delta, \epsilon$ and μ are to be determined. Equations (2.6) and (2.7) give

$$\begin{aligned} &\alpha + \beta + \gamma = 0, \\ &\alpha\lambda_1^2 + \beta\lambda_2^2 + \gamma\lambda_3^2 = 0, \\ &\alpha\lambda_1^4 + \beta\lambda_2^4 + \gamma\lambda_3^4 = 0, \\ &\alpha(p_i n_i)^2 + \beta(q_i n_i)^2 + \gamma(r_i n_i)^2 + 2\delta(p_i n_i q_j n_j) + 2\epsilon(q_i n_i r_j n_j) + 2\mu(r_i n_i p_j n_j) = 0, \\ &\alpha\lambda_1^2 (p_i n_i)^2 + \beta\lambda_2^2 (q_i n_i)^2 + \gamma\lambda_3^2 (r_i n_i)^2 + \delta(\lambda_1^2 + \lambda_2^2)(p_i n_i q_j n_j) \\ &+ \epsilon(\lambda_2^2 + \lambda_3^2)(r_i n_i) q_j n_j + \mu(\lambda_3^2 + \lambda_1^2)(p_i n_i)(r_j n_j) = 0. \end{aligned}$$

We now consider the following three cases:

(i) $\lambda_1^2 \neq \lambda_2^2 \neq \lambda_3^2$; (ii) $\lambda_1^2 = \lambda_2^2 \neq \lambda_3^2$; (iii) $\lambda_1^2 = \lambda_2^2 = \lambda_3^2$

For case (i), the only solution of equations $(2.8)_1$, $(2.8)_2$ and $(2.8)_3$ is

$$\alpha = \beta \quad \gamma = 0. \tag{2.10}$$

Thus equations $(2.8)_4$ and $(2.8)_5$ simplify to

$$\begin{aligned} \delta(p_i n_i)q_j n_j + \epsilon(q_i n_i)r_j n_j + \mu(r_i n_i)p_j n_j &= 0, \\ (\lambda_1^2 + \lambda_2^2)\delta(p_i n_i)q_j n_j + (\lambda_2^2 + \lambda_3^2)\epsilon(r_i n_i)q_j n_j + (\lambda_3^2 + \lambda_1^2)\mu(p_i n_i)r_j n_j &= 0. \end{aligned} \tag{2.11}$$

The problem of finding universal relations reduces to that of finding solutions of equation (2.3) for all choices of δ, ϵ and μ that satisfy equations (2.11). Alternatively, introducing Lagrange multipliers Λ_1 and Λ_2, the following equation (2.12) must hold for all choices of δ, ϵ and μ:

$$\begin{aligned} & 2\delta T_{ij} p_i q_j + 2\epsilon T_{ij} q_i r_j + 2\mu T_{ij} r_i p_j \\ & \Lambda_1[\delta(p_i n_i)q_j n_j + \epsilon(q_i n_i)r_j n_j + \mu(r_i n_i)p_j n_j] \\ & \Lambda_2[(\lambda_1^2 + \lambda_2^2)\delta(p_i n_i)q_j n_j + (\lambda_2^2 + \lambda_3^2)\epsilon(r_i n_i)q_j n_j \\ + & (\lambda_3^2 + \lambda_1^2)\mu(p_i n_i)r_j n_j] = 0. \end{aligned} \tag{2.12}$$

Thus

$$\begin{aligned} 2T_{ij} p_i q_j - \Lambda_1 p_i n_i q_j n_j - \Lambda_2(\lambda_1^2 + \lambda_2^2)p_i n_i q_j n_j &= 0, \\ 2T_{ij} q_i r_j - \Lambda_1 q_i n_i r_j n_j - \Lambda_2(\lambda_2^2 + \lambda_3^2)r_i n_i q_j n_j &= 0, \\ 2T_{ij} r_i p_j - \Lambda_1 r_i n_i p_j n_j - \Lambda_2(\lambda_3^2 + \lambda_1^2)p_i n_i r_j n_j &= 0. \end{aligned} \tag{2.13}$$

When one of the eigenvectors of $\mathbf{B}$, say $\mathbf{p}$, is parallel to $\mathbf{n}$, then

$$q_i n_i = r_i n_i = 0,$$

and equations (2.13) yield

$$T_{ij} p_i q_j = T_{ij} q_i r_j = T_{ij} r_i p_j = 0.$$

Hence $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are eigenvectors of $\mathbf{T}$ or eigenvectors of $\mathbf{B}$ are also eigenvectors of $\mathbf{T}$; a property already observed above. In this case, equations (2.15) are the three universal relations.

If none of the eigenvectors of $\mathbf{B}$ is parallel to $\mathbf{n}$, then $p_i n_i \neq 0$, $q_i n_i \neq 0$, $r_i n_i \neq 0$. We can eliminate Λ_1 and Λ_2 from equations (2.13) to arrive at the following universal relations:

$$\begin{aligned} &\frac{1}{(\lambda_2^2-\lambda_1^2)}\left[\frac{T_{ij}\,q_i r_j}{(q_k n_k)(r_\ell n_\ell)} - \frac{T_{ij}\,r_i p_j}{(r_k n_k)(p_\ell n_\ell)}\right] \\ &\frac{1}{(\lambda_3^2-\lambda_2^2)}\left[\frac{T_{ij}\,r_i p_j}{(r_k n_k)(p_\ell n_\ell)} - \frac{T_{ij}\,p_i q_j}{(p_k n_k)(q_\ell n_\ell)}\right] \\ &\frac{1}{}\left[\frac{T_{ij}\,p_i q_j}{(p_k n_k)(q_\ell n_\ell)} - \frac{T_{ij}\,q_i r_j}{(q_k n_k)(r_\ell n_\ell)}\right] \end{aligned} \tag{2.16}$$

Substitution from constitutive relation (2.1) into (2.16) reveals that each term equals γ_5. Thus relations (2.16) provide a way to determine γ_5 from the experimental data.

