Thermodynamics of simple materials
of differential type

par

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ABSTRACT. — Simple materials of differential type are studied according to an entropy inequality proposed by Green and Laws. Propagation of weak waves in the theory linearized about a uniform temperature distribution is investigated. It is shown that the necessary condition in order that weak waves may propagate in materials whose constitutive quantities depend upon the rate of change of temperature gradient is opposite of that required for materials whose constitutive quantities do not depend upon the rate of change of temperature gradient.

RÉSUMÉ. — On étudie les milieux matériellement simples de type différentiel à partir d'une inégalité d'entropie proposée par Green et Laws. On envisage la propagation d'ondes de discontinuité faible dans la théorie linéarisée autour d'une distribution uniforme de la température. On montre que la condition nécessaire pour que les ondes puissent se propager dans les matériaux dont les grandeurs constitutives dépendent du taux du gradient de température est l'opposée de celle exigée pour les matériaux dont les grandeurs constitutives ne dépendent pas de ce taux.

1. Introduction

Materials of differential type are simple materials whose local state depends upon the present value of the constitutive variables and of their time derivatives up to a certain order [1]. In thermomechanics of simple materials the constitutive variables usually considered are the deformation gradient $\mathbf{F}$, the absolute temperature θ and the spatial gradient $\mathbf{g}$ of θ. For general simple materials the local state can depend upon the histories of $\mathbf{F}$, θ and $\mathbf{g}$; but for materials of differential type, the local state depends upon $\mathbf{F}, \overset{.}{\mathbf{F}}, \ldots, \overset{(L)}{\mathbf{F}}, \theta, \overset{.}{\theta}, \ldots, \overset{(M)}{\theta}, \mathbf{g}, \overset{.}{\mathbf{g}}, \ldots, \overset{(N)}{\mathbf{g}}$. Here $\overset{(n)}{f}$ stands for the

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n^{th} material time derivative of f and $\overset{(0)}{f} = f$. We shall call such a material a material of differential type of complexity (L, M, N). With this definition the materials of differential type studied in [2], [3] and [4] are, respectively, of complexity (0, 1, 0), (1, 1, 0) and (N, 0, 0).

In this paper we study materials of differential type of complexity (L, M, 0) and (1, 1, 1) according to the following entropy inequality proposed by Green and Laws [5]:

$$(1) \qquad \rho_0 \dot{\eta} + \left(\frac{q_A}{\varphi} \right)_{,A} - \rho_0 \frac{r}{\varphi} \geqq 0.$$

Here η is the specific entropy, ρ_0 is the mass density in the reference configuration, r is the supply per unit mass of the internal energy, q is the heat flux per unit surface area in the reference configuration, $\varphi > 0$ is a constitutive quantity and A stands for differentiation with respect to rectangular Cartesian coordinates X_A of a particle in the reference configuration. The inequality (1) is more general than the Clausius-Duhem inequality in which $\varphi = \theta$. Green and Laws proposed an inequality more general than (1) which for homogeneous bodies becomes (1). We study herein homogeneous bodies only and therefore use (1).

We show that for materials of differential type of complexity (L, M, 0) φ is a function of θ and $\dot{\theta}$ and that

$$\textit{either} \qquad \frac{\partial \varphi}{\partial \dot{\theta}} = 0 \qquad \textit{or} \qquad \frac{\partial \mathbf{q}}{\partial \overset{(L)}{\mathbf{F}}} = \frac{\partial \mathbf{q}}{\partial \overset{(M)}{\theta}} = \mathbf{0},$$

provided $L \geqq 1$, $M > 1$. However, for materials of differential type of complexity (1, 1, 1) we prove that $\varphi = \varphi(\theta, \dot{\theta})$ and

$$\textit{either} \qquad \frac{\partial \varphi}{\partial \dot{\theta}} = 0 \qquad \textit{or} \qquad \frac{\partial \eta}{\partial \dot{g}} = \frac{\partial \eta}{\partial \dot{F}} = \mathbf{0}.$$

Thus a necessary condition for the propagation of thermal disturbances with finite speed in these materials is that $\partial \varphi / \partial \dot{\theta} = 0$. This disagrees with the result, obtained below, that for materials of differential type of complexity (1, 1, 0), thermal disturbances may propagate with finite speed in the theory linearized about a uniform temperature distribution provided $\partial \varphi / \partial \dot{\theta} \neq 0$. When $\partial \varphi / \partial \dot{\theta} = 0$, the inequality (1) reduces to the Clausius-Duhem inequality.

