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Decay of the kinetic and the thermal energy of compressible viscous fluids

by

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ABSTRACT. — We consider a heat conducting compressible viscous fluid at rest in a rigid container and show that the thermal and the kinetic energy of arbitrary disturbances of the rest state eventually decays.

RÉSUMÉ. — On considère un fluide compressible, visqueux et conducteur de la chaleur au repos dans un réservoir rigide. On montre que l'énergie thermique et cinétique de perturbations arbitraires est amortie.

. Introduction

In [1], [2] I extended Kampé de Fériet's [3] result concerning the decay of the energy of incompressible viscous fluids filling a closed rigid container to heat conducting incompressible viscous fluids and also to the case when the fluid does not fill the vessel. Here I study the corresponding problem for heat conducting compressible Navier-Stokes fluids and show that the thermal and the kinetic energy of arbitrary disturbances of the rest state eventually decays.

It would seem that for the mechanical problem, one should be able to obtain decay rate of the kinetic energy from the analysis of Shahinpoor and Ahmadi [4]. However, these authors make an assumption which for Navier-Stokes fluids implies that after a certain time, the rate at which body forces do work balances the rate of work done by the hydrostatic pressure. I do not make any such assumption and obtain a slightly weaker result.

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2. Formulation of the problem

Assume that the fluid when at rest occupies a bounded region R with a boundary ∂R which is smooth enough to apply the divergence theorem, the Poincaré inequality [5] and the Korn inequality [5]. The density ρ, the temperature θ and the components of velocity v_i satisfy the system of equations

$$\dot{\rho}+\rho v_{i,i}=0, \tag{1}$$

$$\rho \dot{v}_i = t_{ij,j}-\rho\Omega_{,i}, \tag{2}$$

$$\rho\dot{\varepsilon} = -q_{i,i}+t_{ij}v_{(i,j)}, \tag{3}$$

with

$$\begin{cases} \varphi=\varphi(\rho,\theta), \quad \varepsilon=\varepsilon(\rho,\theta), \quad \eta=\eta(\rho,\theta), \\ \varphi=\varepsilon-\eta\theta, \\ t_{ij}=-p\,\delta_{ij}+\lambda\, v_{k,k}\,\delta_{ij}+2\mu\, v_{(i,j)}, \\ q_i=-\varkappa\theta_{,i}, \\ v_{(i,j)}\equiv\frac{1}{2}(v_{i,j}+v_{j,i}). \end{cases} \tag{4}$$

In (1)-(4) φ, ε, η denote, respectively, the specific free energy, the specific internal energy and the specific entropy of the fluid. Ω is the potential of the body forces and is assumed to be a non-negative bounded function of the position only. The pressure p, the shear viscosity μ, the bulk viscosity λ and the thermal conductivity $\varkappa$ are, in general, functions of ρ and θ. We have employed the conventional indical notation so that repeated subscripts imply summation over the range 1, 2, 3. Moreover a comma followed by a subscript i denotes partial differentiation with respect to the space variable x_i and a superposed dot indicates material time differentiation. Assume that the initial disturbances are such that the classical solution of (1)-(4) exists subject to these initial conditions and the following boundary conditions

$$\begin{cases} v_i = 0 & \text{on } \chi(\partial_1 \mathrm{R}, t), \\ t_{ij}\eta_j = -p_0\eta_i & \text{on } \chi(\partial_1^c \mathrm{R}, t), \\ \theta = \theta_0 & \text{on } \chi(\partial_2 \mathrm{R}, t), \\ q_i\eta_i = b(\theta,\theta_0)(\theta-\theta_0) & \text{on } \chi(\partial_2^c \mathrm{R}, t), \end{cases} \tag{5}$$

Here $\partial_1 \mathrm{R}$ and $\partial_1^c \mathrm{R}$ are complementary subsets of $\partial \mathrm{R}$, $\chi(\partial_1 \mathrm{R}, t)$ is the present configuration of $\partial_1 \mathrm{R}$ at time t and η_i is the ith component of an outward directed unit normal to $\partial \mathrm{R}$. The boundary condition $(5)_2$ states that the part $\chi(\partial_1^c \mathrm{R}, t)$ of the boundary of the fluid is subjected to a uniform hydrostatic pressure p_0. For the case when the rigid container is completely filled with the viscous fluid and the fluid adheres to the walls, $\partial_1^c \mathrm{R}$ would be zero. I assume that $b \geqq 0$ is a bounded function of θ and θ_0. The thermodynamical restrictions imposed by the Clausius-Duhem inequality are

$$(6) \qquad \begin{cases} p = \rho^2 \dfrac{\partial \varphi}{\partial \rho}, \qquad \eta = -\dfrac{\partial \varphi}{\partial \theta}, \\ \mu \geqq 0, \qquad \varkappa \geqq 0, \qquad 3\lambda + 2\mu \geqq 0. \end{cases}$$