For case (ii) of equations (2.9), the solution of equations (2.8_1)–(2.8_3) is

$$\alpha+\beta \quad 0, \qquad \nu \quad 0,$$

and equations (2.8_4) and (2.8_5) reduce to

$$\begin{aligned} &\alpha[(p_i n_i)^2-(q_i n_i)^2]+2\delta(p_i n_i)q_j n_j \\ +\;& 2\epsilon(q_k n_k)r_i n_i+2\mu(r_i n_i)(p_j n_j) = 0, \\ &\alpha\lambda_1^2((p_i n_i)^2-(q_i n_i)^2)+2\delta\lambda_1^2(q_i n_i)p_j n_j \\ +\;& \epsilon(\lambda_1^2+\lambda_3^2)(r_i n_i)(q_j n_j)+\mu(\lambda_3^2+\lambda_1^2)(n_i p_i)r_j n_j = 0. \end{aligned}$$

The analogue of equation (2.12) is

$$\begin{aligned} &\alpha T_{ij}(p_i p_j - q_i q_j)+2\delta T_{ij}\,p_i q_j+2\epsilon T_{ij}\,q_i r_j+2\mu T_{ij}\,r_i p_j \\ &\Lambda_1[\alpha((p_i n_i)^2-(q_i n_i)^2)+2\delta(p_i n_i)(q_j n_j) \\ +\;& 2\epsilon(q_i n_i)(r_j n_j)+2\mu(r_i n_i)(p_j n_j)] \\ &\Lambda_2[\alpha\lambda_1^2((p_i n_i)^2-(q_i n_i)^2)+2\delta\lambda_1^2(q_i n_i)(p_j n_j) \\ &\epsilon(\lambda_1^2+\lambda_3^2)(r_i n_i)(q_j n_j)+\mu(\lambda_3^2+\lambda_1^2)(n_i p_i)(r_j n_j)] = 0. \end{aligned}$$

The necessary and sufficient conditions for equation (2.19) to hold for all choices of α, δ, μ and ϵ are

$$\begin{aligned} T_{ij}(p_i p_j - q_i q_j)-\Lambda_1[(p_i n_i)^2-(q_i n_i)^2]-\Lambda_2\lambda_1^2[(p_i n_i)^2-(q_i n_i)^2] &= 0, \\ T_{ij}\,p_i q_j-\Lambda_1(p_k n_k)(q_j n_j)-\Lambda_2\lambda_1^2(q_i n_i)(p_j n_j) &= 0, \\ 2T_{ij}\,q_i r_j-2\Lambda_1(q_k n_k)(r_i n_i)-\Lambda_2(\lambda_1^2+\lambda_3^2)(q_i n_i)(r_j n_j) &= 0, \\ 2T_{ij}\,r_i p_j-2\Lambda_1(r_k n_k)(p_\ell n_\ell)-\Lambda_2(\lambda_3^2+\lambda_1^2)(n_\ell p_\ell)(r_i n_i) &= 0. \end{aligned} \tag{2.20}$$

If eigenvector $\mathbf{r}$ corresponding to the distinct eigenvalue λ_3^2 of $\mathbf{B}$ is parallel to $\mathbf{n}$, then $p_\ell n_\ell = 0$, $q_\ell n_\ell = 0$, and equations (2.20) give

$$\begin{aligned} T_{ij}\, p_i q_j &= 0, \qquad T_{ij}\, q_i r_j = 0, \qquad T_{ij}\, r_i p_j = 0, \\ T_{ij}\,(p_i p_j - q_i q_j) &\quad 0. \end{aligned}$$

That is, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are eigenvectors of $\mathbf{T}$ and eigenvalues corresponding to eigenvectors $\mathbf{p}$ and $\mathbf{q}$ are equal. However, if eigenvector $\mathbf{p}$ corresponding to the repeated eigenvalue λ_1 is parallel to $\mathbf{n}$ then $q_i n_i = 0$, $r_i n_i = 0$, and equations (2.20) reduce to

$$T_{ij}\,(p_i p_j - q_i q_j \quad - \Lambda_1 - \Lambda_2 \lambda_1^2 = 0, \;\; T_{ij}\, p_i q_j = 0, \;\; T_{ij}\, q_i r_j \quad 0, \;\; T_{ij}\, r_i p_j = 0.$$

That is, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are eigenvectors of $\mathbf{T}$ but eigenvalues of $\mathbf{T}$ corresponding to eigenvectors $\mathbf{p}$ and $\mathbf{q}$ are not equal for all transversely isotropic elastic materials.