2. Restrictions from the entropy inequality

We study materials for which the following balance laws hold:

$$(2) \quad \begin{cases} \rho J = \rho_0, \quad J = \det F, \quad F_{iA} = x_{i,A}, \\ \rho_0 \dot{v}_i = S_{iA,A} + \rho_0 b_i, \\ \rho_0 \dot{\varepsilon} = -q_{A,A} + S_{iA} \dot{x}_{i,A} + \rho_0 r. \end{cases}$$

Here $x = x(X, t)$ gives the present position of the material particle that occupied place X in the reference configuration, ρ is the mass density at time t, b is the specific body force, S_{iA} is the Piola-Kirchoff stress tensor and ε is the specific internal energy. Introducing the definition

$$(3) \quad \psi = \varepsilon - \eta \varphi,$$

and eliminating r from $(2)_5$ and (1) we obtain the following inequality

$$(4) \quad \rho_0 (\dot{\psi} + \eta \dot{\varphi}) - S_{iA} \dot{x}_{i,A} + \frac{q_A \varphi_{,A}}{\varphi} \leqq 0.$$

2.1. MATERIALS OF DIFFERENTIAL TYPE OF COMPLEXITY (L, M, O)

In (2) and (4), S, q, ψ, η and φ are constitutive quantities. For materials of differential type of complexity (L, M, 0), these constitutive quantities are assumed to be smooth functions of the following local state variables :

$$F, \dot{F}, \ldots, \overset{(L)}{F}; \theta, \dot{\theta}, \ldots, \overset{(M)}{\theta}; g.$$

Here $g_A = \theta_{,A}$. In order to distinguish between a function and its value we denote the function by a superposed caret. That is, we use the notation

$$(5) \quad \varepsilon(X, t) = \hat{\varepsilon}(F, \dot{F}, \ldots, \overset{(L)}{F}; \theta, \dot{\theta}, \ldots, \overset{(M)}{\theta}; g).$$

Substitution for S, q and ε into $(2)_{4,5}$ gives field equations for x and θ.

Referring the reader to [2], [5] for details we use an argument due to Coleman and Noll [6] and conclude that the following are necessary and sufficient conditions for every solution of the field equations to satisfy (4) :

$$(6)_{1-2} \quad \frac{\partial \hat{\psi}}{\partial \overset{(L)}{F}} + \hat{\eta} \frac{\partial \hat{\varphi}}{\partial \overset{(L)}{F}} = 0, \quad L \geqq 1, \quad \frac{\partial \hat{\psi}}{\partial \overset{(M)}{\theta}} + \hat{\eta} \frac{\partial \hat{\varphi}}{\partial \overset{(M)}{\theta}} = 0,$$

$$(6)_{3\text{-}6}\begin{cases}\hat{q}_{A)}\dfrac{\partial\hat{\varphi}}{\partial F_{i\,(B}^{(l)}}=0,\qquad l=0,1,2,\ldots,L,\\[2ex]\hat{q}_{A}\dfrac{\partial\hat{\varphi}}{\partial\overset{(m)}{\theta}}=0,\qquad m=2,3,\ldots,M,\\[2ex]\hat{q}_{A)}\dfrac{\partial\hat{\varphi}}{\partial g_{(B}}=0,\\[2ex]\dfrac{\partial\hat{\varphi}}{\partial\dot{\theta}}\,\hat{q}_{A}+\rho_0\,\hat{\varphi}\left(\dfrac{\partial\hat{\psi}}{\partial g_A}+\hat{\eta}\,\dfrac{\partial\hat{\varphi}}{\partial g_A}\right)=0,\end{cases}$$

$$(7)\qquad\left[\rho_0\left(\dfrac{\partial\hat{\psi}}{\partial F_{iA}}+\hat{\eta}\,\dfrac{\partial\hat{\varphi}}{\partial F_{iA}}\right)-\hat{S}_{iA}\right]\dot{F}_{iA}$$

$$+\sum_{l=1}^{L-1}\rho_0\left(\dfrac{\partial\hat{\psi}}{\partial F_{iA}^{(l)}}+\hat{\eta}\,\dfrac{\partial\hat{\varphi}}{\partial F_{iA}^{(l)}}\right)\overset{(l+1)}{F_{iA}}$$

$$+\sum_{m=0}^{M-1}\rho_0\left(\dfrac{\partial\hat{\psi}}{\partial\overset{(m)}{\theta}}+\hat{\eta}\,\dfrac{\partial\hat{\varphi}}{\partial\overset{(m)}{\theta}}\right)\overset{(m+1)}{\theta}+\dfrac{1}{\hat{\varphi}}\,\hat{q}_A\dfrac{\partial\hat{\varphi}}{\partial\theta}g_A\leqq 0.$$

