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Decay of the kinetic and the thermal energy of incompressible micropolar fluids

by

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ABSTRACT. — We consider a heat conducting incompressible micropolar fluid at rest and filling a closed rigid container. We account for the heat generated due to viscous dissipation and show that the kinetic and the thermal energy of arbitrary disturbances of the rest state decays exponentially. The rate of decay depends upon the viscosity coefficients, heat conduction coefficients and the size and shape of the container.

RÉSUMÉ. — Nous considérons ici un fluide en repos, incompressible et conducteur de la chaleur, et remplissant un récipient solide et fermé. Nous prenons en compte la chaleur engendrée par la dissipation visqueuse et nous montrons que l'énergie cinétique et thermique de perturbation arbitraire de l'état de repos tend vers zéro exponentiellement. Le décrément dépend des coefficients de viscosité, des coefficients de conduction de la chaleur, et de la taille et de la forme du récipient.

1. Introduction

The problem of the decay of the energy of incompressible micropolar fluids and compressible micropolar fluids has been studied by Lakshmana Rao [1] and, Shahinpoor and Ahmadi [2]. These authors employ the linear theory of micropolar fluids due to Eringen [3]. Ahmadi [4] has recently studied the universal stability of thermo-magneto-micropolar fluid motions. His results when specialized to thermomechanical deformations show that the kinetic and the thermal energy of arbitrary disturbances of the rest state of a micropolar fluid contained in a closed rigid container decays exponentially. However, Ahmadi neglects the heat generated due to viscous dissipation and assumes that the specific heat is constant. Thus the energy equation becomes a linear parabolic equation. His analysis does not appear to be applicable to the case when the energy

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equation is non linear and includes the term representing heat generation due to dissipation. We note that whereas Ahmadi assumes the Boussinesq equation of state, that is, the mass density depends linearly upon the change of temperature, we assume that the mass density stays constant. This assumption is consistent with the theory proposed by Eringen [3]. The interaction between the flow and the temperature field is manifested through the appearance of viscous dissipation term in the energy equation, temperature in the equation governing the rate of change of microrotation, and the dependence of viscosity coefficients upon the temperature.

We remark that the region within the container is assumed to have smooth boundary so that the divergence theorem [5], the Poincaré inequality [6] and the Korn inequality [6] are applicable.

2. Formulation of the problem

The field equations governing the thermomechanical deformations of a linear micropolar fluid are [3]:

$$(1)\quad \begin{cases} v_{i,i} = 0, \\ \rho \dot{v}_i = t_{ki,k} - \rho \Omega_{,i}, \\ \rho j \dot{\nu}_r = m_{kr,k} + \varepsilon_{rks} t_{ks} + \rho l_r, \\ \rho \dot{\varepsilon} = t_{ks} b_{ks} + m_{ks} \nu_{s,k} + q_{k,k} + \rho h, \end{cases}$$

in which

$$(2)\quad \begin{cases} t_{ks} = -p\,\delta_{ks} + \mu (v_{k,s} + v_{s,k}) + \bar{\eta}\, b_{ks}, \\ m_{ks} = \alpha \nu_{r,r} \delta_{ks} + \beta \nu_{k,s} + \gamma \nu_{s,k} + \bar{\alpha} \varepsilon_{ksr} \theta_{,r}, \\ q_k = \varkappa \theta_{,k} + \bar{\beta} \varepsilon_{krs} \nu_{r,s}, \\ b_{ks} = v_{s,k} - \varepsilon_{ksr} \nu_r. \end{cases}$$

Throughout this paper we use rectangular Cartesian coordinates and the Cartesian tensor notation wherever convenient. Thus in the preceeding equations δ_{ks} is the Kronecker delta, ε_{ksr} is the alternating tensor or the permutation symbol, a comma followed by an index k indicates partial differentiation with respect to x_k, and repeated subscripts imply summation over the range 1, 2, 3 of indices. We have used spatial description of the motion. In (1) and (2), $\mathbf{v}$ stands for the velocity of a fluid particle that presently is at $\mathbf{x}$, ρ is the mass density, t_{ks} is the Cauchy stress tensor,

Ω is the potential of body forces and is assumed to be a function of $\mathbf{x}$ only, j is the microinertia, $\mathbf{v}$ is the microrotation of the flow, $\mathbf{l}$ is the supply density of the microrotation, ε is the internal energy density, $\mathbf{q}$ is the heat flux per unit present area, $\theta > 0$ is the present temperature of the fluid particle that currently occupies the place $\mathbf{x}$, and a superimposed dot indicates material time derivative. Furthermore, p is the arbitrary hydrostatic pressure and is not determined by the flow; μ, $\bar{\eta}$, α, β and γ are the viscosity coefficients and $\varkappa$, $\bar{\alpha}$ and $\bar{\beta}$ are the heat conduction coefficients. We presume that the fluid is homogeneous and that the viscosity and the heat conduction coefficients are temperature dependent. The assumption of homogeneity implies that the mass density ρ is constant which we take to be positive.