When none of the eigenvectors of $\mathbf{B}$ is parallel to $\mathbf{n}$ and $\mathbf{p}\cdot\mathbf{n} \neq \mathbf{q}\cdot\mathbf{n}$, then the elimination of Λ_1 and Λ_2 from equations (2.20) gives

$$\begin{aligned} \frac{T_{ij}\, p_i r_j}{p_k n_k} &= \frac{T_{ij}\, q_i r_j}{q_\ell n_\ell}, \\ \frac{T_{ij}\, q_i p_j}{q_k n_k\, p_\ell n_\ell} &= \frac{T_{ij}\,(p_i p_j - q_i q_j)}{(p_k n_k)^2 - (q_\ell n_\ell)^2} \end{aligned}$$

For $\mathbf{p}\cdot\mathbf{n} = \mathbf{q}\cdot\mathbf{n}$, the universal relation is

$$p_i T_{ij}\, p_j \quad q_i T_{ij}\, q_j\,.$$

If desired, γ_5 can be evaluated from the following relation:

$$\gamma_5 = \frac{1}{(\lambda_3^2 - \lambda_1^2)}\left[\frac{r_i T_{ij}\, q_j}{r_k n_k q_\ell n_\ell} - \frac{p_i T_{ij}\, q_j}{p_k n_k q_\ell n_\ell}\right] \tag{2.24}$$

For case (iii) of equation (2.9), equations (2.8_1)–(2.8_3) have the solution

$$\alpha + \beta + \gamma = 0. \tag{2.25}$$

Thus equations (2.8_4) and (2.8_5) yield

$$\begin{aligned} \alpha[(p_i n_i)^2 - (r_j n_j)^2] + \beta[(q_i n_i)^2 - (r_j n_j)^2] &= 0, \\ \delta(p_i n_i) q_j n_j + \epsilon(q_i n_i) r_j n_j + \mu(r_i n_i) p_j n_j &= 0. \end{aligned} \tag{2.26}$$

The equation which is analogous to equation (2.12) is

$$
\begin{aligned}
& \alpha T_{ij}\,(p_i p_j - r_i r_j) + \beta T_{ij}\,(q_i q_j - r_i r_j) + \delta T_{ij}\,(p_i q_j + q_i p_j) \\
+ \; & \epsilon\, T_{ij}\,(q_i r_j + q_j r_i) + \mu T_{ij}\,(r_i p_j + r_j p_i) \\
& \Lambda_1[\alpha((p_i n_i)^2 - (r_j n_j)^2) + \beta((q_i n_i)^2 - (r_j n_j)^2)] \\
& \Lambda_2[\delta(p_i n_i) q_j n_j + \epsilon\,(q_i n_i) r_j n_j + \mu (r_i n_i) p_j n_j] = 0.
\end{aligned}
$$

The necessary and sufficient conditions for this equation to hold for all choices of α, β, δ, ϵ and μ are

$$
\begin{aligned}
T_{ij}\,(p_i p_j - r_i r_j) - \Lambda_1((p_i n_i)^2 - (r_i n_i)^2) &= 0, \\
T_{ij}\,(q_i q_j - r_i r_j) - \Lambda_1((q_i n_i)^2 - (r_i n_i)^2) &= 0, \\
2T_{ij}\, p_i q_j - \Lambda_2 (p_i n_i) q_j n_j &= 0, \\
2T_{ij}\, q_i r_j - \Lambda_2 (q_i n_i) r_j n_j &= 0, \\
2T_{ij}\, r_i p_j - \Lambda_2 (r_i n_i) p_j n_j &= 0.
\end{aligned}
$$

Since $\mathbf{B}$ is a spherical tensor, every vector is an eigenvector of $\mathbf{B}$. We choose three orthonormal vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ as eigenvectors of $\mathbf{B}$. When $\mathbf{p}$ is parallel to $\mathbf{n}$, then $p_i n_i = 1$, $q_i n_i = 0$, $r_i n_i = 0$, and equations (2.28) reduce to

$$
\begin{aligned}
T_{ij}\,(p_i p_j - r_i r_j) - \Lambda_1 \quad & 0, \\
T_{ij}\,(q_i q_j - r_i r_j) &= 0, \\
T_{ij}\, p_i q_j = 0, \qquad T_{ij}\, q_i r_j &= 0, \qquad T_{ij}\, r_i p_j = 0.
\end{aligned}
$$

That is, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are eigenvectors of $\mathbf{T}$, and eigenvalues corresponding to eigenvectors $\mathbf{q}$ and $\mathbf{r}$ are equal, but these differ from the eigenvalue corresponding to the eigenvector $\mathbf{p}$. The difference between the eigenvalues of $\mathbf{T}$ corresponding to eigenvectors $\mathbf{p}$ and $\mathbf{q}$ is material dependent and hence not universal. For the case of $\mathbf{p}$ not parallel to $\mathbf{n}$ and $\mathbf{p}\cdot\mathbf{n} \neq \mathbf{q}\cdot\mathbf{n} \neq \mathbf{r}\cdot\mathbf{n}$, the elimination of Λ_1 and Λ_2 from equations (2.28) yields the following universal relations:

$$
\begin{aligned}
\frac{T_{ij}\,(p_i p_j - q_i q_j)}{(p_k n_k)^2 - (q_\ell n_\ell)^2} & \quad \frac{T_{ij}\,(q_i q_j - r_i r_j)}{(q_k n_k)^2 - (r_\ell n_\ell)^2} = \frac{T_{ij}\,(r_i r_j - p_i p_j)}{(r_k n_k)^2 - (p_\ell n_\ell)^2}, \\
\frac{T_{ij}\, p_i q_j}{(p_k n_k)(q_\ell n_\ell)} &= \frac{T_{ij}\, q_i r_j}{(q_k n_k)(r_\ell n_\ell)} = \frac{T_{ij}\, r_i p_j}{(r_k n_k)(p_\ell n_\ell)}.
\end{aligned} \tag{2.30$_1$}
$$