Following Ericksen [6] I introduce a finite Taylor expansion in the temperature for φ, obtaining thereby

$$\varepsilon - \theta_0 \eta = \psi(\rho) + \mathrm{K}(\theta - \theta_0)^2$$

$$(8) \qquad \psi(\rho) \equiv \varphi(\rho, \theta_0), \qquad \mathrm{K} = -\frac{1}{2} \frac{\partial^2 \varphi}{\partial \theta^2}(\rho, \theta^*) = \frac{c(\rho, \theta^*)}{2\theta^*}.$$

Here c is the specific heat and θ^* is a value of the temperature between θ and θ_0. Using (4) and (6), one can verify that

$$(9) \qquad (\dot{\varepsilon} - \theta_0 \dot{\eta}) = \left(1 - \frac{\theta_0}{\theta}\right) \dot{\varepsilon} + \frac{\theta_0}{\theta} \frac{p}{\rho^2} \dot{\rho}.$$

Taking the inner product of (2) with v_i, substituting for $\dot{\varepsilon}$ from (3) into the right-hand side of (9), integrating these equations over R, adding the respective sides of these equations and then simplifying by use of the divergence theorem, (1), (4), (5), and (7), we obtain

$$\dot{\mathrm{E}}_1 + \dot{\mathrm{E}}_2 = -\int \frac{\theta_0}{\theta} \{\lambda v_{k,k} v_{l,l} + 2\mu v_{(k,l)} v_{(k,l)}\} \, d\mathrm{V}$$
$$\int \frac{\theta_0 \varkappa}{\theta^2} \theta_{,i} \theta_{,i} \, d\mathrm{V} - \int \frac{b}{\theta} (\theta - \theta_0)^2 \, d\mathrm{A},$$

where

$$(11)\qquad \begin{cases} E_1(t) \equiv \displaystyle\int \rho\left[\frac{v_i v_i}{2} + K(\theta-\theta_0)^2\right] dV, \\ E_2(t) = \displaystyle\int \rho\left[\psi(\rho)+\Omega+\frac{p_0}{\rho}\right] dV. \end{cases}$$

Note that E_1 is a mesure of the kinetic energy and the temperature deviation of the fluid particles from that in the rest state. E_1 equals zero implies that the fluid is at rest and the temperature is uniform throughout the fluid and is θ_0. In the following, initial disturbances are taken to be those for which $E_1(0)+E_2(0)$ is finite.

With the assumptions

$$(12)\qquad \begin{aligned} & c_1 \equiv \sup_{\rho,\theta} K < \infty, \qquad c_2 \equiv \theta_0 \inf_{\rho,\theta} \frac{\varkappa}{\theta^2} > 0, \\ & \text{either } \bigcap_{t>0} \partial_2 R(t) \neq \varnothing \quad \text{or} \quad c_3 \equiv \inf_{\theta} \frac{b}{\theta} > 0, \\ & \rho_0 = \sup_{\substack{X\in R\\ t>0}} \rho < \infty, \qquad c_4 = \theta_0 \inf_{\rho,\theta} \frac{\lambda+(2/3)\mu}{\theta^2} > 0, \\ & c_5 = \theta_0 \inf_{\rho,\theta} \frac{\mu}{\theta^2} > 0, \qquad \bigcap_{t>0} \partial_1 R(t) \neq \varnothing. \end{aligned}$$

I prove below the following result

$$(13)\qquad E_1(t) \to 0 \quad \textit{as } t\to\infty \qquad \textit{and} \qquad \lim_{t\to\infty} E_2(t) \textit{ exists.}$$

The assumption $(12)_7$ states that there is a material subsurface of the walls of the container to which the fluid always adheres. The conditions (12) can be sharpened by taking the infimum or the supremum over those values of ρ and θ which are ever realized at any fluid particle. That this set of values of ρ and θ is bounded for the case when $\partial_1 R = \partial R$ follows from the results of Hill and Knops [7].