Assume that the initial disturbances are such that a classical solution of (1) satisfying these initial conditions and the following boundary conditions exists

$$(3)\quad \begin{cases} \mathbf{v}(\mathbf{x}, t) = \mathbf{0} \quad \text{on } \partial\mathrm{R}, \quad t > 0, \\ \mathbf{v}(\mathbf{x}, t) = \mathbf{0} \quad \text{on } \partial\mathrm{R}, \quad t > 0, \\ \theta = \theta_0 \quad \text{on } \partial_1 \mathrm{R}(t), \quad t > 0, \\ q_i n_i = -b(\theta, \theta_0)(\theta - \theta_0) \quad \text{on } \partial_2 \mathrm{R} \equiv \partial\mathrm{R} - \partial_1 \mathrm{R}, \quad > 0. \end{cases}$$

Here R is the region of space occupied by the fluid, $\partial_1 \mathrm{R} \subset \partial\mathrm{R}$ and the possibilities $\partial_1 \mathrm{R} = \emptyset$ or $\partial_1 \mathrm{R} = \partial\mathrm{R}$ are not excluded. In order that heat may flow out of the fluid when it is at a temperature higher than that of the environment, b should be non-negative. We assume that b is also a bounded function of θ and θ_0. The boundary condition $(3)_1$ implies that the fluid adheres to the walls of the container and that once the fluid is given the initial disturbance, the container is held at rest subsequently. Thus the fluid always occupies the same region R of space.

The thermodynamical restrictions imposed by the Clausius-Duhem inequality are [3]:

$$(4)\quad \begin{cases} 2\mu + \bar{\eta} \geqq 0, \quad \bar{\eta} \geqq 0, \quad \varkappa \geqq 0, \\ 3\alpha + \beta + \gamma \geqq 0, \quad \gamma - |\beta| \geqq 0, \\ \eta = -\dfrac{\partial\varphi}{\partial\theta}, \quad \varphi = \varepsilon - \eta\theta. \end{cases}$$

Here η is the entropy density, and φ is the Helmholtz free energy. It is shown in [3] that for incompressible micropolar fluids, ε, η and φ are functions of θ only. Expanding $\varphi(\theta_0)$ in terms of finite Taylor series around θ, we obtain

$$\varphi_0 = \varphi(\theta) - \eta(\theta_0 - \theta) - K(\theta - \theta_0)^2, \tag{5}$$

where

$$\varphi_0 \equiv \varphi(\theta_0), \qquad K = -\frac{1}{2}\frac{\partial^2 \varphi}{\partial \theta^2}(\theta^*) = \frac{c(\theta^*)}{2\theta^*},$$

c is the specific heat and θ^* is a value of the temperature between θ and θ_0. Substitution for $\varphi(\theta)$ from (5) into $(4)_7$ gives

$$\varepsilon - \theta_0 \eta = \varphi_0 + K(\theta - \theta_0)^2.$$

Also from $(4)_{6,7}$ one gets $\partial\varepsilon/\partial\theta = \theta(\partial\eta/\partial\theta)$ and hence one can conclude that

$$\begin{aligned} \frac{d}{dt}[K(\theta - \theta_0)^2] &= \dot{\varepsilon} - \theta_0 \dot{\eta} = \left(1 - \frac{\theta_0}{\theta}\right)\dot{\varepsilon} \\ &= \frac{1}{\rho}\left(1 - \frac{\theta_0}{\theta}\right)\left[t_{ks}\, b_{ks} + m_{ks}\, \nu_{s,k} + q_{k,k} + \rho\, h\right]. \end{aligned} \tag{6}$$

In order to arrive at the last equation, we have substituted for $\dot{\varepsilon}$ from $(1)_4$.