Throughout the paper the round parantheses around the indices indicate symmetrization about the indices A, B, *etc.* (7) is called the residual inequality. Assuming that $\mathbf{q}\neq\mathbf{0}$ we conclude from $(6)_{3,4,5}$ that

$$(8)\qquad\qquad\qquad\varphi=\hat{\varphi}(\theta,\dot{\theta})$$

and now from $(6)_{1,2,6}$ that

$$(9)\quad\dfrac{\partial\hat{\psi}}{\partial\overset{(L)}{F}}=\mathbf{0},\quad L\geqq 1;\qquad\dfrac{\partial\hat{\psi}}{\partial\overset{(M)}{\theta}}=0,\quad M\geqq 2;\qquad\dfrac{\partial\hat{\varphi}}{\partial\dot{\theta}}\,\hat{q}_A+\rho_0\,\hat{\varphi}\,\dfrac{\partial\hat{\psi}}{\partial g_A}=0.$$

By differentiating $(9)_3$ with respect to $\overset{(L)}{F}$ and $\overset{(M)}{\theta}$ and using $(9)_{1,2}$ we arrive at the following:

$$(10)\qquad Either\quad\dfrac{\partial\hat{\varphi}}{\partial\dot{\theta}}=0\quad or\quad\dfrac{\partial\hat{q}}{\partial\overset{(L)}{F}}=\dfrac{\partial\hat{q}}{\partial\overset{(M)}{\theta}}=\mathbf{0},$$

provided $M\geqq 2$, $L\geqq 1$.

We note that the left-hand side of (7) has the maximum value namely zero in a process in which

$$\dot{F}=\ddot{F}=\ldots=\overset{(L)}{F}=\mathbf{0},\qquad\dot{\theta}=\ddot{\theta}=\ldots=\overset{(M)}{\theta}=0,\qquad g=\mathbf{0},$$

usually called equilibrium. The necessary conditions for this maximum are

$$\frac{\partial \hat{\psi}}{\partial \overset{(l)}{\mathbf{F}}}\Bigg|_E = 0, \qquad l = 1, 2, \ldots, L-1,$$

$$\hat{S}_{iA}\Big|_E = \rho_0 \frac{\partial \hat{\psi}}{\partial F_{iA}}\Bigg|_E,$$

$$\frac{\partial \hat{\psi}}{\partial \theta}\Bigg|_E + \hat{\eta}\Big|_E \frac{\partial \hat{\varphi}}{\partial \theta}\Bigg|_E = 0, \qquad \frac{\partial \hat{\psi}}{\partial \dot{\theta}}\Bigg|_E + \hat{\eta}\Big|_E \frac{\partial \hat{\varphi}}{\partial \dot{\theta}}\Bigg|_E = 0,$$

$$\frac{\partial \hat{\psi}}{\partial \overset{(m)}{\theta}}\Bigg|_E = 0, \qquad m = 2, 3, \ldots, M-1; \quad M > 2,$$

$$\hat{q}_A\Big|_E \frac{\partial \hat{\varphi}}{\partial \theta}\Bigg|_E = 0, \qquad \frac{\partial \hat{\varphi}}{\partial \theta}\Bigg|_E \frac{\partial \hat{q}_{(A}}{\partial g_{B)}}\Bigg|_E \leqq 0,$$

