Additional Universal Relations for Transversely Isotropic Elastic Materials

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Abstract: For the case of distinct eigenvalues of the left Cauchy–Green tensor universal relations for a transversely isotropic elastic material are expressed in a compact form. Also, two new universal relations are derived for an incompressible transversely isotropic elastic material.

Key Words: universal manifold, incompressible materials, simple shear, annular wedge

1. INTRODUCTION

A universal relation is an equation that holds for every material in a specified class. Universal relations are basic tools in the analysis of continuum theories of the mechanical behavior of materials [1,2].

Recently, Batra [3] adopted the Hayes and Knops [4] approach to derive universal relations for finite homogeneous and inhomogeneous deformations of a transversely isotropic elastic material. The first part of this note discusses relationships between results in [3] and those in [5] and [6]. Moreover, we give additional universal relations for incompressible transversely isotropic elastic materials which cannot be found by using the Hayes and Knops method. They are found by employing the universal manifold method of Pucci and Saccomandi [7,8] which generalizes the Hayes and Knops technique. A brief historical development of universal relations is given in [7]. For unconstrained transversely isotropic elastic materials, Rivlin [9] has derived universal relations for three classes of simple shearing deformations superimposed on a homogeneous deformation.

2. BASIC EQUATIONS AND THE MAIN UNIVERSAL RELATION

The constitutive equation for a transversely isotropic elastic material with the axis of transverse isotropy along the unit vector $\mathbf{n}$ in the current configuration is

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$$\mathbf{T}=\gamma_1\mathbf{I}+\gamma_2\mathbf{B}+\gamma_3\mathbf{B}^2+\gamma_4\mathbf{n}\otimes\mathbf{n}+\gamma_5\left(\mathbf{n}\otimes\mathbf{B}\mathbf{n}+\mathbf{B}\mathbf{n}\otimes\mathbf{n}\right), \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{I}$ the identity tensor, and $\mathbf{B}$ the left Cauchy–Green strain tensor. The response coefficients γ_i $(i=1,\ldots,5)$ are functions of the five principal invariants $I_1, I_2, \ldots, I_5$ defined as

$$I_1=\mathrm{tr}\mathbf{B},\quad I_2=\mathrm{tr}\mathbf{B}^2,\quad I_3=\mathrm{tr}\mathbf{B}^3,\quad I_4=\mathbf{n}\cdot\mathbf{B}\mathbf{n},\quad I_5=\mathbf{n}\cdot\mathbf{B}^2\mathbf{n}. \tag{2}$$

The tensor product $\otimes$ between vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as $(\mathbf{a}\otimes\mathbf{b})\,\mathbf{c} = (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}$ for all vectors $\mathbf{c}$. For an incompressible transversely isotropic elastic material $\mathbf{B}$ must satisfy $\det\mathbf{B}=1$ and γ_1 in (1) is replaced by an arbitrary pressure $-p$.

When $\gamma_4=\gamma_5=0$ in (1) then eigenvectors of $\mathbf{B}$ are also eigenvectors of $\mathbf{T}$. For $\gamma_2+I_3\gamma_3>0$ and $\gamma_3<0$, Batra's [10] theorem implies that eigenvectors of $\mathbf{T}$ are also eigenvectors of $\mathbf{B}$. When the eigenvalues λ_1^2, λ_2^2 and λ_3^2 of $\mathbf{B}$ corresponding to eigenvectors $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are such that $\lambda_1^2\neq\lambda_2^2\neq\lambda_3^2$ all universal relations may be found from the relation

$$\mathbf{TB}-\mathbf{BT}=\mathbf{0}, \tag{3}$$

as has been discussed in detail by Beatty [11]. Isotropic materials are a special case of transversely isotropic materials. For an isotropic material, $\gamma_4=\gamma_5=0$ and γ_1, γ_2 and γ_3 are functions of I_1, I_2 and I_3.