If $\mathbf{p} \cdot \mathbf{n} = \mathbf{q} \cdot \mathbf{n}$, then the universal relation is

$$p_i T_{ij} p_j = q_i T_{ij} q_j \tag{2.30$_2$}$$

and similar relations hold when $\mathbf{q} \cdot \mathbf{n} = \mathbf{r} \cdot \mathbf{n}$ or $\mathbf{p} \cdot \mathbf{n} = \mathbf{r} \cdot \mathbf{n}$. Thus for the choice of basis vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ with $\mathbf{p} \cdot \mathbf{n} = \mathbf{q} \cdot \mathbf{n} = \mathbf{r} \cdot \mathbf{n}$, a spherical left Cauchy–Green tensor $\mathbf{B}$ will result in a spherical Cauchy stress tensor $\mathbf{T}$ for a transversely isotropic elastic material. For $\mathbf{q} \cdot \mathbf{n} \neq \mathbf{r} \cdot \mathbf{n}$, γ_5 is given by

$$\gamma_5 \quad \frac{1}{\lambda_1^2}\left[\frac{T_{ij}\,(q_i q_j - r_i r_j)}{(q_k n_k)^2 - (r_\ell n_\ell)^2} - \frac{T_{ij}\, p_i r_j}{}\right] \tag{2.31}$$

We note that the universal relations derived above for unconstrained transversely isotropic elastic materials also apply to incompressible transversely isotropic elastic materials provided that deformations considered are isochoric.

3. UNIVERSAL RELATIONS FOR SIMPLE DEFORMATIONS

3.1. Controllable homogeneous deformations

We first study three classes of homogeneous deformations. Thus the stress tensor in a homogeneous transversely isotropic elastic body will be a constant and the balance of linear momentum without body forces will be identically satisfied. These deformations are controllable in the sense that they can be produced by the action of surface tractions alone.

3.1.1. Triaxial stretches

Consider a rectangular block made of a transversely isotropic elastic material subjected to triaxial homogeneous deformations

$$x_1 = bX_1, \qquad x_2 = cX_2, \qquad x_3 = dX_3,$$

where b, c and d are constants. For this deformation

$$\lambda_1 = b, \qquad \lambda_2 = c, \qquad \lambda_3 = d, \qquad \mathbf{p} = \mathbf{e}_1, \qquad \mathbf{q} = \mathbf{e}_2, \qquad \mathbf{r} = \mathbf{e}_3.$$

Let the unit vector $\mathbf{p}$ be parallel to $\mathbf{n}$. For $b \neq c \neq d$, equations (2.15) give

$$T_{12} = T_{23} = T_{31} = 0$$

as the three universal relations. The constitutive relation (2.1) implies that the eigenvalues of $\mathbf{T}$ are distinct. For $b \neq c = d$, equations (2.21) imply that (3.2) and $T_{22} = T_{33}$ are the universal relations. However, when $\mathbf{q}$ rather than $\mathbf{p}$ is parallel to $\mathbf{n}$, then equations (3.2) still hold but T_{22} need not equal T_{33}. For $b = c = d$, we again get (3.2) and for $\mathbf{p}$ parallel to $\mathbf{n}$, $T_{22} = T_{33}$ also as the universal relations.

Let the axis $\mathbf{n}$ of transverse isotropy be not aligned along any one of the coordinate axes. For $b \neq c \neq d$, equations (2.16) imply that

$$\frac{1}{c^2 - b^2}\left(\frac{T_{23}}{n_2 n_3} - \frac{T_{31}}{n_3 n_1}\right) = \frac{1}{d^2 - c^2}\left(T_{31} \qquad T_{12} \right) = \frac{1}{b^2 - d^2}\left(\frac{T_{12}}{n_1 n_2} - \frac{T_{23}}{n_2 n_3}\right) \tag{3.3}$$

are the three universal relations. When $b = c \neq d$ and $n_1 \neq n_2$, then it follows from equations (2.23) that

$$\frac{T_{23}}{n_2} = \frac{T_{31}}{n_1}$$

$$T_{21} \qquad (T_{11} - T_{22})$$

are the universal relations. For $b = c = d$ and $n_1 \neq n_2 \neq n_3$, equations (2.30) imply that the universal relations are

$$\frac{T_{11} - T_{33}}{} = \frac{T_{22} - T_{33}}{} = \frac{T_{22} - T_{11}}{}$$

$$\frac{T_{23}}{n_2 n_3}$$

For incompressible transversely isotropic materials, the deformation (3.1) must satisfy the relation

$$bcd = 1.$$

Thus under the conditions stated for equations (3.2), (3.3) and (3.4) and with b, c and d satisfying (3.6), universal relations (3.2), (3.3), and (3.4) hold for incompressible transversely isotropic elastic materials.