(11) $$\frac{\partial \hat{S}_{iA}}{\partial \dot{F}_{jB}}\Bigg|_E \geqq 0, \qquad \frac{\partial^2 \hat{\psi}}{\partial \overset{(l)}{F}_{iA} \partial \overset{(l+1)}{F}_{jB}}\Bigg|_E \leqq 0, \qquad l = 1, 2, \ldots, L-1,$$

$$\frac{\partial^2 \hat{\psi}}{\partial \overset{(m)}{\theta} \partial \overset{(m+1)}{\theta}}\Bigg|_E \leqq 0, \qquad m = 2, 3, \ldots, M-2; \quad M > 2,$$

$$\frac{\partial^2 \hat{\psi}}{\partial \dot{\theta} \partial \overset{..}{\theta}}\Bigg|_E + \frac{\partial \hat{\eta}}{\partial \overset{..}{\theta}}\Bigg|_E \frac{\partial \hat{\varphi}}{\partial \dot{\theta}}\Bigg|_E \leqq 0, \qquad M > 2,$$

$$\frac{\partial \hat{\eta}}{\partial \overset{..}{\theta}}\Bigg|_E \frac{\partial \hat{\varphi}}{\partial \dot{\theta}}\Bigg|_E \leqq 0, \qquad M = 2,$$

$$\frac{\partial \hat{\eta}}{\partial \dot{\theta}}\Bigg|_E \frac{\partial \hat{\varphi}}{\partial \theta}\Bigg|_E \leqq \frac{\partial \hat{\eta}}{\partial \theta}\Bigg|_E \frac{\partial \hat{\varphi}}{\partial \dot{\theta}}\Bigg|_E.$$

Here the index E signifies that the quantity is evaluated in an equilibrium process. We can sharpen these results further by setting $\hat{\varphi}\big|_E = \theta$.

2.2. MATERIALS OF DIFFERETNIAL TYPE OF COMPLEXITY (1, 1, 1)

For these materials S, q, ψ, η and φ are smooth functions of F, $\dot{\text{F}}$, θ, $\dot{\theta}$, g and $\dot{\text{g}}$. We denote the function defining any of these by the same symbol with a surperposed tilde. Proceeding in the usual manner (*e. g. see*

[2], [4], [5], [6]), we obtain the following restrictions imposed by the entropy inequality on the constitutive functions:

(12)
$$\left\{ \begin{array}{l} \dfrac{\partial \tilde{\psi}}{\partial \dot{\mathbf{F}}} + \tilde{\eta} \dfrac{\partial \tilde{\varphi}}{\partial \dot{\mathbf{F}}} = 0, \qquad \dfrac{\partial \tilde{\psi}}{\partial \dot{\theta}} + \tilde{\eta} \dfrac{\partial \tilde{\varphi}}{\partial \dot{\theta}} = 0, \qquad \dfrac{\partial \tilde{\psi}}{\partial \dot{\mathbf{g}}} + \tilde{\eta} \dfrac{\partial \tilde{\varphi}}{\partial \dot{\mathbf{g}}} = 0, \\[3mm] \tilde{q}_{A)} \dfrac{\partial \tilde{\varphi}}{\partial g_{(B}} = 0, \qquad \tilde{q}_{A)} \dfrac{\partial \tilde{\varphi}}{\partial \dot{g}_{(B}} = 0, \qquad \tilde{q}_{A)} \dfrac{\partial \tilde{\varphi}}{\partial F_{i\,(B}} = 0, \qquad \tilde{q}_{A)} \dfrac{\partial \tilde{\varphi}}{\partial \dot{F}_{i\,(B}} = 0, \end{array} \right.$$

(13)
$$\rho_0 \left[\left(\dfrac{\partial \tilde{\psi}}{\partial \theta} + \tilde{\eta} \dfrac{\partial \tilde{\varphi}}{\partial \theta} \right) \dot{\theta} + \left(\dfrac{\partial \tilde{\psi}}{\partial g_A} + \tilde{\eta} \dfrac{\partial \tilde{\varphi}}{\partial g_A} \right) \dot{g}_A \right]$$
$$+ \left[\rho_0 \left(\dfrac{\partial \tilde{\psi}}{\partial F_{iA}} + \tilde{\eta} \dfrac{\partial \tilde{\varphi}}{\partial F_{iA}} \right) - \tilde{S}_{iA} \right] \dot{F}_{iA} + \dfrac{\tilde{q}_A}{\tilde{\varphi}} \left(\dfrac{\partial \tilde{\varphi}}{\partial \theta} g_A + \dfrac{\partial \tilde{\varphi}}{\partial \dot{\theta}} \dot{g}_A \right) \leqq 0.$$

Assuming that $\mathbf{q} \neq \mathbf{0}$, we conclude from $(12)_{4,5,6,7}$ that $\varphi = \tilde{\varphi}\,(\theta, \dot{\theta})$. Therefore equations $(12)_{1,3}$ simplify to