Results in [3] for $\lambda_1^2\neq\lambda_2^2\neq\lambda_3^2$ may be obtained directly by following the approach of [5] and [6]. From (1) we get

$$\begin{aligned}\mathbf{TB}-\mathbf{BT} &= \gamma_4\left[(\mathbf{n}\otimes\mathbf{n})\,\mathbf{B}-\mathbf{B}\,(\mathbf{n}\otimes\mathbf{n})\right]\\ &+\gamma_5\left[(\mathbf{n}\otimes\mathbf{B}\mathbf{n}+\mathbf{B}\mathbf{n}\otimes\mathbf{n})\,\mathbf{B}-\mathbf{B}\,(\mathbf{n}\otimes\mathbf{B}\mathbf{n}+\mathbf{B}\mathbf{n}\otimes\mathbf{n})\right].\end{aligned} \tag{4}$$

Because $\mathbf{B}=\mathbf{B}^T$ we have $(\mathbf{n}\otimes\mathbf{n})\,\mathbf{B}=\mathbf{n}\otimes\mathbf{B}\mathbf{n}$. Thus we write (4) as

$$\mathbf{TB}-\mathbf{BT}=\gamma_4\left[\mathbf{n}\otimes\mathbf{B}\mathbf{n}-\mathbf{B}\mathbf{n}\otimes\mathbf{n}\right]+\gamma_5\left[\mathbf{n}\otimes\mathbf{B}^2\mathbf{n}-\mathbf{B}^2\mathbf{n}\otimes\mathbf{n}\right]. \tag{5}$$

Here we omit the trivial case of $\mathbf{n}$ being an eigenvector of $\mathbf{B}$ because in this case universal relations are given by (3).

Recall that an antisymmetric tensor $\mathbf{W}$ corresponds to an axial vector $\mathbf{W}_\times\equiv\mathbf{u}$, such that for any vector $\mathbf{v}$, we have $\mathbf{W}\mathbf{v}=\mathbf{u}\times\mathbf{v}$. Moreover, for vectors $\mathbf{a}$ and $\mathbf{b}$, we have the identity $(\mathbf{a}\otimes\mathbf{b}-\mathbf{b}\otimes\mathbf{a})_\times=-\mathbf{a}\times\mathbf{b}$. Using these, Equation (5) simplifies to

$$(\mathbf{TB}-\mathbf{BT})_\times=-\mathbf{n}\times\left(\gamma_4\mathbf{B}\mathbf{n}+\gamma_5\mathbf{B}^2\mathbf{n}\right). \tag{6}$$

Therefore the universal relation we are searching for is

$$(\mathbf{TB}-\mathbf{BT})_\times\cdot\mathbf{n}=0, \tag{7}$$

which is Equation (2.16) of [3]. The compact form (7) allows for a more direct comparison with the results of [5] and [6]. Moreover, whereas for an isotropic material the meaning of the three general universal relations is the coaxiality between the Cauchy stress and the left Cauchy–Green strain tensor here the meaning of the only general universal relation is the orthogonality between the axial vector associated with $\mathbf{TB} - \mathbf{BT}$ and the axis $\mathbf{n}$ of transverse isotropy. It also provides a way to find the axis of transverse isotropy for these materials.

We now use the universal manifold method [7] to prove that (7) is the only universal relation. We consider a universal solution and the corresponding strain $\mathbf{B}$, and fix a point $\mathbf{X}$ in the reference configuration. Introduce an 11-dimensional space $\mathcal{S}$ with coordinates

$$\boldsymbol{\tau} = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6\}, \quad \boldsymbol{\gamma} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\},$$

where

$$\tau_1 = T_{11},\ \tau_2 = T_{12},\ \tau_3 = T_{13},\ \tau_4 = T_{22},\ \tau_5 = T_{23},\ \tau_6 = T_{33}.$$