3.1.2. Simple shear

For simple shear deformations in the x_1–x_2 plane,

$$x_1 = X_1 + \kappa X_2, \qquad x_2 = X_2, \qquad x_3 = X_3,$$

where $\mathbf{x}$ is the position of the material particle that occupied place $\mathbf{X}$ in the reference configuration. For the deformation (3.7),

$$\mathbf{B} = \begin{bmatrix} 1+\kappa^2 & \kappa & 0 \\ \kappa & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
\lambda_1^2 \quad & 1+\frac{1}{2}\kappa^2+\kappa\left(1+\frac{1}{4}\kappa^2\right)^{\frac{1}{2}} \quad \lambda_2^{-2}, \qquad \lambda_3 = 1, \\
\lambda_2^2 \quad & 1+\frac{1}{2}\kappa^2-\kappa\left(1+\frac{1}{4}\kappa^2\right)^{\frac{1}{2}} \\
\mathbf{p}, \mathbf{q} = & \frac{\mathbf{e}_1+\mathbf{e}_2(-\frac{1}{2}\kappa \pm (1+\frac{1}{4}\kappa^2)^{\frac{1}{2}})}{(2+\frac{1}{2}\kappa^2 \mp \kappa\,(1+\frac{1}{4}\kappa^2)^{\frac{1}{2}})^{\frac{1}{2}}}, \qquad \mathbf{r} = \mathbf{e}_3.
\end{aligned} \tag{3.8}$$

Thus $\lambda_1^2 \neq \lambda_2^2 \neq \lambda_3^2$. When either $\mathbf{p}$ or $\mathbf{q}$ or $\mathbf{r}$ is parallel to $\mathbf{n}$, then $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are eigenvectors of $\mathbf{T}$, and equations (2.15) give

$$T_{13} = T_{23} = 0, \qquad T_{11} - T_{22} = \kappa\, T_{12} \tag{3.9}$$

as the three well known universal relations which are the same as those for an isotropic material. For $\mathbf{r} = \mathbf{n}$, relations (3.9) are intuitive since the material in the x_1–x_2 plane is isotropic. However, when the axis $\mathbf{n}$ of transverse isotropy is parallel to either the x_1-axis or the x_2-axis, then from equations (2.13) we conclude that

$$T_{13} = 0, \qquad T_{23} = 0, \tag{3.10}$$

and only (3.9_1) and (3.9_2) are the universal relations. For the general case of $\mathbf{n}$ not parallel to $\mathbf{p}$, $\mathbf{q}$, or $\mathbf{r}$, we obtain from (2.16) the following universal relations for a transversely isotropic elastic material:

$$\begin{aligned}
& \frac{1}{(-2\kappa\,\bar\kappa)}\left[\frac{T_{13}-T_{23}(\frac{\kappa}{2}+\bar\kappa)}{(n_1-n_2(\frac{\kappa}{2}+\bar\kappa))n_3} - \frac{T_{13}-T_{23}(\frac{\kappa}{2}-\bar\kappa)}{(n_1-n_2(\frac{\kappa}{2}-\bar\kappa))n_3}\right] \\
& \frac{1}{(-\frac{\kappa^2}{2}+\kappa\,\bar\kappa)}\left[\frac{T_{13}-T_{23}(\frac{\kappa}{2}-\bar\kappa)}{(n_1-n_2(\frac{\kappa}{2}-\bar\kappa))n_3} - \frac{T_{11}-T_{22}-\kappa\,T_{12}}{(n_1^2-n_2^2-\kappa\,n_1n_2)}\right] \\
& \frac{1}{(\frac{\kappa^2}{2}+\kappa\,\bar\kappa)}\left[\frac{T_{11}-T_{22}-\kappa\,T_{12}}{(n_1^2-n_2^2-\kappa\,n_1n_2)} - \frac{T_{13}-T_{23}(\frac{\kappa}{2}+\bar\kappa)}{(n_1-n_2(\frac{\kappa}{2}+\bar\kappa))n_3}\right]
\end{aligned} \tag{3.11}$$

where

$$\bar\kappa^2 = 1+\kappa^2/4. \tag{3.12}$$

It is implicit in equations (3.11) that terms in the denominator do not vanish. The simple shear deformation (3.7) is isochoric. Thus universal relations (3.9), (3.10) and (3.11) also hold for incompressible transversely isotropic elastic materials under the conditions stated for them.

3.1.3. *Simultaneous shear and extension*

We consider a uniform shear and extension of a block defined by

$$x_1 = DX_1 + EX_2, \qquad x_2 = FX_2, \qquad x_3 = GX_3, \tag{3.13}$$

where D, E, F and G are constants. This deformation is isochoric when $DFG =$ For the deformation (3.13),

$$\mathbf{B} \qquad \begin{matrix} D^2+E^2 & EF & 0 \\ EF & F^2 & 0 \\ 0 & 0 & G^2 \end{matrix} \Bigg]$$

$$\begin{aligned}
2\lambda_{1,2}^2 \quad & D^2+E^2+F^2 \pm H^2 \qquad \lambda_3^2 = G^2, \\
\mathbf{p} \quad & \frac{EF}{H_1}\mathbf{e}_1 + \frac{F^2+H^2}{H_1}\mathbf{e}_2, \\
\mathbf{q} &= \frac{EF}{H_2}\mathbf{e}_1 + \frac{F^2-H^2}{H_2}\mathbf{e}_2, \\
\mathbf{r} &= \mathbf{e}_3, \\
H^2 &= ((D^2+E^2-F^2)^2 + 4E^2F^2)^{\frac{1}{2}}, \\
H_1 &= (E^2F^2 + (F^2+H^2)^2)^{\frac{1}{2}}, \\
H_2 &= (E^2F^2 + (F^2-H^2)^2)^{\frac{1}{2}}.
\end{aligned}$$

For $\mathbf{p}$, $\mathbf{q}$ or $\mathbf{r}$ parallel to $\mathbf{n}$, it follows from equations (2.15) that $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are eigenvectors of $\mathbf{T}$, and

$$T_{13} = 0, \qquad T_{23} = 0, \qquad T_{11} - T_{22} = MT_{12}; \qquad M = (D^2+E^2-F^2)/EF \tag{3.15}$$