(14)
$$\dfrac{\partial \tilde{\psi}}{\partial \dot{\mathbf{F}}} = \dfrac{\partial \tilde{\psi}}{\partial \dot{\mathbf{g}}} = 0.$$

Differentiation of $(12)_2$ with respect to $\dot{\mathbf{F}}$ and $\dot{\mathbf{g}}$ and the use of (14) gives equations which imply that

(15)   *either* $\dfrac{\partial \tilde{\varphi}}{\partial \dot{\theta}} = 0$ *or* $\dfrac{\partial \tilde{\eta}}{\partial \dot{\mathbf{F}}} = \dfrac{\partial \tilde{\eta}}{\partial \dot{\mathbf{g}}} = \mathbf{0}.$

We simplify (13) by substituting $\varphi = \tilde{\varphi}\,(\theta, \dot{\theta})$ into it and derive the following restrictions on the equilibrium values (*i. e.* when $\dot{\mathbf{F}} = \dot{\theta} = \mathbf{g} = \dot{\mathbf{g}} = \mathbf{0}$) of the constitutive functions:

(16)
$$\left\{ \begin{array}{ll} \dfrac{\partial \tilde{\psi}}{\partial \theta} \bigg|_E + \tilde{\eta}\,|_E \dfrac{\partial \tilde{\varphi}}{\partial \theta} \bigg|_E = 0, & \tilde{S}_{iA}\,|_E = \rho_0 \dfrac{\partial \tilde{\psi}}{\partial F_{iA}} \bigg|_E, \\[4mm] \rho_0\,\tilde{\varphi}\,|_E \dfrac{\partial \tilde{\psi}}{\partial g_A} \bigg|_E + \tilde{q}_A\,|_E \dfrac{\partial \tilde{\varphi}}{\partial \dot{\theta}} \bigg|_E = 0, & \dfrac{\partial \tilde{S}_{iA}}{\partial \dot{F}_{jB}} \bigg|_E \geqq 0, \\[4mm] \tilde{q}_A\,|_E \dfrac{\partial \tilde{\varphi}}{\partial \dot{\theta}} \bigg|_E = 0, & \dfrac{\partial \tilde{q}_{(A}}{\partial \dot{g}_{B)}} \bigg|_E \dfrac{\partial \tilde{\varphi}}{\partial \dot{\theta}} \bigg|_E \leqq 0, \\[4mm] \dfrac{\partial \tilde{q}_{(A}}{\partial g_{B)}} \bigg|_E \dfrac{\partial \tilde{\varphi}}{\partial \dot{\theta}} \bigg|_E \leqq 0, & \dfrac{\partial \tilde{\eta}}{\partial \dot{\theta}} \bigg|_E \dfrac{\partial \tilde{\varphi}}{\partial \dot{\theta}} \bigg|_E \leqq \dfrac{\partial \tilde{\eta}}{\partial \theta} \bigg|_E \dfrac{\partial \tilde{\varphi}}{\partial \dot{\theta}} \bigg|_E. \end{array} \right.$$

3. Propagation of weak waves

We now study the process of heat conduction in a stationary body made of material of differential type of complexity $(1, 1, 1)$. The relevant field equation obtained from the balance of the internal energy is

$$(17) \qquad \rho_0 \left[\frac{\partial \tilde{\varepsilon}}{\partial \theta} \dot{\theta} + \frac{\partial \tilde{\varepsilon}}{\partial \dot{\theta}} \ddot{\theta} + \frac{\partial \tilde{\varepsilon}}{\partial g_A} \dot{g}_A + \frac{\partial \tilde{\varepsilon}}{\partial \dot{g}_A} \ddot{g}_A \right]$$

$$= - \left[\frac{\partial \tilde{q}_A}{\partial \theta} g_A + \frac{\partial \tilde{q}_A}{\partial \dot{\theta}} \dot{g}_A + \frac{\partial \tilde{q}_{(A}}{\partial g_{B)}} g_{B, A} + \frac{\partial \tilde{q}_{(A}}{\partial \dot{g}_{B)}} \dot{g}_{B, A} \right] + \rho_0 \, r.$$