In this space (1) represents a linear homogeneous manifold $\mathcal{V}$ of dimension three parametrized by γ. This manifold may be projected in the subspace $\mathcal{S}^*(\boldsymbol{\tau})$ of $\mathcal{S}$ on the manifold $\Pi(\mathcal{V})$ whose dimension k equals the rank of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ B_{11} & B_{12} & B_{13} & B_{22} & B_{23} & B_{33} \\ B^2_{11} & B^2_{12} & B^2_{13} & B^2_{22} & B^2_{23} & B^2_{33} \\ n_1^2 & n_1 n_2 & n_1 n_3 & n_2^2 & n_2 n_3 & n_3^2 \\ \mathbb{B}_{11} & \mathbb{B}_{12} & \mathbb{B}_{13} & \mathbb{B}_{22} & \mathbb{B}_{23} & \mathbb{B}_{33} \end{pmatrix}, \tag{8}$$

where $\mathbb{B} = \mathbf{n} \otimes \mathbf{Bn} + \mathbf{Bn} \otimes \mathbf{n}$, and B^2_{ij} are components of the tensor $\mathbf{B}^2$. The *universal* manifold $\Pi(\mathcal{V})$ is linear and homogeneous and may be represented by a set of linear homogeneous equations (the universal relations). The number of these equations is $6 - k$. Since the rank of the matrix (8) is 5, therefore we have only one universal relation.

The rank of the matrix (8) may be lower (i.e. $k < 5$) because either $\lambda_i^2 = \lambda_j^2$ for $i \neq j$ or $\mathbf{n}$ is an eigenvector of $\mathbf{B}$. In these cases new universal relations are possible and these can be found by the Hayes and Knops method. However, universal relations that are valid when the structure of the space $\mathcal{S}$ changes cannot be found by the Hayes and Knops approach. This holds for special deformations that introduce a relationship among the γ_i or for a special class of materials, e.g. constrained materials. On the other hand, universal relations may be a powerful tool to understand quantitatively the significance of a constitutive parameter. For example, let us consider a transversely isotropic material with $\gamma_5 = 0$. For this class of materials a new universal relation, namely

$$(\mathbf{TB} - \mathbf{BT})_\times \cdot \mathbf{Bn} = 0, \tag{9}$$

is valid because the structure of $\mathcal{S}$ has changed. If experimental data for a transversely isotropic elastic material do not satisfy (9), then clearly $\gamma_5 \neq 0$.

3. INCOMPRESSIBLE TRANSVERSELY ISOTROPIC MATERIALS

For an incompressible transversely isotropic material, γ_1 in Equation (1) is replaced by an arbitrary pressure $-p$. To show that for these materials additional universal relations can be found, we deduce them for a homogeneous deformation and a nonhomogeneous deformation; universal relations may be found for other isochoric deformations.

3.1. Simple Shear

We consider the simple shear deformation

$$x = X + KY, \quad y = Y, \quad z = Z, \tag{10}$$

of a block bounded by the pair of surfaces $X = \pm X_0$, $Y = \pm Y_0$ and $Z = \pm Z_0$ in the reference configuration, and the axis $\mathbf{n}$ of transverse isotropy in the XZ (or xz) plane. Here (x, y, z) are the current rectangular Cartesian coordinates of the material point initially at (X, Y, Z) in a common Cartesian frame $\varphi = \{O; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. A straightforward calculation gives

$$(\mathbf{TB} - \mathbf{BT})_\times = K\{T_{13}\mathbf{i} - (KT_{13} + T_{23})\,\mathbf{j} + (KT_{12} + T_{22} - T_{11})\,\mathbf{k}\}. \tag{11}$$

The universal relation (7) becomes

$$T_{13}n_1 - (KT_{13} + T_{23})\,n_2 + (KT_{12} + T_{22} - T_{11})\,n_3 = 0. \tag{12}$$

To simplify the computations, we set $\mathbf{n} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{k}$ and obtain from (12)

$$T_{13}\cos\alpha + (KT_{12} + T_{22} - T_{11})\sin\alpha = 0. \tag{13}$$

For the simple shear deformation (10), the referential plane $X = X_1$ is deformed into the current plane $x - Ky = X_1$. The normal vector to this surface is $\mathbf{m} = \cos\tau\mathbf{i} - \sin\tau\mathbf{j}$ where $\tau = \tan^{-1} K$ is the angle of shear. The normal traction, $\mathcal{N}$, and the shear traction, $\mathcal{T}$, on this inclined plane are given by

$$\mathcal{N} = \frac{T_{11} + K^2T_{22} - 2T_{12}K}{1 + K^2}, \quad \mathcal{T} = \frac{K(T_{11} - T_{22}) + T_{12}(1 - K^2)}{1 + K^2}. \tag{14}$$