Taking the inner product of $(1)_2$ with $\mathbf{v}$, of $(1)_3$ with $\mathbf{v}$, setting $\mathbf{l} = \mathbf{0}$, integrating the resulting equations over the region occupied by the fluid, simplifying by using $(1)_1$, the divergence theorem and the boundary conditions (3), and adding twice the result to respective sides of (6), we get

$$\dot{E} = -\int\left(1 + \frac{\theta_0}{\theta}\right)(t_{ks}\, b_{ks} + m_{ks}\, \nu_{s,k})\, dV + \int\left(1 - \frac{\theta_0}{\theta}\right)(q_{k,k} + \rho\, h)\, dV, \tag{7}$$

in which

$$E \equiv \int \rho\,[v^2 + j\,\nu^2 + K(\theta - \theta_0)^2]\, dV. \tag{8}$$

Thus E equal to zero implies that the fluid is at rest, there is no micro-rotation of the fluid and the temperature of the fluid is uniform and equals θ_0.

We now state the theorem we wish to prove below.

THEOREM. — *For every initial disturbance of the rest state of the thermo-micropolar fluid for which* E (0) *is finite and there exists a classical solution* *of* (1) *satisfying these initial conditions and the boundary conditions* (3), *the energy* E *satisfies the inequality*

$$\text{(9)} \qquad E(t) \leqq E(0)\, e^{-dt}$$

provided that

$$\left\{\begin{array}{l} c_1 \equiv \inf\limits_{\theta} 2\mu > 0, \\ c_2 \equiv \inf\limits_{\theta}\left[\alpha + \dfrac{\beta}{3} + \dfrac{\gamma}{3},\ \gamma+\beta\right] > 0, \\ c_3 \equiv \inf\limits_{\theta} \dfrac{\theta_0}{\theta^2}\varkappa > 0, \\ c_4 \equiv \inf\limits_{\theta} \dfrac{b(\theta, \theta_0)}{\theta} > 0, \\ c_5 \equiv \sup\limits_{\theta} K < \infty, \\ \gamma - |\beta| \geqq 0, \qquad \bar{\eta} \geqq 0, \qquad h(\theta-\theta_0) \leqq 0. \end{array}\right.$$

A comparison of (10) with the requirements on the various coefficients needed to prove a similar but somewhat weaker result for heat conducting compressible micropolar fluids [7] reveals that only the definitions of c_1 and c_2 are slightly different. Whereas the infimum here is taken of the values of viscosity coefficients, in [7] the infimum is taken of the values of viscosity coefficients divided by the temperature θ. The definitions (10) of various constants can be sharpened by taking the infimum or the supremum over those values of θ which are ever realized at any fluid particle. We note that $(10)_8$ is satisfied even when there is no supply of internal energy, i. e. $h = 0$.

Proof of the Theorem. – Recalling (2) and using the definitions

$$v_{(k,s)} \equiv \frac{1}{2}(v_{k,s} + v_{s,k}),$$

$$v_{[k,s]} \equiv \frac{1}{2}(v_{k,s} - v_{s,k}),$$

$$v^d_{k,s} \equiv v_{k,s} - \frac{1}{3} v_{r,r}\,\delta_{ks}$$

we note that

$$(12)\quad \begin{aligned} & t_{ks}b_{ks}+m_{ks}\nu_{s,k} \\ & = 2\mu v_{(k,s)}v_{(k,s)}+\left(\alpha+\frac{\beta}{3}+\frac{\gamma}{3}\right)\nu_{r,r}\nu_{s,s} \\ & \quad +(\gamma+\beta)\nu^d_{(k,s)}\nu^d_{(k,s)}+(\gamma-\beta)\nu_{[k,s]}\nu_{[k,s]}+\bar{\alpha}\varepsilon_{ksr}\theta_{,r}\nu_{s,k} \\ & \geqq c_1 v_{(s,k)}v_{(s,k)}+c_2\nu_{(k,s)}\nu_{(k,s)}+\bar{\alpha}\varepsilon_{ksr}\theta_{,r}\nu_{s,k}. \end{aligned}$$