Assuming that θ, $\dot{\theta}$, g, $\ddot{\theta}$, $\dot{g}$, $g_{A,B}$ and r are continuous across a singular surface and third order derivatives of θ are discontinuous, the jumps $[\dddot{\theta}]$ $[g]$, $[\ddot{\theta}]$, $[\ddot{g}]$, $[g_{A,B}]$ across the wave vanish and since (17) holds on both sides of the wave,

$$(18) \qquad \rho_0 \frac{\partial \tilde{\varepsilon}}{\partial \dot{g}_A} [\ddot{g}_A] = - \frac{\partial \tilde{q}_{(A}}{\partial \dot{g}_{B)}} [\dot{g}_{B, A}].$$

From the theory of moving singular surfaces ([7], p. 506), we have

$$(19) \qquad \begin{cases} [\ddot{g}_A] = c u_n^2 \, n_A, \\ [\dot{g}_{B, A}] = - c u_n n_A n_B, \end{cases}$$

where $c \equiv [\theta_{, ABC}] \, n_A \, n_B \, n_C \neq 0$, n_A is the unit wave normal and u_n is the wave speed. Substitution of (19) into (18) yields

$$(20) \qquad \rho_0 \, c \frac{\partial \tilde{\varepsilon}}{\partial \dot{g}_A} n_A u_n^2 = \frac{\partial \tilde{q}_{(A}}{\partial \dot{g}_{B)}} c u_n n_A n_B.$$

If $\partial \tilde{\varphi} / \partial \dot{\theta} \neq 0$, then by (15), (14) and (3), $\partial \tilde{\varepsilon} / \partial \dot{g} = 0$ and, therefore, from (20) $u_n = 0$. Thus weak discontinuities in θ do not propagate in these materials if $\partial \tilde{\varphi} / \partial \dot{\theta} \neq 0$. However, when $\partial \tilde{\varphi} / \partial \dot{\theta} = 0$, $\partial \tilde{\varepsilon} / \partial \dot{g}_A$ and $\partial \tilde{q}_{(A} / \partial \dot{g}_{B)}$ are not restricted by the entropy inequality and (20) may have a non-zero solution for u_n.

In a theory linearized about a uniform temperature distribution, the speed of propagation of weak disturbances is given by

$$(21) \qquad \rho_0 \frac{\partial \tilde{\varepsilon}}{\partial \dot{g}_A} \bigg|_E n_A u_n^2 = u_n \frac{\partial \tilde{q}_{(A}}{\partial \dot{g}_{B)}} \bigg|_E n_A n_B.$$

For isotropic materials, $\tilde{\varepsilon}$ is an isotropic scalar function of its variables. Therefore, $\partial\tilde{\varepsilon}/\partial\mathbf{g}$ is an isotropic vector and $\partial\tilde{\varepsilon}/\partial\dot{\mathbf{g}}|_E = 0$. However, $\partial\tilde{q}_{(A}/\partial\dot{g}_{B)}|_E$ may not be zero for isotropic materials. Thus in the theory linearized about a uniform temperature distribution, weak waves do not propagate in isotropic materials of differential type of complexity (1, 1, 1).

Specializing the results of section 2.1 to materials of differential type of complexity (1, 1, 0), we investigate the speed of propagation of thermal disturbances. The pertinent equation is

$$(22) \qquad \rho_0 \frac{\partial\bar{\varepsilon}}{\partial\dot{\theta}}[\ddot{\theta}] = -\left(\rho_0 \frac{\partial\bar{\varepsilon}}{\partial g_A} + \frac{\partial\bar{q}_A}{\partial\dot{\theta}}\right)[\dot{g}_A] - \frac{\partial\bar{q}_{(A}}{\partial g_{B)}}[g_{B,A}],$$

where we have assumed that θ, $\dot{\theta}$, $\mathbf{g}$ are continuous across the singular surface and have denoted the constitutive functions by a superimposed bar. Substituting for various jumps from the theory of moving singular surfaces, we obtain

$$(23) \qquad \rho_0 \frac{\partial\bar{\varepsilon}}{\partial\dot{\theta}} u_n^2 D = +\left(\rho_0 \frac{\partial\bar{\varepsilon}}{\partial g_A} + \frac{\partial\bar{q}_A}{\partial\dot{\theta}}\right) n_A u_n D - \frac{\partial\bar{q}_{(A}}{\partial g_{B)}} D n_A n_B,$$

where $D = [\theta_{,AB}] \, n_A \, n_B$. The two solutions u_n of this equation are either both real or both complex. In the theory linearized about a uniform temperature distribution (23) becomes

$$(24) \qquad \rho_0 u_n^2 \frac{\partial\bar{\varepsilon}}{\partial\dot{\theta}}\bigg|_E = \left(\rho_0 \frac{\partial\bar{\varepsilon}}{\partial g_A}\bigg|_E + \frac{\partial\bar{q}_A}{\partial\dot{\theta}}\bigg|_E\right) n_A u_n - \frac{\partial\bar{q}_{(A}}{\partial g_{B)}}\bigg|_E n_A n_B.$$