The universal relation (3.15) is independent of G, and is the same as that derived by Beatty [2] for an isotropic elastic material. Universal relations (3.15) will hold for an incompressible transversely isotropic elastic material if $DFG = 1$. When the axis $\mathbf{n}$ of transverse isotropy is parallel to either the x_1- or the x_2-axis, then (3.15$_1$) and (3.15$_2$) are the only universal relations. For the case of $\mathbf{n}$ not parallel to $\mathbf{p}$, $\mathbf{q}$ or $\mathbf{r}$, the following universal relation follows from equation (2.16):

$$\frac{1}{H^2}\left[\frac{2T_{11}EF - T_{23}(D^2+E^2-F^2+H^2)}{n_3(2EFn_1 + (F^2-H^2-D^2-E^2)n_2)}\right.$$
$$\left.\frac{2T_{13}EF - T_{23}(D^2+E^2-F^2-H^2)}{n_3(2EFn_1 + (F^2+H^2-D^2-E^2)n_2)}\right]$$
$$\frac{2}{2G^2+H^2-(D^2+E^2+F^2)}\left[\frac{2T_{13}EF - T_{23}(D^2+E^2-F^2-H^2)}{n_3(2EFn_1 + (F^2+H^2-D^2-E^2)n_2}\right.$$

$$\frac{T_{11}E^2F^2 + T_{22}(F^2+H^2-D^2-E^2)(F^2-H^2-D^2-E^2)/4}{(EFn_1+n_2(F^2+H^2-D^2-E^2)/2)(EFn_1+n_2(F^2-H^2-D^2-E^2)/2)}$$
$$\left. + \frac{T_{12}EF(F^2-D^2-E^2)/2}{(EFn_1+n_2(F^2+H^2-D^2-E^2)/2)(EFn_1+n_2(F^2-H^2-D^2-E^2)/2)}\right]. \quad (3.16)$$

For the sake of brevity, we have not written out the expression corresponding to the second equality in (2.16).

3.2. *Nonhomogeneous deformations*

We now consider five families of nonhomogeneous deformations that may or may not be controllable in every homogeneous transversely isotropic elastic material. Irrespective of whether or not special body forces are required to produce these deformations, universal relations will hold at a point. We use cylindrical coordinates (r, θ, z) or spherical coordinates (r, θ, ϕ) as needed for specifying the deformations; lower-case letters indicate coordinates of a point in the present configuration and upper-case letters coordinates of the same point in the reference configuration. The parameters A, B, C, D, E and F are constants; those appearing in the denominator must not vanish. These families of deformations have been described in detail by Truesdell and Noll [1, section 57]. Families 1, 2 and 4 are isochoric, thus these deformations are possible in both compressible and incompressible transversely isotropic elastic materials. Families 3 and 5 are isochoric only when constants appearing in them satisfy the constraint $\det \mathbf{F} = 1$. Below, we use physical components of $\mathbf{B}$ and $\mathbf{T}$.

Family 1: Bending, stretching and shearing of a rectangular block is described by

$$r = \sqrt{2AX_1}, \qquad \theta = BX_2, \qquad z = \frac{X_3}{AB} - BCX_2. \quad (3.17)$$

Thus

$$\begin{aligned}
\mathbf{B} &= \begin{bmatrix} \frac{A^2}{r^2} & 0 & 0 \\ 0 & B^2r^2 & -B^2Cr \\ 0 & -B^2Cr & B^2C^2 + \frac{1}{A^2B^2} \end{bmatrix} \\
\lambda_1^2 &= A^2/r^2, \\
2\lambda_{2,3}^2 &= B^2r^2 + B^2C^2 + \frac{1}{A^2B^2} \pm H, \\
\mathbf{p} &= \mathbf{e}_r, \\
\mathbf{q} &= \frac{B^2Cr}{H_2}\mathbf{e}_\theta + \frac{B^2r^2-\lambda_2^2}{H_2}\mathbf{e}_z, \\
\mathbf{s} &= \frac{B^2Cr}{H_3}\mathbf{e}_\theta + \frac{B^2r^2-\lambda_3^2}{H_3}\mathbf{e}_z, \\
H^2 &= (B^2r^2 - B^2C^2 - \frac{1}{A^2B^2})^2 + 4B^4C^2r^2, \\
H_{2,3}^2 &= (\lambda_{2,3}^2 - B^2r^2)^2 + B^4C^2r^2
\end{aligned} \quad (3.18)$$

We have denoted eigenvectors of $\mathbf{B}$ by $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{s}$ rather than by $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ in order not to confuse the last one with the radial coordinate r.

When the axis $\mathbf{n}$ of transverse isotropy is parallel to $\mathbf{e}_r$, equations (2.15) imply that $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{s}$ are eigenvectors of $\mathbf{T}$, and the universal relations are

$$T_{r\theta} = 0, \qquad T_{rz} = 0, \qquad T_{\theta\theta} - T_{zz} = T_{\theta z}\frac{1 + A^2B^2(C^2 - r^2)}{A^2B^4Cr}. \tag{3.19}$$

However, when the axis $\mathbf{n}$ of transverse isotropy is not aligned along $\mathbf{e}_r$, $\mathbf{e}_\theta$ or $\mathbf{e}_z$, then universal relations are given by equations (2.16) with λ_1^2, λ_2^2, λ_3^2 and the corresponding eigenvectors given by equations (3.18).