We have used (10) to obtain the preceeding inequality (12). Multiplying both sides of (12) by $[1+(\theta_0/\theta)]$, and then integrating it over the region occupied by the fluid and noting that

$$\int \bar{\alpha}\varepsilon_{ksr}\theta_{,r}\nu_{s,k}\,dV = \int \bar{\alpha}\varepsilon_{ksr}\theta_{,r}\nu_s n_k\,dA = 0,$$

in view of $(3)_2$, we conclude the following

$$(13)\quad \int\left(1+\frac{\theta_0}{\theta}\right)(t_{ks}b_{ks}+m_{ks}\nu_{s,k})\,dV \geqq \left[c_1\int v_{(s,k)}v_{(s,k)}\,dV+c_2\int \nu_{(s,k)}\nu_{(k,s)}\,dV\right].$$

The use of Poincaré's inequality [6]:

$$\int v_{k,s}v_{k,s}\,dV \geqq p_1\int v^2\,dV,$$

and Korn's inequality [6]:

$$(15)\quad \int v_{(k,s)}v_{(k,s)}\,dV \geqq p_2\int v_{k,s}v_{k,s}\,dV,$$

in (13) gives the following

$$(16)\quad \int\left(1+\frac{\theta_0}{\theta}\right)(t_{ks}b_{ks}+m_{ks}\nu_{s,k})\,dV \geqq \min\left(c_1,\frac{c_2}{j}\right)\frac{p_1p_2}{\rho}\int \rho(v^2+j\nu^2)\,dV.$$

Here p_1 and p_2 are constants whose values depend upon the region R. By using the divergence theorem we obtain

$$\int\left(1-\frac{\theta_0}{\theta}\right)q_{k,k}\,dV = \int\left(1-\frac{\theta_0}{\theta}\right)q_k n_k\,dA - \int\frac{\theta_0}{\theta^2}q_k\theta_{,k}\,dV.$$

Substitution of the boundary conditions $(3)_{3,4}$ and the constitutive relation $(2)_3$ for q_k into the right-hand side of the preceeding equation, and the observation that

$$\int\frac{\theta_0}{\theta^2}\bar{\beta}\varepsilon_{krs}\nu_{r,s}\theta_{,k}\,dV = -\int \nu_r\left(\frac{\theta_0}{\theta^2}\bar{\beta}\varepsilon_{krs}\theta_{,k}\right)_{,s}dV,$$

enables us to conclude the following

$$\int\left(1-\frac{\theta_0}{\theta}\right) q_{k,k}\, dV = -\int \frac{b}{\theta}(\theta-\theta_0)^2\, dA - \int \frac{\theta_0}{\theta^2}\varkappa(\theta_0-\theta)_{,s}(\theta_0-\theta)_{,s}\, dV. \tag{17}$$

When $\bigcap\limits_{t>0} \partial_1 R(t) > 0$, that is, there is a material subsurface of the boundary surface ∂R on which the temperature is prescribed to be θ_0, the Poincaré inequality (14) holds with p_1 now being a real positive valued function of time t. Denoting its infimum also by p_1 and assuming that it is positive, we conclude that

$$\int\left(1-\frac{\theta_0}{\theta}\right) q_{k,k}\, dV \leqq -\frac{a}{c_5}\int K(\theta_0-\theta)^2\, dV, \tag{18}$$

in which $a = c_3 p_1$. However, if $\partial_1 R = \emptyset$, then we can use Poincaré's inequality in the form ([8], p. 355):

$$\int(\theta-\theta_0)^2\, dA + \int \theta_{,i}\theta_{,i}\, dV \geqq p_1 \int(\theta-\theta_0)^2\, dV,$$

and again conclude (18) from (17) with $a = \min(c_3, c_4)\, p_1$. Putting (16) and (18) into (7) and recalling $(10)_8$, we arrive at (9) with

$$d = \min\left(\frac{c_1 p_1 p_2}{\rho}, \frac{c_2 p_1 p_2}{\rho j}, \frac{a}{\rho c_5}\right)$$

Since p_1 and p_2 depend upon the region R, i. e, the size and the shape of the container and the values of c_1, c_2, etc. depend upon the heat conduction and viscosity coefficients, the value of d depends upon the size and the shape of the container, the density ρ, the microinertia j, and the heat conduction and the viscosity coefficients.

3. Remarks

We have assumed here the existence of a classical solution of (1) to keep the analysis simple. Otherwise, one can suitably define a weak solution and arrive at (9) essentially by using the same arguments (*cf.* [9] and [10]). The problem of existence of solutions of (1) has not been studied so far. Recently, Lange [11] studied the existence of solutions of initial-boundary value problems for equations which describe the homothermal flow of incompressible micropolar fluids.