Recalling $(11)_7$ we note that (24) has real roots when $\partial\bar{\varepsilon}/\partial\dot{\theta}|_E > 0$. But for $\partial\bar{\varepsilon}/\partial\dot{\theta}|_E$ to be positive it is necessary that $\partial\bar{\varphi}/\partial\dot{\theta}|_E \neq 0$. [cf. $(11)_{13}$]. Thus in this case thermal disturbances may propagate with finite speed. We remark that for isotropic materials the first term on the right-hand side of (24) vanishes.

4. Remarks

The function $\bar{\varphi}(\theta, \dot{\theta})$ for materials of differential type of complexity (1, 1, 0) need not be the same as $\tilde{\varphi}(\theta, \dot{\theta})$. However, if we make an assump-

tion, proposed by Müller [3], that across any singular surface $q_A\, n_A/\varphi$ is continuous whenever $q_A\, n_A$ and θ are continuous, then we can show that $\bar\varphi = \tilde\varphi$. Indeed, consider a material singular surface separating bodies made of materials of differential type of complexity (1, 1, 0) and (1, 1, 1) and assume that θ is continuous across the material singular surface. Since the balance of internal energy requires that the normal component of heat flux be continuous, the preceeding assumption gives that the normal component of the entropy flux is also continuous. Hence $\varphi\,(\theta,\dot\theta) = \tilde\varphi\,(\theta,\dot\theta)$. In this case the condition $\partial\tilde\varphi/\partial\dot\theta \neq 0$ allows for the possibility of propagation of thermal disturbances with finite speed in the linear theory of materials of differential type of complexity (1, 1, 0) but disallows the propagation of thermal waves in materials of differential type of complexity (1, 1, 1).

In [3], Müller studied fluids of differential type of complexity (1, 1, 0) according to his own inequality for supply free bodies and showed that in these materials thermal disturbances can propagate with finite speed in the theory linearized about a uniform temperature distribution provided the coldness function $\Lambda\,(\theta,\dot\theta)$ depends upon $\dot\theta$. If we study isotropic materials of differential type of complexity (1, 1, 1) according to Müller's theory we can show that entropy flux = (heat flux) (coldness Λ), the coldness Λ is a universal function of θ and $\dot\theta$ in the sense that $\Lambda\,(\theta,\dot\theta)$ is the same function for materials of differential type of complexity (1, 1, 0) and (1, 1, 1) and that a necessary condition for the propagation of thermal disturbances with finite speed in isotropic materials of differential type of complexity (1, 1, 1) is $\partial\Lambda/\partial\dot\theta = 0$.

When $\partial\tilde\varphi/\partial\dot\theta = 0$, then $\varphi = \tilde\varphi\,(\theta) = \tilde\varphi\,(\theta)\,|_E = \theta$ and the entropy inequality (1) becomes the Clausis-Duhem inequality. Thus the Clausius-Duhem inequality does allow the propagation of thermal disturbances with finite speed in materials of differential type of complexity (1, 1, 1).

In [8], Batra studied non-simple elastic materials whose local state depends upon F, $F_{i\,A,B}$, θ, $\dot\theta$, g. It is shown therein that whereas Clausius-Duhem inequality allows the dependence of the constitutive functions upon $F_{i\,A,B}$, the inequality (1) does not provided we assume that φ in (1) does depend upon $\dot\theta$.

That φ in (1) is not a function of θ and $\dot\theta$ for all materials is clear from the non-simple rigid heat conductors studied by Batra [9]. For these

materials the local state is assumed to depend upon θ, $\dot{\theta}$, $\ddot{\theta}$, $\mathbf{g}$, $\dot{\mathbf{g}}$, $g_{A,B}$. It is proved in [9] that for isotropic homogeneous bodies made of such a material φ is a function of θ, $\dot{\theta}$ and $\mathbf{g}$.

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