Family 2: Straightening, stretching and shearing of a sector of a hollow cylinder is given by the isochoric deformation field

$$x_1 = \frac{1}{2}AB^2R^2, \qquad x_2 = \frac{\Theta}{AB}, \qquad x_3 = \frac{Z}{B} + \frac{C\Theta}{AB} \tag{3.20}$$

From (3.20) we obtain

$$\mathbf{B} \qquad \begin{matrix} A^2B^4R^2 & 0 & 0 \\ 0 & \frac{1}{R^2A^2B^2} & \frac{C}{R^2A^2B^2} \\ 0 & \frac{C}{R^2A^2B^2} & \frac{1}{B^2} \end{matrix}$$

$$\begin{aligned}
\lambda_1^2 &= A^2B^4R^2 \\
2\lambda_{2,3}^2 &= \frac{1}{R^2A^2B^2} \quad \frac{1}{B^2} \pm H, \\
\mathbf{p} &= \mathbf{e}_1, \\
\mathbf{q} &= \frac{C}{H_2R^2A^2B^2}\mathbf{e}_2 + \frac{\lambda_2^2 - \frac{1}{R^2A^2B^2}}{H_2}\mathbf{e}_3 \\
\mathbf{s} &= \frac{C}{H_3R^2A^2B^2}\mathbf{e}_2 + \frac{\lambda_3^2 - \frac{1}{R^2A^2B^2}}{H_3}\mathbf{e}_3 \\
H^2 &= \left(\frac{1}{R^2A^2B^2} - \frac{1}{B^2}\right)^2 + \frac{4C^2}{R^4A^4B^4}, \\
H_{2,3}^2 &= \left(\lambda_{2,3}^2 - \frac{1}{R^2A^2B^2}\right)^2 + \frac{C^2}{R^4A^4B^4}
\end{aligned}$$

For the axis $\mathbf{n}$ of transverse isotropy aligned along the x_1-coordinate axis, universal relations deduced from equations (2.15) are

$$T_{12} = 0, \qquad T_{13} = 0, \qquad T_{22} - T_{33} = T_{23}\frac{B^2(1 - C^2) - 2Ax_1}{B^2C} \tag{3.21}$$

Equations (2.16) give the universal relations when $\mathbf{n}$ is not parallel to $\mathbf{p}$, $\mathbf{q}$ or $\mathbf{s}$.

Family 3: Inflation, bending, torsion, extension and shearing of an annular wedge is defined by

$$r = \sqrt{AR^2 + B}, \qquad \theta = C\Theta + DZ, \qquad z = E\Theta + FZ.$$

The deformation is isochoric if $A(CF - DE) = 1$. From equation (3.22) we obtain

$$\mathbf{B} = \begin{bmatrix} \frac{r^2}{R^2} & 0 & 0 \\ 0 & r^2(\frac{C^2}{R^2} + D^2) & r(\frac{EC}{R^2} + DF) \\ 0 & r(\frac{EC}{R^2} + DF) & \frac{E^2}{R^2} + F^2 \end{bmatrix}$$

$$\lambda_1^2 = \frac{r^2}{R^2},$$

$$2\lambda_{2,3}^2 \quad \left(\frac{E^2}{R^2} + F^2\right) + r^2\left(\frac{C^2}{R^2} + D^2\right) \pm H,$$

$$\mathbf{p} \quad \mathbf{e}_r,$$

$$\mathbf{q} = \frac{r\left(\frac{EC}{R^2} + DF\right)}{H_2}\mathbf{e}_\theta + \frac{\left(\lambda_2^2 - r^2\left(\frac{C^2}{R^2} + D^2\right)\right)}{H_2}\mathbf{e}_z,$$

$$\frac{r\left(\frac{EC}{R^2} + DF\right)}{H_3}\mathbf{e}_\theta + \frac{\lambda_3^2 - r^2\left(\frac{C^2}{R^2} + D^2\right)}{H_3}\mathbf{e}_z,$$

$$H^2 \quad \left(r^2\left(\frac{C^2}{R^2} + D^2\right) - \left(\frac{E^2}{R^2} + F^2\right)\right)^2 + 4r^2\left(\frac{EC}{R^2} + DF\right)^2$$

$$H_{2,3}^2 = \left[\lambda_{2,3}^2 - r^2\left(\frac{C^2}{R^2} + D^2\right)\right]^2 + r^2\left(\frac{EC}{R^2} + DF\right)^2$$

For the axis $\mathbf{n}$ of transverse isotropy aligned along a radial direction, equations (2.15) give

$$T_{r\theta} = 0, \qquad T_{rz} = 0, \qquad T_{\theta\theta} - T_{zz} = T_{\theta z}\frac{r^2(C^2 + D^2R^2) - (E^2 + F^2R^2)}{r(CE + DFR^2)} \tag{3.24}$$

as the universal relations. For $\mathbf{n}$ not aligned along any one of the eigenvectors of $\mathbf{B}$, equations (2.16) give the universal relations.