3. Proof of (13)

From $(12)_{1,2,3}$ and by using Poincaré's inequality we conclude that (Cf. [1]) :

$$\left[\int \theta_0 \frac{\varkappa}{\theta^2} \theta_{,i} \theta_{,i} \, dV + \int \frac{b}{\theta} (\theta - \theta_0)^2 \, dA\right] \leqq -c_6 \frac{p_1}{c_1} \int K (\theta - \theta_0)^2 \, dV,$$

where $c_6 = \min(c_2, c_3)$ and p_1 is a positive-valued function of R and ∂_2 R. For the case when $\partial_2 \mathrm{R} \neq \partial \mathrm{R}$, p_1 varies with time t and I assume that it is bounded above and denote its supremum also by p_1. A similar remark holds for the function p_2 appearing below in (15). Now by using $(12)_{4,5,6,7}$,

$$\begin{aligned}&\lambda v_{k,k} v_{i,i} + 2\mu v_{(i,j)} v_{(i,j)} \\ &= \left(\lambda + \frac{2}{3}\mu\right) v_{i,i} v_{k,k} + 2\mu \left(v_{(i,j)} - \frac{v_{k,k}}{3}\delta_{ij}\right)\left(v_{(i,j)} - \frac{v_{k,k}}{3}\delta_{ij}\right),\end{aligned}$$

Poincaré's inequality and Korn's inequality (cf. [1]), we obtain

$$\int \frac{\theta_0}{\theta} \{\lambda v_{k,k} v_{i,i} + 2\mu v_{(i,j)} v_{(i,j)}\} \, dV \geqq \frac{2 c_7 p_2}{\rho_0} \int \frac{\rho}{2} v_i v_i \, dV,$$

where $c_7 = \min(c_4, c_5)$ and p_2 is a positive valued function of R and ∂_1 R. (14) and (15) when combined with (10) give

$$\dot{\mathrm{E}}_1 + \dot{\mathrm{E}}_2 \leqq -c_8 \mathrm{E}_1 \tag{16}$$

with $c_8 = \min(2 c_7 p_2/\rho_0, c_6 p_1/c_1 \rho_0)$. It follows from (16) that

$$\begin{cases} \mathrm{E}_1(t) + \mathrm{E}_2(t) \leqq \mathrm{E}_1(0) + \mathrm{E}_2(0), \\ \lim\limits_{t \to \infty} \mathrm{E}_1(t) + \mathrm{E}_2(t) \text{ exists.} \end{cases} \tag{17}$$

Further, integration of (16) over (0, T), T being an arbitrary real positive number, gives

$$c_8 \int_0^{\mathrm{T}} \mathrm{E}_1(t) \, dt \leqq [\mathrm{E}_1(0) + \mathrm{E}_2(0)].$$

Since the right-hand side of this inequality is independent of T, we conclude that

$$\mathrm{E}_1(t) \in \mathrm{L}^1(0, \infty).$$

Now using $E_2(t) \leqq E_1(0)+E_2(0)$, we obtain from (16) the following

$$\dot{E}_1(t) \in L^1(0, \infty).$$

(18) and (19) imply $(13)_1$ and $(13)_2$ now follows from $(17)_2$.

4. Remarks

For the special case when there exists a time t_0 such that for $t \geqq t_0$

$$\dot{E}_2 = 0,$$

(16) reduces to

$$\dot{E}_1 \leqq -c_8 E_1, \qquad t \geqq t_0, \tag{21}$$

and hence

$$E_1(t) \leqq E_1(t_0) e^{-c_8 t} \leqq [E_1(0)+E_2(0)] e^{-c_8 t}. \tag{22}$$

In [4] Shahinpoor and Ahmadi study the decay of the kinetic energy of compressible micropolar fluids and make an assumption which, for the present problem, reduces to (20). It is not entirely clear under what circumstances (20) holds. Should (20) hold, then the kinetic and the thermal energy of the fluid would decay monotonically and the rate of decay would depend upon the thermomechanical properties of the fluid, the size and shape of the container and the boundary conditions.

The assumption that a classical solution of (1)-(3) under boundary conditions (5) and suitable initial conditions exits is made here to keep the analysis simple. Otherwise one can assume the existence of a suitably defined weak solution (e. g., *see* [1], [8]) and prove (13) by using essentially the same arguments.

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