We now choose the pressure so that the normal traction $\mathcal{N}$ vanishes on the inclined plane. That is,

$$p = \left(1 + K^2\right)^{-1}\left[\gamma_2 + \gamma_3 + (\gamma_4 + 2\gamma_5)\cos^2\alpha\right]. \tag{15}$$

In this case the structure of the space $\mathcal{S}$ has changed because the pressure p cannot be considered as an independent constitutive parameter; p is related to γ_i through Equation (15). In the new space $\mathcal{S}$ we introduce coordinates

$$\boldsymbol{\tau} = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6\}, \quad \boldsymbol{\gamma} = \{\gamma_2, \gamma_3, \gamma_4, \gamma_5\}, \tag{16}$$

and the linear homogeneous manifold $\mathcal{V}$ may be projected on to the manifold $\Pi(\mathcal{V})$ whose dimension k is given by the rank of the (6×4) matrix

$$\begin{pmatrix} B_{11} + \frac{1}{1+K^2} & B_{12} & B_{13} & B_{22} + \frac{1}{1+K^2} & B_{23} & B_{33} + \frac{1}{1+K^2} \\ B_{11}^2 + \frac{1}{1+K^2} & B_{12}^2 & B_{13}^2 & B_{22}^2 + \frac{1}{1+K^2} & B_{23}^2 & B_{33}^2 + \frac{1}{1+K^2} \\ n_1^2 + \frac{\cos^2\alpha}{1+K^2} & n_1 n_2 & n_1 n_3 & n_2^2 + \frac{\cos^2\alpha}{1+K^2} & n_2 n_3 & n_3^2 + \frac{\cos^2\alpha}{1+K^2} \\ \mathbb{B}_{11} + \frac{2\cos^2\alpha}{1+K^2} & \mathbb{B}_{12} & \mathbb{B}_{13} & \mathbb{B}_{22} + \frac{2\cos^2\alpha}{1+K^2} & \mathbb{B}_{23} & \mathbb{B}_{33} + \frac{2\cos^2\alpha}{1+K^2} \end{pmatrix}. \tag{17}$$

Therefore, the following new universal relation not found in [3] exists (because $6 - 4 = 2$):

$$\left(K^2 + 2\right) T_{22} - T_{11} + 2\cot\alpha T_{13} = 0. \tag{18}$$

If a different boundary condition were used to find the hydrostatic pressure p, then we would have obtained a different universal relation.

3.2. Inflation, Extension and Azimuthal Shearing of an Annular Wedge

Another interesting example is the nonhomogeneous deformation

$$r = AR, \quad \theta = C\Theta, \quad z = FZ, \tag{19}$$

which is a particular case of the Singh–Pipkin universal deformations with constant invariants. Here (r, θ, z) are coordinates of a point in the present configuration that occupied the place (R, Θ, Z) in the reference configuration. We assume that the axis of transverse isotropy is perpendicular to the z-axis. Physical components of the left Cauchy–Green tensor are given by

$$\mathbf{B} = \begin{pmatrix} A^2 & 0 & 0 \\ 0 & A^2C^2 & 0 \\ 0 & 0 & F^2 \end{pmatrix}. \tag{20}$$

For an incompressible material constants A, C and F are related by

$$A^2CF = 1. \tag{21}$$

The deformation (19) describes the inflation, extension and azimuthal shearing of an annular wedge and has been studied in [12]. The universal relation in terms of the physical components of $\mathbf{T}$ and $\mathbf{n}$ that can be obtained using the Hayes and Knops method is

$$T_{\theta z}\left(A^2C^2-F^2\right)n_r+T_{rz}\left(F^2-A^2\right)n_\theta+T_{r\theta}\left(A^2-A^2C^2\right)n_z=0. \tag{22}$$

Here we show that the universal manifold method gives a new universal relation.