Family 4: Inflation or eversion of a sector of a spherical shell is given by the deformation field

$$r = (\pm R^3 + A)^{\frac{1}{3}} \qquad \theta \quad \pm\Theta, \qquad \phi = \Phi. \tag{3.25}$$

For the deformation (3.25), the physical components of $\mathbf{B}$ are

$$\mathbf{B} = \begin{bmatrix} \frac{R^4}{r^4} & 0 & 0 \\ 0 & \frac{r^2}{R^2} & 0 \\ 0 & 0 & \frac{r^2}{R^2} \end{bmatrix}$$

If the axis $\mathbf{n}$ of transverse isotropy is parallel to $\mathbf{e}_r$, then we conclude from equations (2.21) that

$$T_{r\theta} = T_{r\phi} = T_{\theta\phi} = 0, \qquad T_{\theta\theta} = T_{\phi\phi}.$$

However, if $\mathbf{n}$ is parallel to either $\mathbf{e}_\theta$ or $\mathbf{e}_\phi$, then only the first three relations in (3.27) hold. When $\mathbf{n}$ is not parallel to $\mathbf{e}_r$, $\mathbf{e}_\theta$ or $\mathbf{e}_\phi$, then we conclude from equations (2.23) that

$$\frac{T_{r\phi}}{n_\phi} \quad \frac{T_{\theta r}}{n_\theta},$$

$$\frac{T_{\theta\phi}}{n_\theta n_\phi} = \frac{(T_{\phi\phi} - T_{\theta\theta})}{n_\phi^2 - n_\theta^2}. \tag{3.28}$$

are the universal relations. Implicit in (3.28) is the assumption that $n_\theta \neq n_\phi$.

Family 5: Inflation, bending, extension and azimuthal shearing of an annular wedge is defined by

$$r = AR, \qquad \theta = B \log \frac{R}{D} + C\Theta, \qquad z = FZ. \tag{3.29}$$

For this deformation to be isochoric, $A^2CF = 1$. The left Cauchy–Green tensor for the deformation (3.29) is

$$\mathbf{B} \quad \begin{bmatrix} A^2 & \frac{AB}{R} & 0 \\ \frac{AB}{R} & \frac{B^2}{R^2} + \frac{C^2r^2}{R^2} & 0 \\ 0 & 0 & F^2 \end{bmatrix} \tag{3.30}$$

Thus

$$\begin{aligned}
\mathbf{p} &= \frac{AB}{RH_1}\mathbf{e}_r + \frac{\lambda_1^2 - A^2}{H_1}\mathbf{e}_\theta, \\
\mathbf{q} &= \frac{AB}{RH_2}\mathbf{e}_r + \frac{\lambda_2^2 - A^2}{H_2}\mathbf{e}_\theta, \\
\mathbf{s} &= \mathbf{e}_z, \\
\lambda_{1,2}^2 &= A^2 + \left(\frac{B^2}{R^2} + \frac{C^2r^2}{R^2}\right) \pm H, \\
\lambda_3^2 & \quad F^2 \\
H^2 &= \left(\frac{B^2}{R^2} + \frac{C^2r^2}{R^2} - A^2\right)^2 + 4\frac{A^2B^2}{R^2} \\
H_{1,2}^2 &= (\lambda_{1,2}^2 - A^2)^2 + \frac{A^2B^2}{R^2}
\end{aligned} \tag{3.31}$$

For the case of $\mathbf{n}$ aligned along either $\mathbf{p}$ or $\mathbf{q}$ or $\mathbf{e}_z$, we conclude from equation (2.15)

$$T_{zr} = 0, \qquad T_{z\theta} = 0, \qquad T_{rr} \quad T_{\theta\theta} = T_{r\theta} \frac{1 - B^2 - C^2}{B} \tag{3.32}$$

When $\mathbf{n}$ is not parallel to $\mathbf{p}$, $\mathbf{q}$ or s, then universal relations can be derived from equations (2.16).

4. REMARKS

The constitutive relation (2.1) is written in terms of the Cauchy stress tensor $\mathbf{T}$ and the left Cauchy–Green tensor $\mathbf{B}$. One could write a similar relation between the second Piola–Kirchhoff stress tensor $\mathbf{S}$, the right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and the unit vector $\mathbf{N}$ along the direction of transverse isotropy in the reference configuration. Universal relations similar to the ones derived above can be deduced in terms of the components of $\mathbf{S}$, and eigenvalues and eigenvectors of $\mathbf{C}$.

Even though we have derived explicit forms of universal relations for simple deformations, universal relations (2.15), (2.16), (2.23) and (2.29) etc are applicable for any deformation. These relations, being characteristic of the constitutive relation (2.1), are also applicable to transversely isotropic fluids for which equation (2.1) applies with $\mathbf{B}$ replaced by the strain-rate tensor $\mathbf{D}$. Then in equations (2.15), (2.16), (2.23) and (2.29) etc, λ_1^2, λ_2^2 and λ_3^2 equal eigenvalues of $\mathbf{D}$, i.e., they are stretchings rather than squares of stretches. Whereas stretches are positive, stretchings need not be positive. However, the positiveness of stretches was not used in the derivation of the universal relations.

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NOTE ADDED IN PROOF

When I presented this work in the Symposium honoring Professor Millard F. Beatty at the 14th U.S. National Congress of Theoretical and Applied Mechanics, Blacksburg, June 23–28, 2002, Dr. P. Saccomandi kindly told me that R.S. Rivlin [11] had recently given universal relations for transversely isotropic elastic materials. Dr. Saccomandi was kind enough to send me a copy of Rivlin's paper. In the terminology of this paper, Rivlin assumed that $n_i = \delta_{i3}$, solved five out of six equations (2.1) for $\gamma_1, \gamma_2, \ldots, \gamma_5$, and substituted for them in the remaining equation to obtain a universal relation. He gave explicit universal relations for three classes of simple shearing deformations superposed on a homogeneous deformation. The approach adopted here is different from but equivalent to Rivlin's. We have studied general deformations, and given explicit universal relations for several deformation fields.