The estimates of constants p_1 and p_2 appearing in the Poincaré inequality and the Korn inequality can be improved by requiring that the functions also satisfy the field equations (1) and (2). But this is rather a hard problem. For estimates of Korn's constants the reader is referred to the work of Bernstein and Toupin [12], Payne and Weinberger [13] and Dafermos [14].

We note that because of the assumed boundary condition $(3)_2$, the heat conduction coefficients $\bar{\alpha}$ and $\bar{\beta}$ do not appear in (13) and (17). Thus for the problem discussed herein the decay rate of the energy does not depend upon $\bar{\alpha}$ and $\bar{\beta}$ and this is why we did not require that $\bar{\alpha}$ and $\bar{\beta}$ satisfy any inequalities. However, should the container be partially filled with the fluid or there be some slipping at the contact surface, $(3)_{1,2}$ will not hold on the entire boundary and terms involving $\bar{\alpha}$ and $\bar{\beta}$ will appear in (13) and (17). We remark that Batra ([9], [15], [16]) has studied the stability of non-polar heat conducting fluids partially filling the container.

REFERENCES

[1] LAKSHMANA RAO S. K., *Decay of the Kinetic Energy of Micropolar Incompressible Fluids* (*Quart. Appl. Math.*, Vol. 27, 1969, pp. 278-280).

[2] SHAHINPOOR M. and AHMADI G., *Decay of the Kinetic Energy of Compressible Micropolar Fluids* (*Int. J. Engng. Sc.*, Vol. 11, 1973, pp. 885-889).

[3] ERINGEN A. C., *Theory of Thermomicrofluids* (*J. Math. Anal. Appl.*, Vol. 38, 1972, pp. 480-496).

[4] AHMADI G., *Universal Stability of Thermo-Magneto-Micropolar Fluid Motions* (*Int. J. Engng. Sc.*, Vol. 14, 1976, pp. 853-859).

[5] KELLOG O. D., *Foundations of Potential Theory* (Dover Publications, Inc., New York, 1954).

[6] CAMPANATO S., *Sui Problemi al Contorno per Sistemi di Equazioni Differenziali Lineari del tipo dell'elasticita*, Part 1 (*Anali Scuola Normale Sup. di Pisa*, Vol. 13, 1959, pp. 223-258).

[7] BATRA R. C., *Decay of the Kinetic and the Thermal Energy of Compressible Micropolar Fluids* (*Acta Mechanica*, in press).

[8] SMIRNOV V. I., *A Course of Higher Mathematics* (Addison-Wesley Publishing Co., Inc., Reading, Vol. 5, 1969 : Translated from Russian by D. E. BROWN).

[9] BATRA R. C., *A Theorem in the Theory of Incompressible Navier-Stokes-Fourier Fluids* (*Istituto Lombardo* (*Rend. Sc.*) A, Vol. 107, 1973, pp. 699-714).

[10] BATRA R. C., *On the Asymptotic Stability of the Rest State of a Viscous Fluid Bounded by a Rigid Wall and an Elastic Membrane* (*Arch. Rational Mech. Anal.*, Vol. 56, 1974, pp. 310-319).

[11] LANGE H., *Die Existenz von Losungen der Gleichungen, Welch die Stromung inkompressibler mikropolarer flüssigkeiten beschreiben* (*Z.A.M.M.*, Vol. 56, 1976, pp. 129-139).

[12] BERNSTEIN B. and TOUPIN R. A., *Korn Inequalities for the Sphere and Circle* (*Arch Rat. Mech. Anal.*, Vol. 8, 1960, pp. 51-64).

[13] PAYNE L. E. and WEINBERGER H. F., *On Korn's Inequality* (*Arch. Rat. Mech. Anal.*, Vol. 8, 1961, pp. 89-98).

[14] DAFERMOS C. M., *Some remarks on Korn's Inequality* (*Z. angew. Math. Phys.*, Vol. 19, 1968, pp. 913-920).

[15] BATRA R. C., *Decay of the Kinetic and the Thermal Energy of Compressible Viscous Fluids* (*J. de Mécanique*, Vol. 13, 1975, pp. 497-503).

[16] BATRA R. C., *On the Asymptotic Stability of an Equilibrium Solution of the Boussinesq Equations* (*Z.A.M.M.*, Vol. 55, 1975, pp. 727-729).

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