For the sake of simplicity we again set $\mathbf{n}=\cos\alpha\mathbf{e}_r+\sin\alpha\mathbf{e}_\theta$, where $\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z$ is the orthonormal basis at a point for the cylindrical coordinates. Therefore, nonzero components of the Cauchy stress tensor are given by

$$\begin{aligned}
T_{rr} &= -p+\gamma_2A^2+\gamma_3A^4+(\gamma_4+2\gamma_5A^2)\cos^2\alpha,\\
T_{\theta\theta} &= -p+\gamma_2A^2C^2+\gamma_3A^4C^4+(\gamma_4+2\gamma_5A^2C^2)\sin^2\alpha,\\
T_{zz} &= -p+\gamma_2F^2+\gamma_3F^4,\\
T_{r\theta} &= (\gamma_4+\gamma_5(A^2+A^2C^2))\sin\alpha\cos\alpha.
\end{aligned} \tag{23}$$

Moreover, the balance of linear momentum implies that

$$p=(T_{rr}-T_{\theta\theta})\log(r/r_0)+2T_{r\theta}\theta+p_0, \tag{24}$$

where p_0 is a constant. The universal relation (22) splits into two trivial universal relations

$$T_{rz}=0,\quad T_{\theta z}=0. \tag{25}$$

Obviously (24) is a relationship among the pressure and the constitutive parameters γ_i. However, it does not give a new universal relation because the dimension of the manifold $\Pi(\mathcal{V})$ is 2 and the complete set of universal relations is given by (25). New universal relations may arise, as has been discussed in [3], if eigenvalues of (20) are repeated, but this is not the only case. Indeed, if we choose the constant p_0 such that the rank of the matrix corresponding to (8) and (17) is lowered, then new universal relations may arise. Substituting for p from (24) in (23) we obtain

$$\begin{aligned}
T_{rr}+2T_{r\theta}\theta &= -(T_{rr}-T_{\theta\theta})\log(r/r_0)-p_0\\
&+\gamma_2A^2+\gamma_3A^4+\left(\gamma_4+2\gamma_5A^2\right)\cos^2\alpha,
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
T_{\theta\theta}+2T_{r\theta}\theta &= -(T_{rr}-T_{\theta\theta})\log(r/r_0)-p_0\\
&+\gamma_2A^2C^2+\gamma_3A^4C^4+\left(\gamma_4+2\gamma_5A^2C^2\right)\sin^2\alpha.
\end{aligned} \tag{27}$$

We now choose

$$p_0 = \gamma_2 A^2 + \gamma_3 A^4 + \left(\gamma_4 + 2\gamma_5 A^2\right) \cos^2 \alpha, \tag{28}$$

and arrive at

$$T_{rr} + 2T_{r\theta}\theta = (T_{\theta\theta} - T_{rr}) \log(r/r_0), \tag{29}$$

and

$$T_{\theta\theta} + 2T_{r\theta}\theta = (T_{\theta\theta} - T_{rr})(\log(r/r_0) + 1). \tag{30}$$

The new universal relation is

$$\frac{T_{rr} + 2T_{r\theta}\theta}{T_{\theta\theta} + 2T_{r\theta}\theta} = \frac{\log(r/r_0)}{\log(r/r_0) + 1}. \tag{31}$$

This universal relation also holds for isotropic elastic materials [8].

4. CONCLUDING REMARKS

The Hayes and Knops method is an efficient and simple way to derive universal relations, but it has two shortcomings. Because it is not coordinate free, some important properties of universal relations derived from it may not be transparent. This is shown in Section 2, where we have derived in a direct and coordinate-free form the main universal relation for transversely isotropic materials. Furthermore, the Hayes and Knops method is not completely general for constrained materials. The new universal relations (18) and (31) cannot be deduced by using this method because to *find* them we need to modify the *structure* of the space where we can read the representation formula (1) as a geometric object.

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